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Generalized Chain Ratio-Product Estimator for Estimating Population Mean With Auxiliary Variate

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ABSTRACT

This paper provides a unified framework for chain ratio-product estimation and proposes a generalized chain ratio-product estimator for estimating population mean in stratified sampling. The bias and variance expressions of the proposed estimator have been derived under large sample approximation. Asymptotic optimum estimator (AOE) and its approximate variance estimator are derived with conditions for allowable departure identified. Some existing estimators in theory are obtained and shown to be particular members of the proposed estimator under certain restrictions. Analytical comparisons of the *AOE* with other existing estimators showed that the *AOE* is substantially more efficient under certain realistic conditions.

Introduction

It is well known in survey sampling that the ratio and product estimators most practically have the limitation of having efficiency not exceeding that of the regression estimator.

To address this problem, most survey statisticians have carried out researches towards the modification of the existing ratio and product estimators to provide better alternatives and improve their precision. Among the authors who have proposed various improved ratio and product estimators include; Kadilar and Cingi [1], Singh [2], Singh and Ruiz Espejo [3], Singh and Vishwakarma [4,5], Singh *et al* [6], Sharma and Tailor [7], Onyeka [8], Tailor [9], Choudhury and Singh [10], Khare and Sinha [11], Sharma, Verma, Sanaullar and Singh [12], Singh and Audu [13], Khare, Srivastava and Kumar [14], Clement [15,16] and Clement and Enang [17,18].

In the progression for more precise and improve ratio and product estimators, this paper aims at providing a unified framework for chain ratio-product estimation and proposes a generalized chain ratio-product estimator for estimating population mean in stratified sampling.

Consider a finite population U of N elements $U = (U_1, U_2, \dots, U_N)$ which consist of H strata with N_h units in the hth stratum from which a simple random sample of size n_h is taken without replacement. The total population size be $N = \sum_{h=1}^{H} N_h$ and the sample size $n = \sum_{h=1}^{H} n_h$, respectively. Associated with the ith element of the hth stratum are y_{hi} and x_{hi} with $x_{hi} > 0$ being the covariate; where y_{hi} is the y value of the ith element in stratum h, and x_{hi} is the x value of the ith element in stratum h, $h = 1, 2, \dots, H$ and $i = 1, 2, \dots, N_h$ where y and x are the study variable and auxiliary variable respectively. For the hth stratum, let $w_h = N_h/N$ be the stratum weights and $f_h = n_h/N_h$, the sample fraction.

Let the hth stratum means of the study variable Y and auxiliary variable X $\left(\bar{y}_h = \sum_{i=1}^{n_h} y_{hi}/n_h\,;\ \bar{x}_h = \sum_{i=1}^{n_h} x_{hi}/n_h\right)$ be the unbiased estimator of the population mean $\left(\bar{Y}_h = \sum_{i=1}^{N_h} y_{hi}/N_h\,;\ \bar{X}_h = \sum_{i=1}^{N_h} x_{hi}/N_h\right)$ of y and x respectively, based on n_h observations.

2. Review of existing estimators

This section gives a review of some existing estimators in the literature with their variance expressions.

2.1 Stratified sampling estimator

The stratified random sampling estimator for population mean $(\bar{y})$ according to Cochran [19] is defined as:

$$\bar{y}_{st} = \sum_{h=1}^{H} w_h \bar{y}_h \tag{1}$$

with its variance estimator given as:

$$Var(\bar{y}_{st}) = \sum_{h=1}^{H} w_h^2 \gamma_h \bar{Y}_h^2\, C_{hy}^2 \tag{2}$$

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2.2 Hansen-Hurwitz-Gurney estimator

Hansen, Hurwitz and Gurney [20] proposed the separate ratio estimator for $(\bar{y})$ as

$$\bar{y}_R^{(S)} = \sum_{h=1}^{H} w_h \bar{y}_h \left(\frac{\bar{X}_h}{\bar{x}_h}\right) \tag{3}$$

with its variance estimator as:

$$\mathrm{Var}\left(\bar{y}_R^{(S)}\right) = \sum_{h=1}^{H} w_h^2 \gamma_h \bar{Y}_h^2 \left[C_{hy}^2 + C_{hx}^2 (1 - 2K_h)\right] \tag{4}$$

2.3 Murthy estimator

Murthy [21] proposed the separate product estimator for $(\bar{y})$ as:

$$\bar{y}_P^{(S)} = \sum_{h=1}^{H} w_h \bar{y}_h \left(\frac{\bar{x}_h}{\bar{X}_h}\right) \tag{5}$$

with its variance estimator as:

$$\mathrm{Var}\left(\bar{y}_R^{(S)}\right) = \sum_{h=1}^{H} w_h^2 \gamma_h \bar{Y}_h^2 \left[C_{hy}^2 + C_{hx}^2 (1 + 2K_h)\right] \tag{6}$$

