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ZZ Transform for Solving Systems of Ordinary Differential Equations

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ABSTRACT

In this paper we introduce some properties and definition of the new integral transform, called ZZ transform. Farther, we use ZZ transform to solve systems of ordinary differential equations. The results indicate these methods to be very effective and simple.

1. Introduction

In recent years, systems of high order ordinary differential equations have been solved intensively by using approximate iterative methods such as variational iteration method [1], differential transformation method [2], Adomian decomposition method [3], the tanh method [4], Homotopy analysis method [5], Homotopy Perturbation Method [6]. In addition to these methods, the spectral methods are also used to solving systems of linear differential equations. Chebyshev collocation method [7] and Taylor collocation method [8] are also applied to solve these systems of differential equations. In order to solve differential equations, several integral transforms were extensively used and applied in theory and application such as the Laplace, Fourier, Mellin, Hankel , Sumudu transforms, Elzaki transform [9-10],and Aboodh Transform[11-13] . Recently, New integral transform, named as ZZ Transformation [14-17] introduce by Zain Ul Abadin Zafar [2016] , ZZ transform was successfully applied to integral equations, ordinary differential equations . In this study, our purpose is to show the applicability of this interesting new transform and its efficiency in solving the linear system of ordinary differential equations.The plane of the paper is as follows: In section 2, we introduce the basic idea of ZZ Transform , In Section3, ZZ transform is demonstrated by applying it on many problems and conclusion is given at the last section, respectively.

2. The ZZ Transform

Definition

Let f(t) be a function defined for all $t \geq 0$. The ZZ transform of $f(t)$ is the function $Z(u,s)$ defined by

$$Z(u,s) = H\{f(t)\} = s\int_0^\infty f(ut)e^{-st}dt$$

provided the integral on the right side exists. The unique function $f(t)$ in (2) is called the inverse transform of $Z(u,s)$ is indicated by

$$f(t) = H^{-1}\{Z(u,s)\}$$

Equation (2) can be written as

$$H\{f(t)\} = \frac{s}{u}\int_0^\infty f(t)e^{-\frac{s}{u}t}dt \tag{1}$$

ZZ transform of some functions :

$$H\{1\} = 1 \quad , \quad H\{t^n\} = n!\frac{u^n}{s^n}, \quad H\{e^{at}\} = \frac{s}{s-ua}$$

$$E(\sin(at)) = \frac{aus}{s^2+a^2u^2} \quad , \quad E(\cos(at)) = \frac{s^2}{s^2+a^2u^2}$$

ZZ transform of derivatives :

1)) let $H\{f(t)\} = Z(u,s)$ then

$$H\{tf^{(n)}(t)\} = \frac{s^n}{u^n}Z(u,s) - \sum_{k=0}^{n-1}\frac{s^{n-k}}{u^{n-k}}f^{(k)}(0)$$

2)) (i) $H\{tf(t)\} = \frac{u^2}{s}\frac{d}{du}\big(Z(u,s)\big) + \frac{u}{s}Z(u,s)$

(ii)) $H\{tf'(t)\} = \frac{u^2}{s}\frac{d}{du}\left(\frac{s}{u}Z(u,s)\right) + Z(u,s)$

(iii) $H\{tf''(t)\} = s\frac{d}{du}\big(Z(u,s)\big) - \frac{s}{u}Z(u,s) + \frac{s}{u}f(0)$

3. System of Ordinary Differential Equations

In this section, the effectiveness and the usefulness of ZZ transform is demonstrated by finding exact solutions for System of Ordinary Differential Equations .

Example (1):

We consider a first –order linear system with constant coefficients

$$\begin{aligned}\frac{dy_1}{dt} &= a_{11}y_1 + a_{12}y_2 + g_1(t)\\ \frac{dy_2}{dt} &= a_{21}y_1 + a_{22}y_2 + g_2(t)\end{aligned} \tag{2}$$

with initial condition $y_1(0) = y_{10}$ and $y_2(0) = y_{20}$

where $a_{11}, a_{12}, a_{21}, a_{22}$ are constants.

