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A Fixed Point Theorem for (ε, δ) Type Contraction Mappings in Metrically Convex Spaces

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ABSTRACT

The aim of this paper is to establish a fixed point theorem for non-self mappings by using the Meir and Keeler (ε, δ) type contraction condition. Our result generalizes completely or partially the result due to Meir and Keeler [5], Rhoades [10] and others.

Introduction

In 1969, Meir and Keeler [5] proved that Banach Contraction Principle remains true for weakly uniformly strict contractions: Given $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$\varepsilon < d(x, y) < \varepsilon + \delta \text{ implies } d(Tx, Ty) < \varepsilon.$$

The result due to Meir and Keeler [5] has been generalized and extended in various ways and there exists a considerable literature for self mappings. To mention a few, we cite [3,5,6,7,8,9].

In this paper, we prove a Meir and Keeler [5] type fixed point theorem for single-valued non-self mappings in metrically convex spaces by using the ideas of Rhoades [10]. In proving the result, we follow the definitions and conventions of Assad [1] and Assad and Kirk [2]. Before formulating our result, for the sake of completeness, we state the following result due to Rhoades [10].

Theorem 1.1. ([10]) Let (X, d) be a complete metrically convex metric space and K be a nonempty closed convex subset of X. Let $T : K \to X$ be a mapping satisfying:

$$d(Tx, Ty) \le M(x, y)$$

where $M(x, y) = \max \{1/2\, d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Ty) + d(y, Tx)}{q} \}$

for all $x, y \in K$, with $x \neq y$, where $0 < h < 1$, $q \geq 1 + 2h$ and $Tx \in K$ for each $x \in \delta K$. Then T has a fixed point in K.

Definition 1.2.

Let K be a nonempty subset of a metric space X and the mapping $T : K \to X$ is said to be (ϵ, δ) contraction if there exists a function $\delta : (0, \infty) \to (0, \infty)$ such that, for any $\epsilon > 0$, $\delta(\epsilon) > \epsilon$ and

$$\epsilon \le M(x, y) = \max \{ d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Ty) + d(y, Tx)}{2} \} < \epsilon(\delta) \qquad (1.1)$$

which implies that $d(Tx, Ty) < \epsilon$.

Definition 1.3. ([2])

A metric space (X, d) is said to be metrically convex if for any $x, y \in X$ with $x \neq y$ there exists a point $z \in X$, $x \neq z \neq y$ such that

$$d(x, z) + d(z, y) = d(x, y).$$

Lemma 1.4. ([2])

Let K be a nonempty closed subset of a metrically convex metric space X. If $x \in K$ and $y \notin K$ then there exists a point $z \in \delta K$ (the boundary of K) such that

$$d(x, z) + d(z, y) = d(x, y).$$

2. Result

The main result runs as follows.

Theorem 2.1.

Let (X, d) be a complete metrically convex metric space and K be a nonempty closed subset of X. The mapping $T : K \to X$ satisfying (1.1) and

(i) for each $x \in K$, $Tx \in K$.

Then T has a unique fixed point in K.

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Proof

Firstly, we proceed to construct two sequences $\{x_n\}$ and $\{y_n\}$ in the following way. Let $x_0 \in K$. Define $y_1 = T x_0$. If $y_1 \in K$ set $y_1 = x_1$. If $y_1 \notin K$, then choose $x_1 \in \delta K$ so that

$$d(x_0, x_1) + d(x_1, y_1) = d(x_0, y_1).$$

If $y_2 \in K$ then set $y_2 = x_2$. If $y_2 \notin K$, then choose $x_2 \in \delta K$ so that

$$d(x_1, x_2) + d(x_2, y_2) = d(x_1, y_2).$$

Thus, repeating the foregoing arguments, one obtains two sequences $\{x_n\}$ and $\{y_n\}$ such that

(ii) $y_{n+1} = T x_n$,

(iii) $y_n = x_n$ if $y_n \in K$,

(iv) if $x_n \in \delta K$ then $d(x_{n-1}, x_n) + d(x_n, y_n) = d(x_{n-1}, y_n)$, where $y_n \notin K$.