2.4 Vishwakarma - Singh estimator

Vishwakarma and Singh [22] proposed a separate ratio-product estimator for population mean in stratified random sampling as:

$$\bar{y}_{RP}^{(S)} = \sum_{h=1}^{H} w_h \bar{y}_h \left\{\alpha_h \frac{\bar{X}_h}{\bar{x}_h} + (1 - \alpha_h) \frac{\bar{x}_h}{\bar{X}_h}\right\} \tag{7}$$

with its variance estimator as:

$$\mathrm{Var}\left(\bar{y}_{RP}^{(S)}\right) = \sum_{h=1}^{H} w_h^2 \gamma_h \bar{Y}_h^2 \left[C_{hy}^2 + C_{hx}^2 (1 - 2\alpha_h)\{(1 - 2\alpha_h) + 2K_h\}\right] \tag{8}$$

2.5 Kadillar and Cingi estimators

Kadillar and Cingi [1] suggested the chain ratio estimator as given by:

$$\bar{y}_R^{(C)} = \sum_{h=1}^{H} w_h \bar{y}_h \left(\frac{\bar{X}_h}{\bar{x}_h}\right)^2 \tag{9}$$

with its variance estimator as:

$$\mathrm{Var}\left(\bar{y}_R^{(C)}\right) = \sum_{h=1}^{H} w_h^2 \gamma_h \bar{Y}_h^2 \left[C_{hy}^2 + 4C_{hx}^2 (1 + K_h)\right] \tag{10}$$

And the chain product estimator as given by:

$$\bar{y}_P^{(C)} = \sum_{h=1}^{H} w_h \bar{y}_h \left(\frac{\bar{x}_h}{\bar{X}_h}\right)^2 \tag{11}$$

with its variance estimator as:

$$\mathrm{Var}\left(\bar{y}_P^{(C)}\right) = \sum_{h=1}^{H} w_h^2 \gamma_h \bar{Y}_h^2 \left[C_{hy}^2 + 4C_{hx}^2 (1 - K_h)\right] \tag{12}$$

3. Suggested estimators

Motivated by Kadilar and Cingi [1] and Vishwakarma and Singh [22], this paper introduces the following new ratio-product-type estimators for estimating population mean $\bar{Y}$ in stratified sampling as:

3.1 Partial chain ratio-product estimator

$$\bar{y}_{RP}^{(PC)} = \sum_{h=1}^{H} W_h \bar{y}_h \left\{\alpha_h \sqrt{\left(\frac{\bar{X}_h}{\bar{x}_h}\right)} + (1 - \alpha_h) \sqrt{\left(\frac{\bar{x}_h}{\bar{X}_h}\right)}\right\} \tag{13}$$

where α_h is a real constant to be determined such that the variance of $\bar{y}_{RP}^{(PC)}$ is minimized.

In order to obtain the first degree of approximation of the bias and variance (or MSE) expressions for the proposed estimators the following definitions are established:

$$\left.\begin{aligned} \bar{y}_h &= \bar{Y}_h (1 + e_{hy}) \\ \bar{x}_h &= \bar{X}_h (1 + e_{hx}) \end{aligned}\right\} \tag{14}$$

$$\left.\begin{aligned} E(e_{hx}^2) &= \gamma_h C_{hx}^2 \\ E(e_{hy}^2) &= \gamma_h C_{hy}^2 \\ E(e_{hx} e_{hy}) &= \gamma_h \rho_{hxy} C_{hx} C_{hy} \\ E(e_{hx}) = E(e_{hy}) &= 0 \\ E(e_{hx} e_{lx}) = E(e_{hx} e_{ly}) &= 0 \\ E(e_{hy} e_{lx}) = E(e_{hy} e_{ly}) &= 0 \end{aligned}\right\} \tag{15}$$

$$\left.\begin{aligned} K_h &= (\rho_{hxy} C_{hy}/C_{hx}) \\ C_{hy}^2 &= (S_{hy}^2/\bar{Y}_h^2) \\ C_{hx}^2 &= (S_{hx}^2/\bar{X}_h^2) \\ S_{hxy} &= \rho_{hxy} S_{hx} S_{hy} \end{aligned}\right\} \tag{16}$$

$$\left.\begin{aligned} S_{hx}^2 &= \frac{1}{N_h-1}\sum_{i=1}^{N_h}(x_{hi}-\bar{X}_h)^2 \\ S_{hy}^2 &= \frac{1}{N_h-1}\sum_{i=1}^{N_h}(y_{hi}-\bar{Y}_h)^2 \\ S_{hxy} &= \frac{1}{N_h-1}\sum_{i=1}^{N_h}(x_{hi}-\bar{X}_h)\ (y_{hi}-\bar{Y}_h) \\ \gamma_h &= \left(\frac{1-f_h}{n_h}\right) = \left(\frac{1}{n_h}-\frac{1}{N_h}\right) \end{aligned}\right\} \tag{17}$$

