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Edge-vertex dominating sets and Edge-vertex domination polynomials of Stars

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ABSTRACT

Let $G=(V,E)$ be a simple graph. A set $S \subseteq E(G)$ is an Edge-Vertex dominating set of G (or simply an *ev*-Dominating set), if for all vertices $v \in V(G)$, there exists an edge $e \in S$ such that e dominates v. Let S_n be the Star graph and let $D_{ev}(S_n,i)$ denote the family of all Edge-Vertex dominating sets of S_n with cardinality i. Let $d_{ev}(S_n,i)=|D_{ev}(S_n,i)|$, be the number of Edge-Vertex dominating sets of S_n with cardinality i. In this paper, we study the concept of Edge-Vertex domination polynomials of Star graph S_n. The Edge-Vertex Domination polynomial of S_n is $D_{ev}(S_n,x)=\sum_{i=1}^{n-1} d_{ev}(S_n,i)x^i$. We obtain some properties of $D_{ev}(S_n,x)$ and its coefficients. Also, we calculate the recursive formula to derive the Edge-Vertex Domination polynomials of Star graph S_n.

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1. Introduction

Let $G = (V, E)$ be a simple graph of order $|V| = n$. A set $S \subseteq V(G)$ is a dominating set of G, if every vertex in $V \setminus S$ is adjacent to at least one vertex in S. For any vertex $v \in V$, the open neighbourhood of v is the set $N(v)=\{u \in V / uv \in E\}$ and the closed neighbourhood of v is the set $N[v]=N(v)\cup\{v\}$. For a set $S \subseteq V$, the open neighbourhood of S is $N(S)=U_{v\in S}N(v)$ and the closed neighbourhood of S is $N[S]=N(S)\cup S$. For a set $S \subseteq V$, the open neighbourhood of S is $N(S)=U_{v\in S}N(v)$ and the closed neighbourhood of S is $N[S]=N(S)\cup S$. Let $S_n, n\geq 2$ be the star with n vertices $V(S_n)=[n]$ and $E(S_n)=\{(1,2),(1,3),\dots,(1,n)\}$.

Definition 1.1

For a graph $G = (V, E)$, an edge $e=uv\in E(G)$, *ev*-dominates a vertex $w\in V(G)$ if

(i) $u = w$ or $v = w$ (w is incident to e) or
(ii) uw or vw is an edge in G (w is adjacent to u or v).

Definition 1.2

A set $S \subseteq E(G)$ is an Edge-Vertex dominating set of G (or simply an *ev*-dominating set), if for all vertices $v\in V(G)$, there exist an edge $e \in S$ such that e dominates v. The Edge-Vertex domination number of a graph G is defined as the minimum size of an Edge-Vertex dominating set of edges in G and it is denoted as $\gamma_{ev}(G)$.

Definition 1.3

Let $D_{ev}(S_n,i)$ be the family of Edge-Vertex dominating sets of a Star graph S_n with cardinality i and let $d_{ev}(S_n,i)=|D_{ev}(S_n,i)|$ be the number of Edge-Vertex dominating sets of S_n. We call the polynomial $D_{ev}(S_n,x)=\sum_{i=1}^{n-1} d_{ev}(S_n,i)x^i$, the Edge-Vertex domination polynomial of the graph S_n

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As usual we use $\lfloor x \rfloor$ for the largest integer less than or equal to *x* and $\lceil x \rceil$ for the smallest integer greater than or equal to *x*. Also, we denote the set $\{e_1, e_2, \cdots, e_n\}$ by $[e_n]$ and the set $\{1, 2, \cdots, n\}$ by $[n]$, throughout this paper.

2. Edge-Vertex Dominating Sets of Stars

Let S_n, $n \geq 2$ be a Star with n vertices $V(S_n) = [n]$ and $E(S_n) = \{e_1, e_2, ..., e_{n-1}\}$. Let $D_{ev}(S_n, i)$ be the family of Edge-Vertex dominating sets of S_n with cardinality i.

Lemma 2.1

The following results hold for all graph G with $|V(G)| = n$ vertices and $|E(G)| = n-1$ edges.

