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Solution of Third Order quasi-linear PDE's Containing Arbitrary Functions by Using Symmetry Group Approach

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ABSTRACT
In this paper the symmetry classification of third order quasi-linear PDE's containing arbitrary function have been studied with derailed examples.

1. Introduction

The studies and analysis of a symmetry properties of a differential equation make sense in solution of differential equations see ([1],[7],[8],[10]) . The symmetry classification of an equation which contain arbitrary function dependent on how we can choice these function , so , the symmetry properties give these selections ([1],[5],[9]).

In this paper , the third order of PDE's that give below with one are more $F_\ell(u)$ for unknown u, we give the sufficient condition to be the admits symmetry is not trivial. Two example sported the idea and procedure .

2. Preliminary results

For the sake of simplicity we consider only second order equations for the unknown function $u = u(x,y)$ of two independent variables x,y (but the extension to more general cases is completely straightforward), and we will deal with quasi-linear PDE's of the following form

$$a_{11}u_{xxx} + a_{12}u_{xxy} + a_{21}u_{yxy} + a_{22}u_{yyy} + b_1u_{xx} + b_2u_{xy} + b_3u_{yy} = \sum_{\ell=1}^{l} \alpha_\ell F_\ell(u) \quad (1)$$

or in a short-hand notation

$$\varepsilon[u] = \alpha_\ell F_\ell(u)$$

where $a_{ij} = a_{ij}(x,y)$, $b_i = b_i(x,y), \alpha_\ell = \alpha_\ell(x,y)$ are given (smooth) function, and $F_\ell(u)$ are L arbitrary (smooth) functions of u (in the examples below we will deal with just one or two functions $F_\ell(u)$). It is understood that no linear relations exist between α_ℓ and between F_ℓ. we will exclude from our consideration the rather trivial case when $F_\ell(u)$ are linear functions of u, which usually can be more simply and conveniently discussed separately by means of direct calculations.

we will denote their Lie generator by

$$X = \xi(x,y,u)\frac{\partial}{\partial x} + \eta(x,y,u)\frac{\partial}{\partial y} + \varphi(x,y,u)\frac{\partial}{\partial u}$$

$$X^3(\Delta)|_{\Delta=0} = 0$$

where $\Delta = \varepsilon[u] - \alpha_\ell F_\ell(u)$ and X^3 is the third prolongation of X .

lemma 1

For any choice of the functions F_ℓ , the coefficients of the Lie point symmetry operators admitted by a PDE of the form (1) satisfy the conditions $\xi_u = \eta_u = 0 , \varphi_{uuu} = 0$ i.e

$$\xi = \xi(x,y), \eta = \eta(x,y), \varphi = A(x,y) + uB(x,y) + \frac{u^2}{2}C(x,y) \quad (2)$$

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now we show that $X^3(\Delta)|_{\Delta=0} = 0$

1. $\xi(x,y)[\varepsilon[u] - \alpha_\ell F_\ell(u)] = -\alpha_\ell F_\ell(u) = -\alpha_\ell \xi_x F_\ell(u) - \xi \alpha_{\ell,x} F_\ell(u)$

2. $\eta(x,y)[\varepsilon[u] - \alpha_\ell F_\ell(u)] = -(\alpha_\ell \eta F_\ell(u)) = -\alpha_\ell \eta_y F_\ell(u) - \eta \alpha_{\ell,y} F_\ell(u)$

3. $A(x,y) + uB(x,y) + \frac{u^2}{2} C(x,y)[\varepsilon[u] - \alpha_\ell F_\ell(u)]$

$$A(x,y)[\varepsilon[u] - \alpha_\ell F_\ell(u)] = \varepsilon[A] - \alpha_\ell A F'_\ell(u)$$

$$uB(x,y)[\varepsilon[u] - \alpha_\ell F_\ell(u)] = u\varepsilon[B] - \alpha_\ell u B F'_\ell(u) - \alpha_\ell B F_\ell(u)$$