Expressing (13) in terms of the e's in (14) gives

$$\bar{y}_{RP}^{(PC)} = \sum_{h=1}^{H} W_h \bar{Y}_h \left(1+e_{hy}\right)\left\{\alpha_h(1+e_{hx})^{-\frac{1}{2}} + (1-\alpha_h)(1+e_{hx})^{\frac{1}{2}}\right\} \tag{18}$$

It is assumed that $|e_{hx}| < 1$ so that expanding $(1+e_{hx})^{-\frac{1}{2}}$ and $(1+e_{hx})^{\frac{1}{2}}$ as a series in power of e_{hx}, multiplying out and retaining terms of the e's to the second degree, gives

$$\bar{y}_{RP}^{(PC)} = \sum_{h=1}^{H} W_h \bar{Y}_h \left[1 + e_{hy} + \tfrac{1}{2}e_{hx} + \tfrac{1}{2}e_{hx}e_{hy} - \tfrac{1}{8}e_{hx}^2 + \alpha_h\left(\tfrac{1}{2}e_{hx}^2 - e_{hx} - e_{hx}e_{hy}\right)\right]$$

$$\bar{y}_{RP}^{(PC)} - \bar{Y} = \sum_{h=1}^{H} W_h \bar{Y}_h \left[\left(e_{hy} + \tfrac{1}{2}e_{hx} + \tfrac{1}{2}e_{hx}e_{hy} - \tfrac{1}{8}e_{hx}^2\right) + \alpha_h\left(\tfrac{1}{2}e_{hx}^2 - e_{hx} - e_{hx}e_{hy}\right)\right] \tag{19}$$

Taking expectation of both sides of (19) and using the results in (15), gives the bias of $\bar{y}_{RP}^{(PC)}$ to the first order of approximation (i.e. to terms of order $o(n_h^{-1})$) as:

$$Bias\left(\bar{y}_{RP}^{(PC)}\right) = \sum_{h=1}^{H} W_h \bar{Y}_h \gamma_h C_{hx}^2\left[k_h\left(\tfrac{1}{2}-\alpha_h\right) + \tfrac{1}{8}(4\alpha_h - 1)\right] \tag{20}$$

Squaring both sides of (19) and retaining terms to the second degree, gives

$$\left(\bar{y}_{RP}^{(PC)} - \bar{Y}\right)^2 = \left[\sum_{h=1}^{H} W_h \bar{Y}_h \left[\left(e_{hy} + \tfrac{1}{2}e_{hx} + \tfrac{1}{2}e_{hx}e_{hy}\right) + \alpha_h\left(\tfrac{1}{2}e_{hx}^2 - e_{hx} - e_{hx}e_{hy}\right)\right]\right]^2$$

$$\left(\bar{y}_{RP}^{(PC)} - \bar{Y}\right)^2 = \sum_{h=1}^{H} W_h^2 \bar{Y}_h^2 \left\{e_{hy} + e_{hx}\left(\tfrac{1}{2}-\alpha_h\right)\right\}^2$$

$$\left(\bar{y}_{RP}^{(PC)} - \bar{Y}\right)^2 = \sum_{h=1}^{H} W_h^2 \bar{Y}_h^2 \left\{e_{hy}^2 + \left(\tfrac{1}{2}-\alpha_h\right)\left[\left(\tfrac{1}{2}-\alpha_h\right)e_{hx}^2 + 2e_{hx}e_{hy}\right]\right\} \tag{21}$$

Taking expectation of both sides of (21) and using the results in (15), gives the variance of $\bar{y}_{RP}^{(PC)}$ to the first order of approximation (i.e. to terms of order $o(n_h^{-1})$) as:

$$Var(\bar{y}_{RP}^{(PC)}) = \sum_{h=1}^{H} W_h^2 \bar{Y}_h^2 \left\{C_{hy}^2 + \left(\tfrac{1}{2}-\alpha_h\right)\left[\left(\tfrac{1}{2}-\alpha_h\right) + 2K_h\right]C_{hx}^2\right\} \tag{22}$$

The $Var(\bar{y}_{RP}^{(PC)})$ in (22) is minimized when

$$\alpha_h = \left(\frac{1+2K_h}{2}\right) = \alpha_{h,opt} \quad \text{(say)} \tag{23}$$