(i) $d_{ev}(G, n-1) = 1$,

(ii) $d_{ev}(G, n-2) = n-1$,

(iii) $d_{ev}(G, i) = 0$ if $i \geq n$,

(iv) $d_{ev}(G, 0) = 0$.

Proof:

Let $G = (V, E)$ be a simple graph of order *n* and size *n* – 1, then

(i) $D_{ev}(G, n-1) = \{G\} = [e_{n-1}]$, therefore $|D_{ev}(G, n-1) = 1|$. Therefore, $d_{ev}(G, n-1) = 1$.

(ii) $D_{ev}(G, n-2) = \{\{G - \{e_i\}\} : \forall\, e_i \in G\}$, therefore $|D_{ev}(G, n-2)| = n-1$. Therefore, $d_{ev}(G, n-2) = n-1$.

(iii) If $i \geq n$, there does not exist $H \subseteq G$ such that $|E(H)| > |E(G)|$. Therefore, $d_{ev}(G, i) = 0$.

(iv) For $i = 0$ there does not exist $H \subseteq G$ such that $|E(H)| = 0$, Φ is not a Edge-Vertex dominating set of G. Therefore, $d_{ev}(G, 0) = 0$.

Lemma 2.2

For all $n \in Z^+$, $D_{ev}(S_n, i) = \Phi$ if and only if $i \geq n$ or $i < 0$.

Theorem 2.3

Let S_n be a Star with vertices $n \geq 2$, then

(i) $d_{ev}(S_n, i) = \binom{n-1}{i}$, if $i \leq n-1$,

(ii)
$$d_{ev}(S_n, i) = \begin{cases} d_{ev}(S_{n-1}, i) + 1, & \text{if } i = 1 \\ d_{ev}(S_{n-1}, i) + d_{ev}(S_{n-1}, i-1), & \\ \text{if } 1 < i \leq n-1 & \end{cases}$$

Proof

(i) Let S_n be a star with *n* vertices and *n* – 1 edges and let $v \in V(S_n)$ be such that *v* is the centre of S_n and let the edges be $\{e_1, e_2, ..., e_{n-1}\}$. Consider an edge e_i. By the definition of Edge-Vertex domination, it covers all the vertices of S_n. Similarly, any other edge of S_n covers all the vertices of S_n. Therefore, the number of Edge-Vertex dominating sets of cardinality 1 is $\binom{n-1}{1}$ Consider any two edges of S_n. These edges cover all the remaining vertices of S_n. Therefore, number of Edge-Vertex dominating sets of cardinality 2 is $\binom{n-1}{2}$. By continuing, we get the number of Edge-Vertex dominating sets of cardinality *i* is $\binom{n-1}{i}$, $i \leq n-1$. Therefore, $d_{ev}(S_n, i) = \binom{n-1}{i}$, $i \leq n-1$.

From Table 1, we have $d_{ev}(S_n, i) = d_{ev}(S_{n-1}, i) + 1$, $i = 1$. For $1 < i \leq n-1$, we have $\binom{n-2}{i-1} + \binom{n-2}{i} = \binom{n-1}{i}$.

Therefore, $d_{ev}(S_n, i) = d_{ev}(S_{n-1}, i) + d_{ev}(S_{n-1}, i-1)$, $1 < i \leq n-1$.

3. Edge-Vertex Domination Polynomials of Stars

In this section, we obtain the Edge-Vertex Domination polynomial $D_{ev}(S_n, x)$ of the Star graph S_n.

Theorem 3.1

$D_{ev}(S_n,x)=(1+x)^{n-1}-1.$

Proof:

Let

$$D_{ev}(S_n,x)=\sum_{i=1}^{n-1} d_{ev}(S_n,i)x^i$$

$$=\sum_{i=1}^{n-1}\binom{n-1}{i}x^i \text{, by theorem 2.3 (i).}$$

$$=\binom{n-1}{1}x+\binom{n-1}{2}x^2+\ldots+\binom{n-1}{n-1}x^{n-1}$$

$$=1+\binom{n-1}{1}x+\binom{n-1}{2}x^2+\ldots+\binom{n-1}{n-1}x^{n-1}-1$$

$$D_{ev}(S_n,x)=(1+x)^{n-1}-1.$$

Theorem 3.2

$D_{ev}(S_n,x)=(1+x)D_{ev}(S_{n-1},x)+x$ with $D_{ev}(S_2,x)=x$ for $n\geq 3$.

