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Perfect Domination in Graph

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ABSTRACT

In this paper we characterized a vertex whose removal increases the perfect domination number of a graph. We also consider the pendent vertices whose removal decreases the perfect domination number.

Introduction

Perfect domination is closely related to Perfect Codes. Perfect Codes have been used in Coding Theory. In this paper we study the effect of removing a vertex from the graph on perfect domination. In particular we characterize those vertices whose removal increases the perfect domination number of a graph.

Preliminaries

Definition-1: Perfect Dominating Set [2].

A subset S of $V(G)$ is said to be a perfect dominating set if for each vertex v not in S, v is adjacent to exactly one vertex of S.

Definition-2: Minimal Perfect Dominating Set.

A perfect dominating set S of the graph G is said to be minimal perfect dominating set if for each vertex v in S, $S-\{v\}$ is not a perfect dominating set.

Definition-3: Minimum Perfect Dominating Set.

A perfect dominating set with smallest cardinality is called minimum perfect dominating set. It is called γ_{pf} set of the graph G.

Definition-4: Perfect Domination Number.

The cardinality of a minimum perfect dominating set is called the perfect domination number of the graph G.It is denoted by $\gamma_{pf}(G)$.

Definition-5: Perfect Private Neighborhood.

Let S be a subset of $V(G)$ and $v \in S$. Then the perfect private neighborhood of v with respect to S is denoted as $P_{pf}[v,S]$ and defined as

$$P_{pf}[v,S] = \{w \in V(G)-S; N(w)\cap S = \{v\}\} \cup \left\{\begin{matrix} v, \text{if } v \text{ is adjacent to no} \\ \text{vertex of } S \\ \text{or} \\ v \text{ is adjacent to at least} \\ \text{two vertices of } S \end{matrix}\right\}$$

Main Results

Theorem-1: A perfect dominating set S of G is minimal perfect dominating set if and only if for each vertex v in S $P_{pf}[v,S]$ is non-empty.

Proof: Suppose S is minimal perfect dominating set of G and $v \in S$. Therefore there is a vertex w not in $S-\{v\}$ such that either w is adjacent to no vertex of $S-\{v\}$ or w is adjacent to at least two vertices of $S-\{v\}$.

If $w = v$ then this implies that $v \in P_{pf}[v,S]$.

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If $w \neq v$ then it is impossible that w is adjacent to at least two vertices of $S-\{v\}$, because S is perfect dominating set. Therefore w is not adjacent to any vertex of $S-\{v\}$. Since S is a perfect dominating set and w is adjacent to only v in S. That is $N(w)\cap S=\{v\}$. Thus, $w \in P_{pf}[v,S]$.

Conversely suppose $v\in S$ and $P_{pf}[v,S]$ contains some vertex of w of G.

If $w=v$ then w is either adjacent to at least two vertices of $S-\{v\}$or w is adjacent to no vertex of $S-\{v\}$. Thus, $S-\{v\}$ is not a perfect dominating set.

If $w\neq v$ then $N(w)\cap S=\{v\}$ implies that w is not adjacent to any vertex of $S-\{v\}$.

Thus, in all cases $S-\{v\}$ is not a perfect dominating set if $v\in S$. Thus, S is minimal perfect dominating set of the graph$G$. ■

Now we define the following symbols.

$$V_{pf}^{+}=\{v\in V(G):\ \gamma_{pf}(G)<\gamma_{pf}(G-v)\}.$$
$$V_{pf}^{-}=\{v\in V(G):\ \gamma_{pf}(G)>\gamma_{pf}(G-v)\}.$$
$$V_{pf}^{0}=\{v\in V(G):\ \gamma_{pf}(G)=\gamma_{pf}(G-v)\}.$$

Note that the above sets are mutually disjoint and their union is $V(G)$.

Lemma-2: Let $v\in V(G)$ and suppose v is a pendent vertex and has a neighbor w of degree at least two. If $v\in V_{pf}^{-}$ then

$$\gamma_{pf}(G-v)=\gamma_{pf}(G)-1.$$

Proof: Let S_1 be a minimum perfect dominating set of $G-\{v\}$. If $w\in S_1$ then S_1 is a perfect dominating set of G with $|S_1|<\gamma_{pf}(G)$.That is $\gamma_{pf}(G)\le|S_1|<\gamma_{pf}(G)$, this is a contradiction. Therefore $w\notin S_1$. Let $S=S_1\cup\{w\}$. Then S is a minimum perfect dominating set of G. Then S is a minimum perfect dominating set of G. Therefore $\gamma_{pf}(G)=|S|=|S_1|+1=\gamma_{pf}(G-v)+1$. ■

Theorem-3: Let v be a vertex of G then $v\in V_{pf}^{+}$ if and only if the following conditions are satisfied.

(1) v belongs to every γ_{pf}set of G.