Substituting the value of $\alpha_{h,opt}$ in (23) for α_h in (13) gives the partial chain ratio-product asymptotically optimum estimator (AOE) for population mean ($\bar{Y}$) in stratified sampling as:

$$\bar{y}_{RP}^{(PC)} = \sum_{h=1}^{H} W_h \bar{y}_h \left\{\left(\frac{1+2K_h}{2}\right)\sqrt{\left(\frac{\bar{X}_h}{\bar{x}_h}\right)} + \left(\frac{1-2K_h}{2}\right)\sqrt{\left(\frac{\bar{x}_h}{\bar{X}_h}\right)}\right\} \tag{24}$$

Similarly, substituting the value of $\alpha_{h,opt}$ in (23) for α_h in (22) gives the variance of asymptotically optimum estimator $(AOE)\left(\bar{y}_{RP}^{(PC)}\right)_{opt}$ (or minimum variance of $\bar{y}_{RP}^{(PC)}$) as:

$$Var\left(\bar{y}_{RP}^{(PC)}\right)_{opt} = \sum_{h=1}^{H} W_h^2 \gamma_h S_{hy}^2\left(1-\rho_{hxy}^2\right) \tag{25}$$

3.2 Chain ratio-product estimator

$$\bar{y}_{RP}^{(C)} = \sum_{h=1}^{H} W_h\,\bar{y}_h\left\{\alpha_h\left(\frac{\bar{X}_h}{\bar{x}_h}\right)^2 + (1-\alpha_h)\left(\frac{\bar{x}_h}{\bar{X}_h}\right)^2\right\} \quad (26)$$

where α_h is a real constant to be determined such that the variance of $\bar{y}_{RP}^{(C)}$ is minimized.

Expressing (26) in terms of the e's in (14) gives

$$\bar{y}_{RP}^{(C)} = \sum_{h=1}^{H} W_h\bar{Y}_h\,(1+e_{hy})\{\alpha_h(1+e_{hx})^{-2} + (1-\alpha_h)(1+e_{hx})^2\} \quad (27)$$

It is assumed that $|e_{hx}| < 1$ so that expanding $(1+e_{hx})^{-2}$ *and* $(1+e_{hx})^2$ as a series in power of e_{hx}, multiplying out and retaining terms of the e's to the second degree, gives

$$\bar{y}_{RP}^{(C)} = \sum_{h=1}^{H} W_h\bar{Y}_h\left[1 + e_{hy} + 2e_{hx} + 2e_{hx}e_{hy} + e_{hx}^2 + \alpha_h\left(2e_{hx}^2 - 4e_{hx} - 4e_{hx}e_{hy}\right)\right]$$

$$\bar{y}_{RP}^{(C)} - \bar{Y} = \sum_{h=1}^{H} W_h\bar{Y}_h\left[\left(e_{hy} + 2e_{hx} + 2e_{hx}e_{hy} + e_{hx}^2\right) + \alpha_h\left(2e_{hx}^2 - 4e_{hx} - 4e_{hx}e_{hy}\right)\right] \quad (28)$$

Taking expectation of both sides of (28) and using the results in (15), gives the bias of $\bar{y}_{RP}^{(C)}$ to the first order of approximation (i.e. to terms of order $o\left(n_h^{-1}\right)$) as:

$$\text{Bias}\left(\bar{y}_{RP}^{(C)}\right) = \sum_{h=1}^{H} W_h\bar{Y}_h\,\gamma_h C_{hx}^2[2k_h + 2\alpha_h(1-2K_h) + 1] \quad (29)$$

Squaring both sides of (28) and retaining terms to the second degree, gives

$$\left(\bar{y}_{RP}^{(C)} - \bar{Y}\right)^2 = \left[\sum_{h=1}^{H} W_h\bar{Y}_h\left[\left(e_{hy} + 2e_{hx} + 2e_{hx}e_{hy} + e_{hx}^2\right) + \alpha_h\left(2e_{hx}^2 - 4e_{hx} - 4e_{hx}e_{hy}\right)\right]\right]^2$$

$$\left(\bar{y}_{RP}^{(C)} - \bar{Y}\right)^2 = \sum_{h=1}^{H} W_h^2\bar{Y}_h^2\left\{e_{hy}^2 + (1-2\alpha_h)\left[(1-2\alpha_h)4e_{hx}^2 + 4e_{hx}e_{hy}\right]\right\} \quad (30)$$

Taking expectation of both sides of (30) and using the results in (15), gives the variance of $\bar{y}_{RP}^{(C)}$ to the first order of approximation (i.e. to terms of order $o\left(n_h^{-1}\right)$) as:

$$\text{Var}(\bar{y}_{RP}^{(C)}) = \sum_{h=1}^{H} W_h^2\bar{Y}_h^2\left\{C_{hy}^2 + (1-2\alpha_h)[(1-2\alpha_h) + K_h]4C_{hx}^2\right\} \quad (31)$$