$$\frac{u^2}{2} C(x,y)[\varepsilon[u] - \alpha_\ell F_\ell(u)] = \frac{u^2}{2}\varepsilon[C] - \alpha_\ell \frac{u^2}{2} C F'_\ell(u) - \alpha_\ell u C F_\ell(u)$$

with $F'_\ell = \frac{dF_\ell}{du}$, *sum over*$(\ell = 1, \dots, L)$

$$p_\ell(x,y) F_\ell(u) + p_{L+1}(x,y) u F_\ell(u) + p_{2L+1}(x,y) F'_\ell(u) + p_{3L+1}(x,y) u F'_\ell(u) + p_{3L+2}(x,y) u^2 F'_\ell + p_{4L+1}(x,y) u^2 + p_{4L+2}(x,y) u + p_{4L+3}(x,y) = 0 \qquad (3)$$

where the coefficient function $p_i(x,y)$ $(i = 1, \dots, 4L+3)$ are given by

$$p_\ell = -(\xi_x + \eta_y)\alpha_\ell - \xi \alpha_{\ell,x} - \eta \alpha_{\ell,y} \quad , p_{L+1} = -\alpha_\ell C \quad , p_{2L+1} = -\alpha_\ell A$$

$$P_{3L+1} = -\alpha_\ell B \quad , P_{3L+2} = -\alpha_\ell \frac{C}{2} \quad , P_{4L+1} = \frac{1}{2}\varepsilon[C]$$

$$P_{4L+2} = \varepsilon[B] \quad , P_{4L+3} = \varepsilon[A] \qquad (4)$$

with $\xi_x = \frac{\partial \xi}{\partial x}$, $\alpha_{\ell,x} = \frac{\partial \alpha_\ell}{\partial x}$ *etc.* $(\ell = 1, \dots, L)$

considering now the determining equation (3) , and observing that the p_i depend only on x,y and not on u, one immediately realizes that , if the 4L+3 functions f_i defined by

$$f \equiv (F_\ell, uF_\ell, F'_\ell, uF'_\ell, u^2 F'_\ell, u^2, u, 1) \quad (\ell = 1, \dots, L) \qquad (5)$$

are linearly independent , then (3) can be satisfied if and only if

$$p_i = 0 \qquad (i = 1, \dots, 4L+3) \qquad (6)$$

Recalling now the definition of kernel of the full (or principal) symmetry groups [1] of eq. (1), i.e. the intersection of all symmetry groups admitted by (1) for any arbitrary choice of $F_\ell(u)$, we can then state the following property.

Lemma 2

consider (6) characterize the kernel of the symmetry groups of equation (1).

Indeed , condition (6), together with the other determining equations (not involving F_ℓ) , determine the functions ξ, η, A, B, C (i.e. the symmetries admitted by equation (1)), which are independent of the choice of the functions F_ℓ. These symmetries may be considered trivial in this context for instance, if all coefficients a,b,α in (1) are independent of y, then such a symmetry operator is $\partial/\partial y$. Some not so obvious examples of symmetries of this type will be presented later .

Therefore, a first conclusion is that , in order to have nontrivial symmetries (i.e. really dependent on the choice of the functions F_ℓ), a necessary condition is the existence of some linear dependence among the functions (5). Another relevant remark which will emerge from our discussion is the important role played also by the coefficient functions $\alpha_\ell(x,y)$ in the determination of the admitted symmetries.

3. Conditions for the existence of symmetries .

Consider the linear space generated by the 4L+3 functions f_i defined in (5) , and , according to our above remarks, now assume that there are some linear relations among these functions. Then the f_i span a space with dimension $k < D = 4L+3$ if this is the case, D coefficients p_i are forced, according to (3), to belong to the orthogonal $(D-K)$-dimensional subspace (with respect to the standard scalar in R^D), and the functions $p_i(x,y)$ turn out to be subjected to k linear conditions. For instance, if there is just one linear relationship between the f_i , say

$$\sum_{i=1}^{D} \lambda_i f_i \qquad (7)$$

where not all the constants λ_i vanish , then $k = D - 1$ and the functions p_i span a 1-dimensinal subspace and must satisfy $D - 1$ equations of the form (assuming that , e.g, $\lambda_D \neq 0$)

$$p_1\lambda_D = \lambda_1 P_D \ ,p_2\lambda_D = \lambda_2 P_D \ ,\dots ,p_{D-1}\lambda_D = \lambda_{D-1}P_D \ . \quad (8)$$

we can then state our main conclusion , which characterizes the crucial determining equation which contains the functions F_ℓ in the following "geometrical" form.