Solution

Let $\overline{y_1} = A(y_1)$, $\overline{\overline{y_2}} = A(y_2)$, $G_1 = A(g_1)$ and ,$G_2 = A(g_2)$

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Introducing the matrices

$$y=\begin{pmatrix}y_1\\y_2\end{pmatrix}\ y_0=\begin{pmatrix}y_{10}\\y_{20}\end{pmatrix},\ g(t)=\begin{pmatrix}g_1(t)\\g_{2(t)}\end{pmatrix}$$

$$A=\begin{pmatrix}a_{11}&a_{12}\\a_{21}&a_{22}\end{pmatrix},\ \frac{dy}{dt}=\begin{pmatrix}\frac{dy_1}{dt}\\\frac{dy_2}{dt}\end{pmatrix} \quad (3)$$

We can write the above system in matrix differential system as

$$\frac{dy}{dt}=Ay+g(t),y_0=y_0 \quad (4)$$

We take ZZ transform of the system with the initial conditions to get

$$a_{21}u(s-a_{11}u)\overline{y_1}-a_{12}a_{21}u^2\overline{y_2}=a_{21}u^2\overline{g_1}(t)+a_{21}s\,u\,y_{10}a_{21}u(s-a_{11}u)\overline{y_1}-(s-a_{11}u)(s-a_{22}u)\overline{y_2}=u(s-a_{11}u)\overline{g_2}(t)+s(s-a_{11}u)\,y_{20}$$

(5)

The solution of this algebraic system is,

$$\overline{y_1}(v)=\frac{\begin{bmatrix}u\,\overline{g_1}(t)+s\,y_{10} & -2a_{12}u\\ u\,\overline{g_2}(t)+s\,y_{20} & 2(s-a_{22}u)\end{bmatrix}}{\begin{bmatrix}s-a_{11}u & ua_{12}\\ ua_{21} & s-a_{22}u\end{bmatrix}}$$

$$\overline{y_2}(v)=\frac{\begin{bmatrix}\overline{g_1}(t)+y_{10} & -(s+u)(s-a_{11}u)\\ \overline{g_2}(t)y_{20} & a_{21}(su+u^2)\end{bmatrix}}{\begin{bmatrix}s-a_{11}u & ua_{12}\\ ua_{21} & s-a_{22}u\end{bmatrix}} \quad (6)$$

Example (2): solve the system of ordinary differential equation

$$\frac{dy}{dt}=-z \quad , \quad \frac{dz}{dt}=y \quad (7)$$

With initial condition

$$z(0)=0,y(0)=1 \quad (8)$$

Solution:

Applying ZZ transform for the equation (7) we have

$$Z\left\{\frac{dy}{dt}\right\}=Z\{-z\} \quad , \quad Z\left\{\frac{dz}{dt}\right\}=Z\{y\}$$

we get

$$\frac{s}{u}y(u,s)-\frac{s}{u}y(0)=-Z(u,s) \quad ,$$

$$\frac{s}{u}Z(u,s)-\frac{s}{u}z(0)=y(u,s)$$

with the initial condition (8) we get

$$\frac{s}{u}y(u,s)-\frac{s}{u}=-Z(u,s),\ \frac{s}{u}Z(u,s)=y(u,s)$$

so

$$\begin{cases}\frac{s}{u}y(u,s)+Z(u,s)=\frac{s}{u}\\ \frac{s}{u}Z(u,s)-y(u,s)=0\end{cases} \quad (9)$$

there fore $Z(u,s)=\frac{su}{s^2+u^2}$, $Z(u,s)=\frac{s^2}{s^2+u^2}$

then $z(t)=sint,\ y(t)=\cos t$ (10)

Example (3)

Consider the system of ordinary differential equation

$$\begin{cases}\frac{dx}{dt}=2x-3y\\ \frac{dy}{dt}=y-2x\end{cases}$$

$$t>0 \quad (11)$$

With the initial condition $x(0)=8$, $y(0)=3$ (12)