Proof:

$$D_{ev}(S_n,x)=\sum_{i=1}^{n-1} d_{ev}(S_n,i)x^i$$

$$=d_{ev}(S_n,1)x+\sum_{i=2}^{n-1} d_{ev}(S_n,i)x^i$$

$$=\binom{n-1}{1}x+\sum_{i=2}^{n-1}\left[d_{ev}(S_{n-1},i)+d_{ev}(S_{n-1},i-1)\right]x^i$$

$$=(n-1)x+\sum_{i=2}^{n-1} d_{ev}(S_{n-1},i)x^i+\sum_{i=2}^{n-1} d_{ev}(S_{n-1},i-1)x^i$$

Consider, $\sum_{i=2}^{n-1} d_{ev}(S_{n-1},i)x^i = d_{ev}(S_{n-1},2)x^2+d_{ev}(S_{n-1},3)x^3+\ldots+d_{ev}(S_{n-1},n-1)x^{n-1}$

$$=d_{ev}(S_{n-1},1)x+d_{ev}(S_{n-1},2)x^2+\ldots+d_{ev}(S_{n-1},n-1)x^{n-1}-d_{ev}(S_{n-1},1)x$$

$$=\sum_{i=1}^{n-1} d_{ev}(S_{n-1},i)x^i-d_{ev}(S_{n-1},1)x$$

$$=D_{ev}(S_{n-1},x)-\binom{n-2}{1}x$$

$$=D_{ev}(S_{n-1},x)-(n-2)x$$

Consider, $\sum_{i=2}^{n-1} d_{ev}(S_{n-1},i-1)x^i = x\sum_{i=2}^{n-1} d_{ev}(S_{n-1},i-1)x^{i-1}$

$$=x\left[d_{ev}(S_{n-1},1)x+d_{ev}(S_{n-1},2)x^2+\ldots+d_{ev}(S_{n-1},n-2)x^{n-2}\right]$$

$$=x\sum_{i=1}^{n-2} d_{ev}(S_{n-1},i)x^i$$

$$=xD_{ev}(S_{n-1},x)$$

$$D_{ev}(S_n,x)=(n-1)x+D_{ev}(S_{n-1},x)-(n-2)x+xD_{ev}(S_{n-1},x)$$

$$=nx-x+D_{ev}(S_{n-1},x)-nx+2x+xD_{ev}(S_{n-1},x)$$

$$=(1+x)D_{ev}(S_{n-1},x)+x$$

Hence the theorem.

Example for Theorem 3.2

Let $D_{ev}(S_n, x)$ be the Edge-Vertex domination polynomial of Star graph S_n. Then,

(i) $D_{ev}(S_3, x) = 2x + x^2$

(ii) $D_{ev}(S_4, x) = 3x + 3x^2 + x^3$

(iii) $D_{ev}(S_5, x) = 4x + 6x^2 + 4x^3 + x$

(iv) $D_{ev}(S_6, x) = 5x + 10x^2 + 10x^3 + 5x^4 + x^5$.

Solution

From Theorem 3.3, $D_{ev}(S_n, x) = (1+x)D_{ev}(S_{n-1}, x) + x$ with $D_{ev}(S_2, x) = x$ for $n \geq 3$.

(i)For $n = 3$, $D_{ev}(S_3, x) = (1+x)D_{ev}(S_2, x) + x$

$= (1+x)x + x$

$= 2x + x^2$

(ii)For $n = 4$, $D_{ev}(S_4, x) = (1+x)D_{ev}(S_3, x) + x$

$= (1+x)(2x + x^2) + x$

$= 3x + 3x^2 + x^3$

(iii)For $n = 5$, $D_{ev}(S_5, x) = (1+x)D_{ev}(S_4, x) + x$

$= (1+x)(3x + 3x^2 + x^3) + x$

$= 4x + 6x^2 + 4x^3 + x^4$

(iv)For $n = 6$, $D_{ev}(S_6, x) = (1+x)D_{ev}(S_5, x) + x$

$= (1+x)(4x + 6x^2 + 4x^3 + x^4) + x$

$= 5x + 10x^2 + 10x^3 + 5x^4 + x^5$.