The $\text{Var}(\bar{y}_{RP}^{(C)})$ in (31) is minimized when

$$\alpha_h = \left(\frac{2+K_h}{4}\right) = \alpha_{h,opt} \quad (\text{say}) \quad (32)$$

Substituting the value of $\alpha_{h,opt}$ in (32) for α_h in (26) gives the partial chain ratio-product asymptotically optimum estimator (AOE) for population mean ($\bar{Y}$) in stratified sampling as:

$$\bar{y}_{RP}^{(C)} = \sum_{h=1}^{H} W_h\,\bar{y}_h\left\{\left(\frac{2+K_h}{4}\right)\left(\frac{\bar{X}_h}{\bar{x}_h}\right)^2 + \left(\frac{2-K_h}{4}\right)\left(\frac{\bar{x}_h}{\bar{X}_h}\right)^2\right\} \quad (33)$$

Similarly, substituting the value of $\alpha_{h,opt}$ in (32) for α_h in (31) gives the variance of asymptotically optimum estimator (AOE) $\left(\bar{y}_{RP}^{(C)}\right)_{opt}$ (or minimum variance of $\bar{y}_{RP}^{(C)}$ as:

$$\text{Var}\left(\bar{y}_{RP}^{(C)}\right)_{opt} = \sum_{h=1}^{H} W_h^2\gamma_h S_{hy}^2\left(1-\rho_{hxy}^2\right) \quad (34)$$

4. Generalized chain ratio-product estimator

The aim of this paper is to provide a unified framework for chain ratio-product estimation in stratified sampling. Consequently, following from section 3, a generalization to the proposed chain ratio-product estimators is established as follows:

Following Kadilar and Cingi [1], let the generalized chain ratio estimator be defined as:

$$\bar{y}_{R}^{(GC)} = \sum_{h=1}^{H} W_h\,\bar{y}_h\left(\frac{\bar{X}_h}{\bar{x}_h}\right)^{\lambda_h} \quad (35)$$

and let the generalized chain product estimator be defined as:

$$\bar{y}_{P}^{(GC)} = \sum_{h=1}^{H} W_h\,\bar{y}_h\left(\frac{\bar{x}_h}{\bar{X}_h}\right)^{\lambda_h} \quad (36)$$

Hence, a generalized chain ratio-product estimator is proposed as given by:

$$\bar{y}_{RP}^{(GC)} = \sum_{h=1}^{H} W_h\,\bar{y}_h\left\{\alpha_h\left(\frac{\bar{X}_h}{\bar{x}_h}\right)^{\lambda_h} + (1-\alpha_h)\left(\frac{\bar{x}_h}{\bar{X}_h}\right)^{\lambda_h}\right\} \quad (37)$$

where α_h is a real constant to be determined such that the variance of $\bar{y}_{RP}^{(GC)}$ is minimized and λ_h is a scalar satisfying the condition $0 \le \lambda_h \le \infty$

4.1 Bias and Variance estimator for the proposed estimator

Expressing (37) in terms of the e's in (14) gives

$$\bar{y}_{RP}^{(GC)} = \sum_{h=1}^{H} W_h \bar{Y}_h \left(1+e_{hy}\right)\left\{\alpha_h(1+e_{hx})^{-\lambda_h} + (1-\alpha_h)(1+e_{hx})^{\lambda_h}\right\} \tag{38}$$

It is assumed that $|e_{hx}| < 1$ so that expanding $(1+e_{hx})^{-\lambda_h}$ and $(1+e_{hx})^{\lambda_h}$ as a series in power of e_{hx}, multiplying out and retaining terms of the e's to the second degree, gives

$$\bar{y}_{RP}^{(GC)} = \sum_{h=1}^{H} W_h \bar{Y}_h \left[1 + e_{hy} + \lambda_h e_{hx} + \lambda_h e_{hx} e_{hy} + \frac{\lambda_h(\lambda_h - 1)}{2} e_{hx}^2 + \alpha_h\left(\lambda_h e_{hx}^2 - 2\lambda_h e_{hx} - 2\lambda_h e_{hx} e_{hy}\right)\right]$$

$$\bar{y}_{RP}^{(GC)} - \bar{Y} = \sum_{h=1}^{H} W_h \bar{Y}_h \left[\left(e_{hy} + \lambda_h e_{hx} + \lambda_h e_{hx} e_{hy} + \frac{(\lambda_h^2 - \lambda_h)}{2} e_{hx}^2\right) + \lambda_h \alpha_h\left(e_{hx}^2 - 2e_{hx} - 2e_{hx} e_{hy}\right)\right] \tag{39}$$