proposition 1

Equation (1) admits nontrivial symmetries only if the $D = 4L + 3$ function (5) are linearly dependent. If this is the case , the D functions p_i(x,y) given by (4) appearing in the determining equation (3) span the subspace orthogonal (with respect to the standard scalar product in R^D) to the k –dimensional $(K < D)$ subspace spanned by the function (5). The admitted symmetries are completely determined by imposing this orthodonality condition to the coefficient functions p_i(x,y), together with the other determining equations not involving the functions F_ℓ(u).

Before considering explicit examples, let us remark that the complete symmetry classification must be accompanied by the determination of the equivalence group [1] , i.e. the group of the transformations which leave invariant the differential structure of the PDE. Standard calculations show easily that , for any fixed choice of the functions $a_{ij}\ ,b_i\ ,\alpha_\ell$ in equation (1), the equivalence group includes in particular , expectedly , the scalings $u \longrightarrow cu\ ,\ F_\ell \longrightarrow c\,F_\ell$ and the translation $u \longrightarrow u + c_0 (c, c_0 = const.)$. Other transformations involving also the variables x,y can appear for particular choices of the functions $a_{ij}\ ,b_i\ ,\alpha_\ell$. The transformations belonging to the equivalence group will play an important role in performing the complete symmetry classification of our equations.

4. First example: a generalized Laplace equation:

To illustrate the main idea and the procedure , and also to clarify some details , we are now going to examples , which can be noteworthy also for their different and interesting peculiarities. We start considering the simplest case , where the r.h.s. of (1) contains only term $\alpha_\ell(x,y)F(u)$

(then D=8).

First of all , let us remark that , independent of the form of the equation arbitrarily. For instance, no relation exists between u^2,u and 1, and also , having excluded the trivial case of linear F , u^2,u ,1 . On the other hand , the necessary linear dependence between the function f_i implies immediately that the presence of some symmetry is possible only if $F(u)$ is an exponential or a power of u. It can also happen that more than one linear relation holds: e.g., if $F = \frac{u^3}{3} + u,\ \ then\ both\ relations$

$$F - \frac{u^3}{3} - u = 0$$

$$F' - u^2 - 1 = 0 \ \ and\ \ 3F - uF' - 2u = 0 \quad (9)$$

let use consider the following generalization of the classical nonlinear Laplace equation

$$\Delta \equiv\ \nabla^3 u - \alpha(x,y)F(u) = 0 \quad (10)$$

where $\alpha(x,y)$ is a given function. In this case , the determining equations not containing F imply in particular

$$\xi_x = \eta_y \qquad ,\xi_y = -\eta_x \qquad , \qquad C = \ const \quad (11)$$

whereas the crucial determining equation (3) involving F is

$$p_1F + p_2uF + p_3F' + p_4uF' + p_5u^2F' + p_6u^2 + p_7u + p_8 = 0 \quad (12)$$

with the coefficient functions $p_i(x,y)$ given by

$$p_1 = -(\xi_x + \eta_y)\alpha_\ell - (\xi\,\alpha_{\ell,x} + \eta\,\alpha_{\ell,y}) \quad , \ p_2 = -\alpha_\ell C \quad ,p_3 = -\alpha_\ell A$$

$$p_4 = -\alpha_\ell B \quad ,p_5 = -\alpha_\ell \frac{C}{2} \quad ,p_6 = \nabla^3 C = 0 \quad ,p_7 = \nabla^3 B$$