Here, one obtains two types of sets we denote as follows:

$$P = \{x_i \in \{x_n\} : x_i = y_i\} \text{ and } Q = \{x_i \in \{x_n\} : x_i \neq y_i\}.$$

One can note that, if $x_n \in Q$ then x_{n-1} and $x_{n+1} \in P$. We wish to estimate $d(x_n, x_{n+1})$. Now, we distinguish the following three cases.

Case 1. If $(x_n, x_{n+1}) \in P$, then $d(x_n, x_{n+1}) = d(T x_{n-1}, T x_n) \leq M(x_{n-1}, x_n)$

$$\leq \max\left\{ d(x_{n-1}, x_n), d(x_{n-1}, T x_{n-1}), d(x_n, T x_n), \frac{d(x_{n-1}, T x_n) + d(x_n, T x_{n-1})}{2} \right\}$$

$$\leq \max\left\{ d(x_{n-1}, x_n), d(x_{n-1}, x_n), d(x_n, x_{n+1}), \frac{d(x_{n-1}, x_{n+1}) + d(x_n, x_n)}{2} \right\}$$

$$\leq \max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\}.$$

If we suppose that $d(x_{n-1}, x_n) \leq d(x_n, x_{n+1})$, then we get $d(x_n, x_{n+1}) \leq d(x_n, x_{n+1})$, which is a contradiction. Otherwise, if $d(x_n, x_{n+1}) \leq d(x_{n-1}, x_n)$ then, one obtains

$$d(x_n, x_{n+1}) \leq M(x_{n-1}, x_n) \leq d(x_{n-1}, x_n).$$

Case 2. If $x_n \in P$ and $x_{n+1} \in Q$, then

$$d(x_n, x_{n+1}) + d(x_{n+1}, y_{n+1}) = d(x_n, y_{n+1})$$

which in turn yields $d(x_n, x_{n+1}) \leq d(x_n, y_{n+1})$. Now, proceeding as Case 1, we have

$$d(x_n, x_{n+1}) \leq M(x_{n-1}, x_n) \leq d(x_{n-1}, x_n).$$

Case 3. If $x_n \in Q$ and $x_{n+1} \in P$. Since $x_n \in Q$ and is a convex linear combination of x_{n-1} and y_n, it follows that

$$d(x_n, x_{n+1}) \leq \max\{d(x_{n-1}, x_{n+1}), d(x_{n+1}, y_n)\}. \tag{2.1}$$

Now, if $d(x_{n-1}, x_{n+1}) \leq d(y_n, x_{n+1})$, then $d(x_n, x_{n+1}) \leq d(y_n, x_{n+1}) = d(x_n, x_{n+1})$.

Proceeding as above, one gets $d(x_n, x_{n+1}) \leq M(x_{n-1}, x_n) \leq d(x_{n-1}, x_n)$.

Next, if $d(y_n, x_{n+1}) \leq d(x_{n-1}, x_{n+1})$, then

$$d(x_n, x_{n+1}) \leq d(x_{n-1}, x_{n+1}) = d(T x_{n-2}, T x_n) = M(x_{n-2}, x_n)$$

$$\leq \max\left\{d(x_{n-2}, x_n), d(x_{n-2}, T x_{n-2}), d(x_n, T x_n), \frac{d(x_{n-2}, T x_n) + d(x_n, T x_{n-2})}{2}\right\}$$

$$\leq \max\left\{d(x_{n-2}, x_n), d(x_{n-2}, x_{n-1}), d(x_n, x_{n+1}), \frac{d(x_{n-2}, x_{n+1}) + d(x_n, x_{n-1})}{2}\right\}.$$

Here, three cases are possible, either $d(x_{n-2}, x_n)$, $d(x_{n-2}, x_{n-1})$ or $\frac{d(x_{n-2}, x_{n+1}) + d(x_n, x_{n-1})}{2}$ will be maximum.

Firstly, we choose $d(x_{n-2}, x_n)$ as maximum then

$$d(x_n, x_{n+1}) \leq d(x_{n-2}, x_n) \leq d(x_{n-2}, x_{n-1}) + d(x_{n-1}, x_n) \leq \max\{d(x_{n-2}, x_{n-1}), d(x_{n-1}, x_n)\}.$$

When $d(x_{n-2}, x_{n-1})$ is maximum then $d(x_n, x_{n+1}) \leq d(x_{n-2}, x_{n-1})$, otherwise

$$d(x_n, x_{n+1}) \leq d(x_{n-1}, x_n).$$

Secondly, we choose $d(x_{n-2}, x_{n-1})$ as maximum then $d(x_n, x_{n+1}) \leq d(x_{n-2}, x_{n-1})$.