Taking expectation of both sides of (39) and using the results in (15), gives the bias of $\bar{y}_{RP}^{(GC)}$ to the first order of approximation (i.e. to terms of order $o(n_h^{-1})$) as:

$$E(\bar{y}_{RP}^{(GC)} - \bar{Y}) = \sum_{h=1}^{H} W_h \bar{Y}_h \, E\left[\left(\lambda_h e_{hx} e_{hy}(1-2\alpha) + \frac{\lambda_h}{2} e_{hx}^2(\lambda_h - 1) + 2\alpha_h\right)\right]$$

$$\mathrm{Bias}\left(\bar{y}_{RP}^{(GC)}\right) = \sum_{h=1}^{H} W_h \bar{Y}_h \gamma_h C_{hx}^2 \left[\lambda_h k_h (1-2\alpha) + \frac{\lambda_h}{2}\{(\lambda_h - 1) + 2\alpha_h\}\right] \tag{40}$$

Squaring both sides of (39) and retaining terms to the second degree, gives

$$\left(\bar{y}_{RP}^{(GC)} - \bar{Y}\right)^2 = \left[\sum_{h=1}^{H} W_h \bar{Y}_h \left[\left(e_{hy} + \lambda_h e_{hx} + \lambda_h e_{hx} e_{hy} + e_{hx}^2\right) + \alpha_h\left(\lambda_h e_{hx}^2 - 2\lambda_h e_{hx} - 2\lambda_h e_{hx} e_{hy}\right)\right]\right]^2$$

$$\left(\bar{y}_{RP}^{(GC)} - \bar{Y}\right)^2 = \sum_{h=1}^{H} W_h^2 \bar{Y}_h^2 \left\{e_{hy}^2 + (1-2\alpha_h)\left[(1-2\alpha_h)\lambda_h^2 e_{hx}^2 + 2\lambda_h e_{hx} e_{hy}\right]\right\} \tag{41}$$

Taking expectation of both sides of (41) and using the results in (15), gives the variance of $\bar{y}_{RP}^{(GC)}$ to the first order of approximation (i.e. to terms of order $o(n_h^{-1})$) as:

$$\mathrm{Var}(\bar{y}_{RP}^{(GC)}) = \sum_{h=1}^{H} W_h^2 \gamma_h \bar{Y}_h^2 \left\{C_{hy}^2 + (1-2\alpha_h) C_{hx}^2\left[(1-2\alpha_h)\lambda_h^2 + 2\lambda_h K_h\right]\right\} \tag{42}$$

4.2 Optimal conditions for the proposed estimator

To investigate the optimality conditions for the proposed estimator, let

$$\frac{\partial \mathrm{Var}(\bar{y}_{RP}^{(GC)})}{\partial \alpha_h} = 0$$

So that

$$\alpha_h = \left(\frac{\lambda_h + K_h}{2\lambda_h}\right)$$

$$= \alpha_{h,opt} \qquad (\text{say}) \tag{43}$$

Substituting the value of $\alpha_{h,opt}$ in (43) for α_h in (37) gives the generalized chain ratio-product asymptotically optimum estimator (AOE) for population mean ($\bar{Y}$) in stratified sampling as:

$$\bar{y}_{RP}^{(GC)} = \sum_{h=1}^{H} W_h \bar{y}_h \left\{\left(\frac{\lambda_h + K_h}{2\lambda_h}\right)\left(\frac{\bar{X}_h}{\bar{x}_h}\right)^{\lambda_h} + \left(\frac{\lambda_h - K_h}{2\lambda_h}\right)\left(\frac{\bar{x}_h}{\bar{X}_h}\right)^{\lambda_h}\right\} \tag{44}$$

Similarly, substituting the value of $\alpha_{h,opt}$ in (43) for α_h in (42) gives the variance of asymptotically optimum estimator (AOE) $\left(\bar{y}_{RP}^{(GC)}\right)_{opt}$ (or minimum variance of $\bar{y}_{RP}^{(GC)}$ as:

$$\mathrm{Var}\left(\bar{y}_{RP}^{(GC)}\right)_{opt} = \sum_{h=1}^{H} W_h^2 \gamma_h S_{hy}^2 \left(1 - \rho_{hxy}^2\right) \tag{45}$$

4.3 Permissible departure

Let K_h^* be defined such that $K_h^* = K_h(1+\beta_h)$, then

$$\alpha_h = \left(\frac{K_h^* + \lambda_h}{2\lambda_h}\right) = \left(\frac{(K_h + \lambda_h) + K_h\beta_h}{2\lambda_h}\right)$$

$$\alpha_h = \left(\frac{K_h + \lambda_h}{2\lambda_h}\right) + \frac{K_h\beta_h}{2\lambda_h}$$

$$\alpha_h = \left(\alpha_{h,opt} + \frac{K_h\beta_h}{2\lambda_h}\right) \tag{46}$$