(2) No subset S of $G-\{v\}$ which is either disjoint from $N[v]$ or intersects $N[v]$ in at least two vertices and $|S|\le\gamma_{pf}(G)$ can be a perfectly dominating set of $G-\{v\}$.

Proof: (1) Suppose $v\in V_{pf}^{+}$. Let S be a γ_{pf} set of G which does not contain v then S is a perfect dominating set of $G-\{v\}$. Therefore $\gamma_{pf}(G-v)\le|S|=\gamma_{pf}(G)$. Thus, $v\notin V_{pf}^{+}$. This is a contradiction. Thus v must belong to every γ_{pf} set of G.

(2) If there is a set S which satisfies the condition stated in (2). Then S is a perfect dominating set of $G-\{v\}$ and therefore $\gamma_{pf}(G-v)\le\gamma_{pf}(G)$.This is a contradiction.

Conversely assume that conditions (1) and (2) hold.

Suppose $v\in V_{pf}^{0}$. Let S be a minimum perfect dominating set of $G-\{v\}$.Then $|S|=\gamma_{pf}(G)$.

Suppose v is not adjacent to any vertex of S. Then S is disjoint from $N[v]$, $|S|=\gamma_{pf}(G)$ and S is a perfect dominating set of $G-\{v\}$.This violates (2).

Suppose v is adjacent to exactly one vertex of S then S is a minimum perfect dominating set of G not containing v which violates (1).

Suppose v is adjacent to at least two vertices of S. Then $S\cap N[v]$ in at least two vertices and S is a perfect dominating set of $G-\{v\}$ with $|S|=\gamma_{pf}(G)$, which again violates (2). Thus, $v\in V_{pf}^{0}$ implies (1) or (2) violated.

Suppose $v\in V_{pf}^{-}$. Let S_1 be a minimum perfect dominating set of $G-\{v\}$. Then $|S_1|<\gamma_{pf}(G)$. If v is not adjacent to any vertex of S_1 then as above (2) is violated. If v is adjacent to exactly one vertex of S_1 then $S_1$is a perfect dominating set of G with $|S_1|<\gamma_{pf}(G)$, which is contradiction.

If v is adjacent to at least two vertices of S_1 then $S_1\cap N[v]$ in at least two vertices, $|S_1|\le\gamma_{pf}(G)$ and $S_1$is a perfect dominating set of $G-\{v\}$, which again violates (2).Thus, $v\in V_{pf}^{-}$ implies that (2) is violated.

Thus, v does not belongs to $V_{pf}^{0}$and V_{pf}^{-}. Hence $v\in V_{pf}^{+}$. ■

Theorem-4: Let v be a pendent vertex which has the neighbor w of degree at least two then $v\in V_{pf}^{-}$ if and only if there is γ_{pf} set S containing w and not containing v such that $P_{pf}[w,S]=\{v\}$.

Proof: Suppose $v\in v_{pf}^{-}$Let S_1 be a minimum perfect dominating set of $G-\{v\}$. Then by Lemma-2, $w\not\in S_1$. Let $S=S_1\cup\{w\}$. Then S is γ_{pf} set containing w.

Since S_1 is a perfect dominating set of $G-\{v\}$, w is adjacent to some vertex of S_1. Therefore $w\not\in P_{pf}[w,S]$. If x is any vertex different from v such that x is adjacent to w then x is also adjacent to some vertex of S_1 because S_1 is a perfect dominating set of $G-\{v\}$. Thus, $x\not\in P_{pf}[w,S]$. Further v is adjacent to only w of S therefore $P_{pf}[w,S]=\{v\}$.

Conversely, suppose there is a γ_{pf} set S containing w such that $P_{pf}[w,S] = \{v\}$. Let $S_1 = S - \{w\}$. Let x be any vertex of $G - \{v\}$which is not in $S - \{v\}$. Since $x \notin P_{pf}[w,S]$, x must be adjacent to some unique vertex of S_1. Thus, S_1 is a minimum perfect dominating set of $G - \{v\}$ with $|S_1| < \gamma_{pf}(G)$. Thus, $v \in v_{pf}^{-}$. ■

Theorem-5: Let S_1 and S_2 be two disjoint perfect dominating sets of Gthen $|S_1| = |S_2|$.

Proof: For every vertex x in S_1 there is a unique vertex $v(x)$ in S_2 which is adjacent to x. Also for every vertex y in S_2there is a unique vertex $u(y)$ in S_1 which is adjacent to y. It may be noted these functions are inverse of each other. Therefore$|S_1| = |S_2|$. ■

Corollary-6: If in a graph G there are perfect dominating sets S_1 and S_2 such that $|S_1| \neq |S_2|$ then $S_1 \cap S_2 \neq \emptyset$. ■

Corollary-7: Let G be a graph with n vertices. If there is a perfect dominating set S with $|S| < \frac{n}{2}$ or $|S| \geq \frac{n}{2}$ then $V(G) - S$ is not a perfect dominating set. ■

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