Finally, if $\frac{d(x_{n-2}, x_{n+1}) + d(x_n, x_{n-1})}{2}$ is maximum, then

$$d(x_n, x_{n+1}) \leq \frac{1}{2}\{d(x_{n-2}, x_{n-1}) + d(x_{n-1}, x_{n+1}) + d(x_n, x_{n-1})\}$$

$$\leq \frac{1}{2}\{d(x_{n-2}, x_{n-1}) + d(x_n, x_{n-1})\} \leq \max\{d(x_{n-2}, x_{n-1}), d(x_n, x_{n-1})\}.$$

Thus in all the cases, we have

$$d(x_n, x_{n+1}) \leq \max\{d(x_{n-1}, x_n), d(x_{n-2}, x_{n-1})\}.$$

Hence

$$d(x_n, x_{n+1}) \leq M(x_{n-1}, x_n) \leq \max\{d(x_{n-1}, x_n), d(x_{n-2}, x_{n-1})\}.$$

It can be easily shown by induction that for $n > 1$, we have

$$d(x_n, x_{n+1}) \leq \max\{d(x_0, x_1), d(x_1, x_2)\}.$$

Thus $d(x_n, x_{n+1})$ is decreasing and tending to 0 as $n \to \infty$. Hence $d(x_n, x_{n+1})$ converges to $t \in [0, \infty)$. If $t = 0$ then conclusion is trivial. So, suppose that $t > 0$. Since $d(x_n, x_{n+1})$ converges to t, the condition (1.1) yields

$t \le M(x_{n-1}, x_n) < \delta(t)$.

Therefore $d(Tx_{n-1}, Tx_n) = d(x_n, x_{n+1}) < t$, this contradicts the fact. Thus

$$d(x_n, x_{n+1}) \to 0 \text{ as } n \to \infty \tag{2.2}$$

Now, we show that for every $\epsilon > 0$ there exists $m, n \ge N$ such that

$\le d(x_n, x_{n-1}) < \delta(\epsilon) \Rightarrow d(Tx_n, Tx_{n-1}) < \epsilon$.

Also, (2.2) implies that there exists an integer N such that $n \ge N$, $d(x_n, x_{n-1}) < \delta'/2$. for all $n \in N$

where $\delta' = \min\{\epsilon/2, (\delta(\epsilon) - \epsilon)/2\}$.

Let $p, q \in N$, where $p = n$ and $q = m - 1$, therefore

$$\epsilon \le d(x_n, x_{m-1})$$

$$\le \max\left\{d(x_n, x_{m-1}), d(x_n, Tx_n), d(x_{m-1}, Tx_{m-1}), \frac{d(x_n, Tx_{m-1}) + d(x_{m-1}, Tx_n)}{2}\right\}$$

$$\le \max\left\{d(x_n, x_{m-1}), d(x_n, x_{n+1}), d(x_{m-1}, x_m), \frac{d(x_n, x_m) + d(x_{m-1}, x_{n+1})}{2}\right\}$$

$$\le \max\left\{d(x_p, x_q), \frac{d(x_p, x_{q+1}) + d(x_q, x_{p+1})}{2}\right\}$$

$$\le \max\left\{d(x_p, x_q), \frac{d(x_p, x_q) + d(x_q, x_{q+1}) + d(x_q, x_p) + d(x_p, x_{p+1})}{2}\right\}$$

$$\le \frac{2d(x_p, x_q) + d(x_q, x_{q+1}) + d(x_p, x_{p+1})}{2}$$

$$\le d(x_p, x_q) + \frac{d(x_q, x_{q+1}) + d(x_p, x_{p+1})}{2}$$

$$\le d(x_p, x_q) + ½(\delta'/2 + \delta'/2) \le \epsilon + \delta' + \delta'/2 < \epsilon + 2\delta' < \epsilon + 2(\delta(\epsilon) - \epsilon)/2 < \delta(\epsilon).$$

Therefore

$\epsilon \le d(x_p, x_q) < \delta(\epsilon)$, yielding thereby $\epsilon \le d(x_n, x_{n-1}) < \delta(\epsilon)$, which implies that $d(Tx_{n-1}, Tx_n) \le \epsilon$.