We obtain $d_{ev}(S_n, i)$ for $2 \leq n \leq 13$ as shown in Table 1.

Table 1. $d_{ev}(S_n, i)$, the number of Edge-Vertex dominating set of S_n with cardinality *i*.

i	1	2	3	4	5	6	7	8	9	10	11	12
n												
2	1											
3	2	1										
4	3	3	1									
5	4	6	4	1								
6	5	10	10	5	1							
7	6	15	20	15	6	1						
8	7	21	35	35	21	7	1					
9	8	28	56	70	56	28	8	1				
10	9	36	84	126	126	84	36	9	1			
11	10	45	120	210	252	210	120	45	10	1		
12	11	55	165	330	462	462	330	165	55	11	1	
13	12	66	220	495	792	924	792	495	220	66	12	1

In the following Theorem, we obtain some properties of $d_{ev}(S_n, i)$.

Theorem 3.3

The following properties hold for the coefficients of $D_{ev}(S_n, x) \; \forall \, n \in Z^+, n \geq 4$.

(i) $d_{ev}(S_n, 1) = n - 1$,

(ii) $d_{ev}(S_n, n-1) = 1$,

(iii) $d_{ev}(S_n, n-2) = n - 1$,

(iv) $d_{ev}(S_n, i) = 0$ if $i \geq n$,

(v) $\gamma_{ev}(S_n) = 1$

(vi) $d_{ev}(S_n, i) = d_{ev}(S_n, n-1-i), 1 \leq i \leq n-2$.

Proof:

(i) We prove this by the method of induction on '*n*'. If *n* = 4, L.H.S = $d_{ev}(S_4,1)=3$ (from table 1). R.H.S = 4 – 1 = 3. Therefore, the result is true for *n* = 4. Assume that the result is true for all *n* < *j*. Therefore, $d_{ev}(S_{j-1},1)=j-2$ is true. Now, we have to prove that the result is true for n = j. $d_{ev}(S_j,1)=d_{ev}(S_{j-1},1)+1=j-2+1$. Therefore, $d_{ev}(S_j,1)=j-1$. Therefore, the result is true for *n* = *j*. Hence, by the principle of induction, the result is true for all *n*, $n\in Z^+$.

(ii) We prove this by the method of induction on '*n*'. If *n* = 4, L.H.S = $d_{ev}(S_4,3)=1$ (from table 1). R.H.S = 1. Therefore, the result is true for *n* = 4. Assume that the result is true for all *n* < *j*. Therefore, $d_{ev}(S_{j-1},j-2)=1$ is true. Now, we have to prove that the result is true for *n* = *j*. $d_{ev}(S_j,j-1)=d_{ev}(S_{j-1},j-1)+d_{ev}(S_{j-1},j-2)=0+1=1$ Therefore, the result is true for *n* = *j*. Hence, by the principle of induction, the result is true for all n, $n\in Z^+$.

(iii) We prove this by the method of induction on '*n*'. If n = 4, L.H.S = $d_{ev}(S_4,2)=3$ (from table 1). R.H.S = 4 – 1 =3. Therefore, the result is true for n = 4. Assume that the result is true for all *n* < *j*. Therefore, $d_{ev}(S_{j-1},j-2)=1$ is true. Now, we have to prove that the result is true for *n* = *j*. $d_{ev}(S_j,j-2)=d_{ev}(S_{j-1},j-2)+d_{ev}(S_{j-1},j-3)$. Therefore, the result

$$=1+j-2$$
$$=j-1$$

is true for *n* = *j*. Hence, by the principle of induction, the result is true for all *n*, $n\in Z^+$.

(iv) From Table 1, we have $d_{ev}(S_n,i)=0$ if $i\geq n$.