$$p_8 = \nabla^3 A \quad (13)$$

Let us first discuss the kernel group. The conditions $p_i = 0\ (i = 1, \dots, 8)$ characterizing the transformations in the kernel group give now $A = B = C = \varphi = 0$ and the condition $p_i = 0$,which now reads $\alpha_\ell(\xi_x + \eta_y) + (\xi\,\alpha_{\ell,x} + \eta\,\alpha_{\ell,y}) = 0$. Introducing a harmonic function $\phi = \phi(x,y)$ such that $\phi_x = \xi \quad ,\phi_y = -\eta$

this condition can be more conveniently transformed into an equation for the single unknown ϕ:

$$\alpha_\ell(\phi_{xx} - \phi_{yy}) + \alpha_{\ell,x}\phi_x - \alpha_{\ell,y}\phi_y = 0 \quad (14)$$

Solution of this equation clearly depend on the choice of the function $\alpha_\ell(x,y)$. For instance , if α_ℓ = const., it gives $\xi_x+\eta_y=0$, which , together with (11) , implies that the symmetries in the kernel group are , as expected , only translations and rotations of the variables x,y . If $\alpha=\exp(2x)$ then $\phi=\exp(-x)(c_1\cos y+c_2\sin y)+\ c_3y+c_4$ and the kernel contains , apart from the translation generated by $\partial/_{\partial y}$, the transformations generated by

$$X_1=\exp(-x)\left(\cos y\frac{\partial}{\partial x}-\sin y\frac{\partial}{\partial y}\right)\ ,\qquad X_2=\exp(-x)\left(\sin y\frac{\partial}{\partial x}+\cos y\frac{\partial}{\partial y}\right).$$

With $\alpha=x^r$, one has that if $r\neq-2$ then the kernel contains only the translation generator $\partial/_{\partial y}$, whereas if $r=-2$ it also contains the transformations generated by two operators

$$X_1=2xy\frac{\partial}{\partial x}-(x^2-y^2)\frac{\partial}{\partial y}\quad and\ \ X_2=x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}.$$

The first transformation describes the kernel group even if $\alpha=x^{-2}\beta(\frac{y}{x^2}+y^2))$, where β is an arbitrary function.

It can be remarked, incidentally, that if we reverse the argument for a moment, one has that: given any harmonic function ϕ (and then any couple of harmonic conjugate functions ξ,η), there are some $\alpha(x,y)$ which solve equation (14), and then, with these functions α, the kernel group contains precisely the symmetry generated by the operator $X=\xi\left(\frac{\partial}{\partial x}\right)+\eta\left(\frac{\partial}{\partial y}\right)$.

Let us now finally perform the symmetry classification of eq.(10). As already remarked, its equivalence group may contain and depending on the specific choice of the function α , other transformations possibly involving also x,y. As we shall see , however, these are not relevant for the symmetry classification of equation (10).

According to our procedure, it is immediately seen that just one linear relation between the eight f_i can exist . For instance, in the case $F=\frac{u^3}{3}+u$ mentioned above, admitting the two linear relations (9),would lead to $p_i=0$, i.e. only the kernel symmetries . We then assume the existence of a single linear relation:

$$\lambda_1F+\lambda_2uF+\lambda_3F'+\lambda_4uF'+\lambda_5u^2F'+\lambda_6u^2+\lambda_7u+\lambda_8=0 \qquad (15)$$

with not all λ_i equal to zero. Observing that $p_7=0$, one has from (8) that λ_7. We now distinguish the cases $\lambda_4\neq0$ and $\lambda_4=0$.