Take ZZ transform of the system with the initial conditions (12) to get

$$Z(\frac{dx}{dt})=Z(2x)-Z(3y) \quad (13)$$

$$Z(\frac{dy}{dt})=Z(y)-Z(2x)$$

we get

$$\begin{cases}\frac{s}{u}X(u,s)-\frac{s}{u}X(0)=2X(u,s)-3Y(u,s)\\ \frac{s}{u}Y(u,s)-\frac{s}{u}y(0)=Y(u,s)-2X(u,s)\end{cases} \quad (14)$$

with the initial condition we get

$$\frac{s}{u}X(u,s)-8\frac{s}{u}=2X(u,s)-3Y(u,s) \quad (15)$$

$$\frac{s}{u}Y(u,s)-3\frac{s}{u}=Y(u,s)-2X(u,s) \quad (16)$$

Solve these equations for *x* and *y* we find:

$$Y(u,s)=\frac{3S^2-22Su}{(s-4u)(S+u)}=\frac{-2s}{(s-4u)}+\frac{s5}{(S+u)} \quad (17)$$

By taking inverse ZZ transform we have:

$$y(t)=5e^{-t}-2e^{4t} \quad (18)$$

Now solve equation (15) and (16) simultaneously from $x(v)$ we have

$$X(u,s)=\frac{8S^2-17Su}{(s-4u)(S+u)}=\frac{3s}{(s-4u)}+\frac{5s}{(S+u)} \quad (19)$$

By taking inverse ZZ transform we have:

$$x(t)=3e^{4t}+5e^{-t} \quad (20)$$

Example (4):

Solve the second order couplet differential system

$$\begin{cases}\frac{d^2x}{dt^2}-3x-4y=0\\ \frac{d^2y}{dt^2}+x+y=0\end{cases} \quad t>0 \quad (21)$$

With the initial condition

$$x(0)=y(0)=0, \quad \frac{dx}{dt}(0)=2, \quad \frac{dy}{dt}(0)=0$$

(22)

Applying ZZ transform for the equation (21) we have

$$Z(\frac{d^2x}{dt^2})-3Z(x)-4Z(y)=Z(0) \quad (23)$$

$$Z(\frac{d^2y}{dt^2})+Z(x)+Z(y)=Z(0)$$

we get

$$\begin{cases}\frac{s^2}{u^2}X(u,t)-\frac{s^2}{u^2}X(0)-\frac{s}{u}X'(0)-3X(u,t)-4Y(u,t)=0\\ \frac{s^2}{u^2}Y(u,t)-\frac{s^2}{u^2}Y(0)-\frac{s}{u}Y'(0)+X(u,t)+Y(u,t)=0\end{cases}$$

with the initial condition (22) we get

$$\frac{s^2}{u^2}X(u,t)-3X(u,t)-4Y(u,t)=2\frac{s}{u}$$

(24)

$$\frac{s^2}{u^2}Y(u,t)+X(u,t)+Y(u,t)=0$$

Solve these equations for *x* and *y* we find:

$$X(u,s)=\frac{2\frac{s}{u}}{\left(\frac{s^2}{u^2}-1\right)^2}=\frac{-2}{(s-4u)}+\frac{5}{(S+u)} \quad , \quad (25)$$

By taking inverse ZZ transform we have

$$x(t)=t(e^t+e^{-t}) \ ,so\ y=\frac{1}{4}(\frac{d^2x}{dt^2}-3x)$$

$$y(t)=\frac{1}{2}(e^t-e^{-t}-te^t-te^{-t}) \quad (26)$$

Example (5)

Solve the second order couplet differential system

$$\begin{cases} y' + z' + y + z = 1 \\ y' + z = e^t \end{cases} \quad (27)$$

With the initial condition

$$y(0) = -1, \quad z(0) = 2 \quad (28)$$

Solution

Applying ZZ transform for the equation (27) we have

$$\begin{cases} Z(y') + Z(z') + Z(y) + Z(z) = Z(1) \\ Z(y') + Z(z) = Z(e^t) \end{cases} \quad (29)$$

we get

$$\begin{cases} \frac{s}{u}Y(u,s) - \frac{s}{u}y(0) + \frac{s}{u}Z(u,s) - \frac{s}{u}Z(0) + Y(u,s) + Z(u,s) = 1 \\ \frac{s}{u}Y(u,s) - \frac{s}{u}y(0) + Z(u,s) = \frac{S}{S-U} \end{cases} \quad (30)$$

with the initial condition (28) we get

$$\begin{cases} \left[\frac{s}{u}+1\right]Y(u,s) + \left[\frac{s}{u}+1\right]Z(u,s) = 1 + \frac{s}{u} \\ \frac{s}{u}Y(u,s) + Z(u,s) = \frac{S}{S-U} - \frac{s}{u} \end{cases} \quad (31)$$