(v) Any edge of S_n is enough to cover all the vertices and edges of S_n. Therefore, the minimum cardinality of the Edge-Vertex dominating set of S_n is 1. Therefore, $\gamma_{ev}(S_n)=1$.

(vi) L.H.S = $d_{ev}(S_n,i)=\binom{n-1}{i}$.

R.H.S = $d_{ev}(S_n,n-1-i)=\binom{n-1}{n-1-i}$

$$=\frac{(n-1)!}{(n-1-i)!(n-1-n+1+i)!}$$
$$=\frac{(n-1)!}{(n-1-i)!i!}$$
$$=\binom{n-1}{i}$$

Therefore, $d_{ev}(S_n,i)=d_{ev}(S_n,n-1-i), 1\leq i\leq n-2$.

Theorem 3.4

The Edge-Vertex dominating roots of the Star graph S_n are $\cos\frac{2(k+1)\pi}{n-1}+i\sin\frac{2(k+1)\pi}{n-1}, k=0,1,...,n-2$

Proof:

The Edge-Vertex domination polynomial of Star graph S_n is $D_{ev}(S_n,x)=(1+x)^{n-1}-1$. To find the Edge-Vertex dominating roots, put $D_{ev}(S_n,x)=0$. Therefore, we get

$$(1+x)^{n-1}-1=0$$
$$(1+x)^{n-1}=1$$
$$(1+x)=(1)^{\frac{1}{n-1}}$$
$$=(\cos 2\pi+i\sin 2\pi)^{\frac{1}{n-1}}$$
$$=[\cos(2k\pi+2\pi)+i\sin(2k\pi+2\pi)]^{\frac{1}{n-1}}$$, where *k* is an integer.

$$= \left[\cos 2(k+1)\pi + i\sin 2(k+1)\pi\right]^{\frac{1}{n-1}},$$
$$k = 0,1,\dots,n-2$$
$$(1+x) = \cos\frac{2(k+1)\pi}{n-1} + i\sin\frac{2(k+1)\pi}{n-1},$$
$$k = 0,1,\dots,n-2$$
$$x = \cos\frac{2(k+1)\pi}{n-1} + i\sin\frac{2(k+1)\pi}{n-1} - 1,$$
$$k = 0,1,\dots,n-2$$

Therefore, the Edge-Vertex dominating roots of the Star graph S_n are

$$\cos\frac{2(k+1)\pi}{n-1} + i\sin\frac{2(k+1)\pi}{n-1},\ k = 0,1,\dots,n-2.$$

Theorem 3.5

$$\frac{d^n}{dx^n} D_{ev}(S_n, x) = (n-1)!.$$

Proof:

The Edge-Vertex domination polynomial of Star graph S_n is $D_{ev}(S_n, x) = (1+x)^{n-1} - 1$.

Differentiating with respect to *x* we get, $\frac{d}{dx}\left[D_{ev}(S_n, x)\right] = (n-1)(1+x)^{n-2}$.

Again differentiating with respect to x we get, $\frac{d^2}{dx^2}\left[D_{ev}(S_n, x)\right] = (n-1)(n-2)(1+x)^{n-3}$.

Continuing this way we get n^{th} derivative,

$$\frac{d^n}{dx^n}\left[D_{ev}(S_n, x)\right] = (n-1)(n-2)\dots\big((n-1)-(n-2)\big)(1+x)^{n-n}$$
$$= (n-1)(n-2)\dots(n-1-n+2)(1+x)^0$$
$$= (n-1)(n-2)\dots 2.1$$
$$= (n-1)!$$

Theorem 3.6

Let S_n be the Star graph with *n* vertices then, $D_{ev}(S_n, -1) = -1$.

Proof:

The Edge-Vertex domination polynomial of Star graph S_n is $D_{ev}(S_n, x) = (1+x)^{n-1} - 1$.

$$D_{ev}(S_n, -1) = (1-1)^{n-1} - 1 = 0 - 1 = -1.$$

4. Conclusion

In this paper we obtain the Edge-Vertex dominating sets and Edge-vertex domination polynomial of Stars. Similarly we can find Edge-Vertex dominating sets and Edge-vertex domination polynomial of some specified graphs.

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