Let $\lambda_4\neq0$. It is not restrictive to put $\lambda_4=1$, and (up to a translation of u) $\lambda_3=0$, which implies $\lambda_8p_3=0$. If $\lambda_8\neq0$, then $p_3=0$ would imply A = 0 and also $p_8=0=p_4\lambda_8$, now, if $p_2=\ p_6=0$ it remains only $p_1\neq0$ and from $\lambda_8p_1=\lambda_1p_8=0$ one concludes that $\lambda_8=0$. Therefore, we get $\lambda_1F+uF'=o,$ i.e.

$$F(u)=u^k with\ \ k=-\lambda_1\qquad(k\neq0,1).$$

Using now (8) , which become $p_1+kp_4=0,\ p_2=p_3=p_5=p_6=p_7=0$, we get $A=C=O$ and

$$k\alpha_\ell B+(\xi_x+\eta_y)\alpha_\ell+(\xi\alpha_{\ell x}+\eta\alpha_{\ell y})=0.$$

The last equation relates the symmetry coefficients ξ,η,B with the specific form of the function $\alpha_\ell(x,y)$. If for instance $\alpha=x^r$, then $kB+\xi_x+\eta_y=0,$ but $B\ most\ be\ \neq0,$ otherwise also $p_4=p_1=0,$ i.e. the kernel group. Therefore , ξ must be proportional to x and equation (10) admits the symmetry operator

$$X=k\left(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}\right)-(r+2)u\frac{\partial}{\partial u}$$

(and obviously the translation of the variable and $y,$ also the translation of x and the rotations of x,y in the case $r=0$, i.e. if $\alpha=const.$).

Let now $\lambda_4=0.$ Then necessarily $\lambda_3\neq0,$ and one can put $\lambda_3=1$ and also $\lambda_1=1$ (possibly up to a scaling of u). Assume first $\lambda_8=0,$ therefore, from (15),

$$F(u)=\exp(-u)$$

and the conditions (8),(13) for the functions p_i become now

$$p_4=-\alpha B=0\ \ ,p_8=\nabla^3u=0\ \ and\ \ p_1=p_3\ \ i.e.$$

$$(\xi_x+\eta_y)\alpha_\ell+(\xi\alpha_{\ell x}+\eta\alpha_{\ell y})=\alpha_\ell A.$$

As before , we can consider some examples. If $\alpha=$ const.the last equation $\xi_x+\eta_y=A$ and then the most general symmetry of the equation $\nabla^3u=\exp(-u)$ is

$$X=\xi\frac{\partial}{\partial x}+\eta\frac{\partial}{\partial y}+(\xi_x+\eta_y)\frac{\partial}{\partial u}$$

where ξ, η are arbitrary harmonic conjugate functions: this is the well known case of the classical Liouville equation , and the translation $\frac{\partial}{\partial x}$. Assume now $\lambda_8 \neq 0$, then

$F(u) = \exp(-u) - \lambda_8$.

Introducing the transformation $u \to u + \tilde{u}$, where $\tilde{u} = \tilde{u}(x,y)$ satisfies the equation $\nabla^3 \tilde{u} + \lambda_8 \alpha = 0$, one obtains the new equation

$\nabla^3 u - \tilde{\alpha}\ \exp(-u) = 0$

where $\tilde{\alpha}(x,y) = \alpha \exp(-\tilde{u})$, which has precisely the same form as the equation considered before. Without repeating details, it can be interesting to provide just one illustrative example . Let

$$\nabla^3 u - \frac{k}{x^3}(\exp(-u) + 1) = 0$$

where $k = const.$ It is easy to see that if $k = 3$ this equation admits the symmetries

$$X = \xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial x} + (\xi_x + \eta_y - \frac{3}{x^2}\xi)\frac{\partial}{\partial u}$$

where ξ, η are arbitrary harmonic conjugate function , if instead $k \neq 3$,

The admitted symmetries are only those in the kernel group.