Now, we show that the sequence $\{x_n\}$ is Cauchy. If this sequence is not Cauchy then there exists $2\epsilon > 0$ such that $d(x_m, x_n) > 2\epsilon$.

For any $j \in [m, n]$, one gets $d(x_m, x_j) \le d(x_m, x_{j+1}) + d(x_{j+1}, x_j)$, which in turn yields

$|d(x_m, x_j) - d(x_m, x_{j+1})| \le d(x_{j+1}, x_j) < \delta'/2$, and $d(x_m, x_n) > 2\epsilon = \epsilon + \epsilon \ge \epsilon + \delta'$,

which implies that there exists a $j \in [m, n]$ with $d(x_m, x_j) \ge \epsilon + \delta'$. However, for m and j, we have

$d(x_m, x_j) \le d(x_m, x_{m+1}) + d(x_{m+1}, x_{j+1}) + d(x_{j+1}, x_j) \le \epsilon + \delta'/2 + \delta'/2 = \epsilon + \delta'$,

which is indeed a contradiction, therefore one, may conclude that the sequence $\{x_n\}$ is Cauchy and it converges to a point z in X.

Now, we assume that there exists a subsequence $\{x_{nk}\}$ of $\{x_n\}$ which is contained in P. Using (1.1), one can write

$d(Tz, z) \le d(Tz, Tx_{nk}) + d(Tx_{nk}, z)$,

$$\le \max\left\{d(z, x_{nk}), d(z, T_z), d(x_{nk}, Tx_{nk}), \frac{d(z, Tx_{nk}) + d(x_{nk}, T_z)}{2}\right\} + d(Tx_{nk}, z).$$

On letting $k \to \infty$, one obtains $d(Tz, z) \le d(Tz, z)$, which is a contradiction, implies that $Tz = z$. This shows that z is a fixed point of T .

To prove the uniqueness of the fixed point z, let z_0 be the another fixed point of T, then

$$d(z, z_0) = d(Tz, Tz_0) \le \max\left\{d(z, z_0), d(z, Tz), d(z_0, Tz_0), \frac{d(z, Tz_0) + d(z_0, Tz)}{2}\right\} \le d(z, z_0)$$

implying there by $z = z_0$. This completes the proof.

Remark 2.2.

By setting $K = X$ in the Theorem 2.1, then we deduce a theorem due to Meir and Keeler [5].

Remark 2.3.

By setting $K = X$ in the Theorem 2.1, then we deduce a partial generalization of theorem due to Rhoades [10] Finally, an example is furnish to establish the existence of the result.

Example 2.4.

Let $X = R$ with Euclidean metric and let $K = \{-1/4\} \cup [0, 1/2]$. Define the mapping $T : K \to X$ as

$Tx = x^2 - 1/4$, if $0 \le x \le ½$ and $Tx = -1/4$, if $x = -1/4$.

Since δK (the boundary of K) $= \{-1/4, 0, 1/2\}$. Also

$x = -1/4$ implies that $T(-1/4) = -1/4 \in K$, $x = 0$ implies that $T(0) = -1/4 \in K$ and

$x = ½$ implies that $T(1/2) = 0 \in K$.

This shows that $x \in \delta K$ implies that $Tx \in K$.

Moreover, if for $x, y \in [0, ½]$, then

$$d(Tx, Ty) = |x^2 - y^2| \le \max\left\{d(x, y), d(x, Tx), d(y, Ty), \frac{d(x,Ty) + d(y,Tx)}{2}\right\}$$

Next, if $x \in [0, 1/2]$ and $y = -1/4$, then

$$d(Tx, Ty) = x^2 = d(y, Tx) \le \max\left\{d(x, y), d(x, Tx), (y, Ty), \frac{d(x,Ty) + d(y,Tx)}{2}\right\}$$

Finally, if $y \in [0, 1/2]$ and $x = -1/4$, then

$$d(Tx, Ty) = y^2 = d(x, Ty) \leq \max\{ d(x, y), d(x, Tx), (y, Ty), \frac{d(x,Ty) + d(y,Tx)}{2} \}$$

This shows that the contraction condition (1.1) is satisfied for every x, y $\in$ K. Thus all the conditions of the Theorem 2.1 are satisfied and (-1/4) is a fixed point of T.

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