Solve these equations for *x* and *y* we find:

$$Y(u,s) = \frac{su}{(S-u)^2} - \frac{s}{s-u} - \frac{u}{s-u} = \frac{su}{(S-u)^2} - \frac{s+u}{s-u}, \quad (32)$$

By taking inverse ZZ transform we have

$$y(t) = 1 - 2e^t + te^t$$

also

$$Z(u,s) = \frac{2\frac{s}{u}}{\frac{s}{u}-1} - \frac{\frac{s}{u}}{\left(\frac{s}{u}-1\right)^2}$$

$$z(t) = 2e^t - te^t \quad (33)$$

Examples 6:

Solve the system of second order differential equations

$$\begin{cases} z'' + y' = \cos x \\ y'' - z = \sin x \end{cases} \quad (34)$$

With the initial condition

$$z(0) = -1, \; z'(0) = -1, \quad y(0) = 1, \; y'(0) = 0 \quad (35)$$

Solution

Applying ZZ transform for the equation (34) we have

$$\begin{cases} Z(z'') + Z(y') = Z(\cos x) \\ Z(y'') - Z(z) = Z(\sin x) \end{cases}$$

(36)

we get

$$\begin{cases} \frac{s^2}{u^2}Z(u,S) - \frac{s^2}{u^2}Z(0) - \frac{s}{u}Z'(0) + \frac{s}{u}Y(u,S) - \frac{s}{u}Y(0) = \frac{S^2}{S^2+u^2} \\ \frac{s^2}{u^2}Y(u,S) - \frac{s^2}{u^2}Y(0) - \frac{s}{u}Y'(0) - Z(u,S) = \frac{Su}{s^2+u^2} \end{cases} \quad (37)$$

with the initial condition (35) we get

$$\begin{cases} \frac{s^2}{u^2}Z(u,S) + \frac{s}{u}Y(u,S) = \frac{S^2}{s^2+u^2} - \frac{s^2}{u^2} \\ \frac{s^2}{u^2}Y(u,S) - Z(u,S) = \frac{Su}{s^2+u^2} + \frac{s^2}{u^2} \end{cases} \quad (38)$$

Solve these equations for *x* and *y* we find:

$$Y(u,s) = \frac{S^2}{s^2+u^2}, \quad (39)$$

By taking inverse ZZ transform we have

$$y(x) = \cos x,$$

$$Z(u,s) = -\left(\frac{S^2}{s^2+u^2} + \frac{Su}{s^2+u^2}\right),$$

$$z(x) = -(\cos x + \sin x) \quad (40)$$

4. Conclusion

In this paper , we use ZZ transform to solve systems of ordinary differential equations , ZZ transform provides powerful method for analyzing derivatives. It is heavily used to solve ordinary differential equations, Partial differential equations and system of ordinary differential equations.

Appendix (ZZ Transform of some Functions)

$f(t)$	$A[f(t)] = k(v)$
1	1
$\sin(at)$	$\frac{aus}{s^2+a^2u^2}$
$\cos(at)$	$\frac{s^2}{s^2+a^2u^2}$
e^{at}	$\frac{s}{s-ua}$
e^{-at}	$\frac{s}{s+ua}$
$\sinh at$	$\frac{aus}{s^2-u^2a^2}$
$\cosh at$	$\frac{s^2}{s^2-u^2a^2}$
te^{at}	$\frac{su}{(s-ua)^2}$
te^{-at}	$\frac{su}{(s+ua)^2}$
$t\sinh at$	$-2s\left(\frac{s^2+u^2a^2}{(s^2-u^2a^2)^2}\right)$
$t\cosh at$	$\frac{-2s^2u\,a^2}{(s^2-u^2a^2)^2}$

Fig. 1

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