The above results concerning eq.(10) can be stated in a complete form as follows.

proposition 2

Give a function $\alpha = \alpha(x,y)$, consider this equation for the harmonic function $\phi = \phi(x,y)$

$$\alpha_\ell(\phi_{xx} + \phi_{yy}) + \alpha_{\ell,x}\phi_x - \alpha_{\ell,y}\phi_y - \alpha_\ell C = 0 \qquad (16)$$

Let $\xi = \phi_x$, $\eta = -\phi_y$. Assume first $C = 0$, for any solution ϕ of (16), the kernel group of the generalized Laplace equation (10) is generated by the symmetry operator

$$X = \xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}.$$

If $F(u) = u^k$, for any solution ϕ of (16) with C = const. $\neq 0$, eq. (10) admits the symmetry operator

$$X = k\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) + C(x,y)\frac{\partial}{\partial u}.$$

In particular , if $\alpha_\ell = const.$ then $C = \xi_x + \eta_y$ where ξ, η are arbitrary harmonic conjugate function , and the case of the standard elliptic Liouville equation is recovered . If finally $F(u) = \exp(-u) + c\ (c = const.)$, the above case is recovered by means of the transformation $u \to u + \tilde{u}$, where $\tilde{u} = \tilde{u}(x,y)$ satisfies the equation $\nabla^3 \tilde{u} - c\alpha_\ell = 0$. This completes the symmetry classification of the PDE (10), apart from the transformations in the equivalence group.

5. An example with two arbitrary functions

We now consider the case of a PDE of the form (1) with two arbitrary function $F_\ell(u)$, i.e. $L = 2$, $D = 13$. To avoid excessive generality, let us restrict our study to a PDE of the following form

$$u_{xxx} + u_{yyy} + \frac{a}{x^2}u_{xx} = \alpha(x,y)F_1(u) + F_2(u) \qquad (17)$$

here $b_1 = \frac{a}{x^2}$, $a \neq 0$ is a constant , $\alpha_1 = \alpha(x,y)$ a given function and $\alpha_2 = 1$. The choice of this equation is motivated and suggested by the theory of plasma physics: it is indeed a generalization of the Grad- Schluter-Shafranov equation (see[6]), which is obtained putting in (17) $a = -1$, $\alpha = x^3$, and describes the magnetohydrodynamic force balance in a magnetically confined toroidal plasma. In this context, u is the so-called magnetic flux variable, x is a radial variable (then $x \geq 0$), while the two arbitrary functions $F_1(u), F_2(u)$ are flux functions related to the plasma pressure and current density profiles.

The determining equations not involving $F_\ell(u)$ give in this case

$$\xi_x = \eta_y , \xi_y = -\eta_x, \quad and \quad B = \frac{3\xi}{x} + b \quad , b = const.$$

First of all , the kernel group is immediately seen to be trivial (apart obviously from the translation generated by $\frac{\partial}{\partial y}$, in the case where α depends only on x ; the possible presence of this symmetry will be tacitly understood in the following). Indeed , from $p_i = 0$(see Lemma 2), one has $A = B = C = 0$, then the above equation implies $\xi = \left(\frac{-b}{3}\right)x$, and the condition $p_3 = 0\ with\ \alpha_2 = 1$ gives finally $\xi = 0$.

Let us now start assuming that there are exactly two linear relations involving the functions $F_1\ and\ F_2$ separately:

$$\lambda_1 F_1 + \lambda_3 u F_1 + \lambda_5 F'_1 + \lambda_7 u F'_1 + \lambda_9 u^2 F'_1 + \lambda_{11} u^2 + \lambda_{12} u + \lambda_{13} = 0$$

$$\lambda_2F_2+\lambda_3uF_2+\lambda_6F'_2+\lambda_8uF'_2+\lambda_{10}u^2F'_2+\mu_{11}u^2+\mu_{12}u+\mu_{13}=0 \quad (18)$$

with not all λ_i,μ_i equal to zero.