Substituting (46) into (42), gives

$$\mathrm{Var}(\bar{y}_{RP}^{(GC)}) = \sum_{h=1}^{H} W_h^2 \gamma_h \bar{Y}_h^2 \{C_{hy}^2 + K_h^2 C_{hx}^2(\beta_h^2 - 1)\}$$

Using the results in (16), gives

$$\mathrm{Var}(\bar{y}_{RP}^{(GC)}) = \sum_{h=1}^{H} W_h^2 \gamma_h S_{hy}^2 [(1-\rho_{hxy}^2)+\beta_h^2 \rho_{hxy}^2]$$

$$\mathrm{Var}\left(\bar{y}_{RP}^{(GC)}\right) = \sum_{h=1}^{H} W_h^2 \gamma_h S_{hy}^2 (1-\rho_{hxy}^2)(1+\Omega_h^2) \quad (47)$$

where

$$\Omega_h^2 = \frac{\beta_h^2 \rho_{hxy}^2}{(1-\rho_{hxy}^2)}$$

Following from (47), it is deduced that

$$\mathrm{Var}\left(\bar{y}_{RP}^{(GC)}\right) - \mathrm{Var}\left(\bar{y}_{RP}^{(GC)}\right)_{opt} = \sum_{h=1}^{H} W_h^2 \gamma_h S_{hy}^2 (1-\rho_{hxy}^2)\,\Omega_h^2 \quad (48)$$

It follows that to ensure only a small increase in the variance $|\theta_h|$ must be in the neighbourhood of zero if ρ_{hxy} is large but can depart substantially from zero if ρ_{hxy} is moderate.

5. Some existing members of the proposed generalized chain ratio-product estimator

In this section it is shown how the existing estimators mentioned in section 2 fit into the proposed estimator. A summary of these estimators with the prescribed restrictions is provided in Table 1.

Table 1: Some existing members of the proposed estimator

S/No.	α_h	λ_h	*Estimator*
1.	0	0	$\sum_{h=1}^{H} W_h \bar{y}_h$ Unbiased stratified sampling
2.	1	1	$\sum_{h=1}^{H} W_h \bar{y}_h \left(\frac{\bar{X}_h}{\bar{x}_h}\right)$ Classical ratio
3.	0	1	$\sum_{h=1}^{H} W_h \bar{y}_h \left(\frac{\bar{x}_h}{\bar{X}_h}\right)$ Classical product
4.	1	2	$\sum_{h=1}^{H} W_h \bar{y}_h \left(\frac{\bar{X}_h}{\bar{x}_h}\right)^2$ Chain ratio
5.	0	2	$\sum_{h=1}^{H} W_h \bar{y}_h \left(\frac{\bar{x}_h}{\bar{X}_h}\right)^2$ Chain product
6.	α_h	1	$\sum_{h=1}^{H} w_h \bar{y}_h \left\{\alpha_h \frac{\bar{X}_h}{\bar{x}_h} + (1-\alpha_h)\frac{\bar{x}_h}{\bar{X}_h}\right\}$ Separate ratio-product
7.	α_h	$\frac{1}{2}$	$\sum_{h=1}^{H} W_h \bar{y}_h \left\{\alpha_h \sqrt{\left(\frac{\bar{X}_h}{\bar{x}_h}\right)} + (1-\alpha_h)\sqrt{\left(\frac{\bar{x}_h}{\bar{X}_h}\right)}\right\}$ Partial chain ratio-product
8.	α_h	2	$\sum_{h=1}^{H} W_h \bar{y}_h \left\{\alpha_h \left(\frac{\bar{X}_h}{\bar{x}_h}\right)^2 + (1-\alpha_h)\left(\frac{\bar{x}_h}{\bar{X}_h}\right)^2\right\}$ Chain ratio-product
9.	1	λ_h	$\sum_{h=1}^{H} W_h \bar{y}_h \left(\frac{\bar{X}_h}{\bar{x}_h}\right)^{\lambda_h}$ Generalized chain ratio
10.	0	λ_h	$\sum_{h=1}^{H} W_h \bar{y}_h \left(\frac{\bar{x}_h}{\bar{X}_h}\right)^{\lambda_h}$ Generalized chain product

Table 1 shows that by altering the values of α_h and λ_h in equation (37), the existing estimators listed in section 2 become special cases as shown in section 5.