Let $\lambda_7\lambda_8\neq 0,$ and put $\lambda_7=\lambda_8=1.$ According to proposition 1, the symmetry coefficients p_i , given by (4), satisfy then the six linear conditions

$$p_1=\lambda_1p_7\,,\ p_2=\lambda_2p_8\,,p_3=\lambda_3p_7\,,p_4=\lambda_4p_8\ ,p_5=\lambda_5p_7\,,p_6=\lambda_6p_8\,, \quad (19)$$
$$p_9=\lambda_9p_7\quad ,p_{10}=\lambda_{10}p_8\ \ ,p_{11}=\lambda_{11}p_7+\mu_{11}p_8\ ,p_{12}=\lambda_{12}p_7+\mu_{12}p_8\ ,p_{13}=\lambda_{13}p_7+\mu_{13}p_8$$

With $\lambda_7\neq 0$, we can put $\lambda_5=0,$ up to a translation of u. conditions (19) and the expression of the coefficients p_i give $p_5=0, A=0, then\ p_6=p_{13}=0$ and therefore also $\lambda_6=\lambda_{13}=\mu_{13}=0$ (indeed $p_7p_8\neq 0,$ otherwise all $p_i=0$); condition $p_2=\lambda_2p_8$ implies that $\xi(x,y)$ must satisfy an equation of the form $\xi_x=k_0\frac{\xi}{x}+k_1$ $(k_0\,,k_1=$ const) which admits harmonic solution only of the form $\xi=cx\,,c=const.$ On the other hand , $\lambda_1=\frac{p_1}{p_7}\,,\lambda_2=\frac{p_2}{p_8}$ imply that α is forced to satisfy

$$\frac{x\alpha_x+y\alpha_y}{\alpha}=r=const. \quad (20)$$

This means that if α does not satisfy this condition , no symmetry is allowed; we then assume for α the form

$$\alpha(x,y)=x^r\beta(\frac{y}{x})$$

where β is arbitrary. Notice that , with α of this form , a new transformation is included in the equivalence group , namely the scaling $x\to cx\ \ ,\ y\to cy\ \ ,F_1\to c^{3-r}F_1\ \ ,F_2\to c^{-3}F_2$. We also deduce $B=const\neq 0$, $p_{12}=0$, which implies in turn $\lambda_{12}=\mu_{12}=0.$ Then we are left with

$$\lambda_1F_1+uF'_1=0\quad ,\lambda_2F_2+uF'_2=0$$

giving (thanks to some scaling-all these transformations belong indeed to the equivalence group) $F_1=u^{-\lambda_1}\,,F_2=u^{-\lambda_2}$ where

$$-\lambda_1=1-\frac{c}{B}(3+r)=1+\frac{2+r}{q}\ \ ,-\lambda_2=1+\frac{3}{q}\ \ with\ b=-cq$$

with admitted symmetry generated by

$$X=x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}-qu\frac{\partial}{\partial u}\,.$$

Let now $\lambda_7=\lambda_8=0.$ then necessarily $\lambda_5\lambda_6\neq 0.$ According to proposition 1 , the orthogonality condition now reads (with $\lambda_5=\lambda_6=1$)

$$p_1=\lambda_1p_5\,,p_2=\lambda_2p_6\,,p_3=\lambda_3p_5\ ,p_4=\lambda_4p_6\,,p_7=p_8=0\,,p_9=\lambda_9p_5\,,p_{10}=\lambda_{10}p_6\,,\quad p_{11}=\lambda_{11}p_5+\mu_{11}p_6\ \ ,\quad p_{12}=\lambda_{12}p_5+\mu_{12}p_6\ \ ,\quad p_{13}=\lambda_{13}p_5+\mu_{13}p_6\,.$$

In this case , one has immediately $B=0$, then $p_{12}=0$ and $\lambda_{12}=\mu_{12}=0,$ and again $\xi=cx$. From $p_2=p_6$ one has $3\xi_x=A=const$., which gives $p_{13}=\lambda_{13}=\mu_{13}=0.$ Up to a scaling of u , one can choose $\lambda_2=1,$ the equations for $F_1\,,F_2$ are then

$$\lambda_1F_1+F'_1=0\quad ,\qquad F_2+F'_2=0$$

giving $F_1=\exp(-\lambda_1u)\quad ,\quad F_2=\exp(-u),$ and finally from $\lambda_1=\frac{p_1}{p_5}$ one deduces the same condition (20) as before for the function $\alpha(x,y),$ and $\lambda_1=1+\left(\frac{r}{3}\right).$

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