6. Comparisons

From (2), (4), (6), (8), (10) and (12), it is established that

(i) $\mathrm{Var}\left(\bar{y}_{RP}^{(GC)}\right)_{opt} < Var(\bar{y}_{st})$ if

$$\sum_{h=1}^{H} w_h^2\gamma_h \bar{Y}_h^2 C_{hy}^2\rho_{hxy}^2 > 0$$

(ii) $\mathrm{Var}\left(\bar{y}_{RP}^{(GC)}\right)_{opt} < Var\left(\bar{y}_{R}^{(S)}\right)$ if

$$\sum_{h=1}^{H} w_h^2\gamma_h \bar{Y}_h^2\left[C_{hx}^2(1-2K_h)+C_{hy}^2\rho_{hxy}^2\right] > 0$$

$\left|C_{hx}^2(1-2K_h)\right| < C_{hy}^2\rho_{hxy}^2$, $K_h > \frac{1}{2}$ or $C_{hx}^2(1-2K_h) \geq 0, 0 < K_h \leq \frac{1}{2}$

Or equivalently,

$$min\left\{0,\frac{1}{2}\right\} < K_h < max\left\{0,\frac{1}{2}\right\}$$

(iii) $\mathrm{Var}\left(\bar{y}_{RP}^{(GC)}\right)_{opt} < Var\left(\bar{y}_{P}^{(S)}\right)$ if

$$\sum_{h=1}^{H} w_h^2\gamma_h \bar{Y}_h^2\left[C_{hx}^2(1+2K_h)+C_{hy}^2\rho_{hxy}^2\right] > 0$$

$\left|C_{hx}^2(1+2K_h)\right| < C_{hy}^2\rho_{hxy}^2$, $K_h < \frac{1}{2}$ or $C_{hx}^2(1+2K_h) \geq 0, \frac{1}{2} < K_h \leq 0$

Or equivalently,

$$\min\left\{\frac{1}{2},0\right\} < K_h < \max\left\{\frac{1}{2},0\right\}$$

(iv) $\mathrm{Var}\left(\bar{y}_{RP}^{(GC)}\right)_{opt} < Var\left(\bar{y}_{R}^{(C)}\right)$ if

$$\sum_{h=1}^{H} w_h^2\gamma_h \bar{Y}_h^2\left[4C_{hx}^2(1+K_h)+C_{hy}^2\rho_{hxy}^2\right] > 0$$

$\left|4C_{hx}^2(1+K_h)\right| < C_{hy}^2\rho_{hxy}^2$, $K_h > 1$ or $4C_{hx}^2(1+K_h) \geq 0, 0 < K_h \leq 1$

Or equivalently,

$$min\{1,0\} < K_h < max\{1,0\}$$

(v) $\mathrm{Var}\left(\bar{y}_{RP}^{(GC)}\right)_{opt} < Var\left(\bar{y}_{P}^{(C)}\right)$ if

$$\sum_{h=1}^{H} w_h^2\gamma_h \bar{Y}_h^2\left[4C_{hx}^2(1-K_h)+C_{hy}^2\rho_{hxy}^2\right] > 0$$

$\left|4C_{hx}^2(1-K_h)\right| < C_{hy}^2\rho_{hxy}^2$, $K_h > 1$ or $4C_{hx}^2(1-K_h) \geq 0, 0 < K_h \leq 1$

Or equivalently,

$$\min\{0,1\} < K_h < \max\{0,1\}$$

(vi) $\mathrm{Var}\left(\bar{y}_{RP}^{(GC)}\right)_{opt} < Var\left(\bar{y}_{RP}^{(S)}\right)$ if

$$\sum_{h=1}^{H} w_h^2\gamma_h \bar{Y}_h^2\left[C_{hx}^2(1-2\alpha_h)\{(1-2\alpha_h)+2K_h\}+C_{hy}^2\rho_{hxy}^2\right] > 0$$

$\left|C_{hx}^2(1-2\alpha_h)\{(1-2\alpha_h)+2K_h\}\right| < C_{hy}^2\rho_{hxy}^2$,

$K_h > 0, \alpha_h = \frac{1}{2}$ or $(1-2\alpha_h) \geq 0,$

$K_h > \alpha_h - \frac{1}{2}$ or $(1-2\alpha_h)+2K_h \geq 0$

Or equivalently,

$$\min\left\{0,\alpha_h-\frac{1}{2}\right\} < K_h < \max\left\{0,\alpha_h-\frac{1}{2}\right\}$$

7. Conclusion

This paper has provided a unified framework for chain ratio-product estimation and proposed a generalized chain ratio-product estimator for estimating population mean in stratified sampling. Analytical comparisons showed that the estimator of variance of the proposed generalized chain ratio-product estimator is more efficient than the estimators of variance of the unbiased stratified sampling estimator $(\bar{y}_{st})$, classical ratio estimator $\left(\bar{y}_R^{(S)}\right)$, classical product estimator $\left(\bar{y}_P^{(S)}\right)$, chain ratio estimator $\left(\bar{y}_R^{(C)}\right)$, chain product estimator $\left(\bar{y}_P^{(C)}\right)$, and the separate ratio-product estimator $\left(\bar{y}_{RP}^{(S)}\right)$ under certain realistic conditions. It is observed that the new estimator is very attractive and should be preferred in practice as it provides consistent and more precise parameter estimates.

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