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Geodetic Dominating Sets and Geodetic Dominating Polynomials of Paths

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ABSTRACT

Let G = (V,E) be a simple graph. A set $S \subseteq V$ is a dominating set of G, if every vertex in V– S is adjacent to atleast one vertex S. Let $D_g(P_n, i)$ be the family of geodetic dominating sets of the graph P_n with cardinality 'i'. Let $dg(P_n, i) = | D_g(P_n, i)|$. In this paper, we obtain a recursive for $d_g(P_n, i)$. Using the recursive formula, we construct the polynomial, $D_g(P_n,x) = \sum_{i=\lceil \frac{n+2}{5} \rceil}^{n} d_g(P_n, i)\, x^i$ which we call geodetic dominating polynomial of P_n and obtain some properties of this polynomial.

1. Introduction

For any graph 'G', the set of vertices is denoted by V(G) and the edge set by E(G). The order and size of G are denoted by p and q respectively. For a vertex $v \in V(G)$, the open neighborhood N(v) is the set of all vertices adjacent to v, and $N[v] = N(v) \cup \{v\}$, is the closed neighborhood of v. The degree d(v) of a vertex v is defined by $d(v) = |N(v)|$.

A subset S of vertices in a graph G in called a geodetic set if every vertex not in S lies on a shortest path between two vertices from S. A subset D of the set of vertices G is called dominating set if every vertex not in D has atleast one neighbour in D.

1.1 Geodetic dominating set

A set of vertices S in a graph G is a geodetic dominating set if S is both a geodetic set and a dominating set. The minimum cardinality of geodetic dominating set of G is its geodetic domination number, and is denoted by $\gamma_g(G)$. A geodetic dominating set of size $\gamma_g(G)$ is said to be a $\gamma_g(G)$– set.

2. Geodetic dominating sets of path

Let $D_g(P_n, i)$ be the family of geodetic dominating sets of P_n with cardinality i. We investigate the geodetic dominating sets of the path P_n. We need the following lemmas to prove our main results in this section.

Lemma 2.1

$$\gamma_g(P_n) = \left\lceil \frac{n+2}{5} \right\rceil$$

By Lemma 2.1, and the definition of domination number , one has the following lemma:

Lemma 2.2

$D_g(P_n, i) = \Phi$ if and only if $i > n$ or $i < \left\lceil \frac{n+2}{5} \right\rceil$.

A simple path is a path in which all internal vertices have degree two.

Lemma 2.3

Let P_n, $n \geq 2$ be the path with $| V (P_n)| = n$

(i) If $D_g(P_{n-1}, -1) = D_g(P_{n-3}, i-1) = \Phi$ then $D_g(P_{n-2}, i-1) = \Phi$.

(ii) If $D_g(P_{n-1}, i-1) \neq \Phi$; $D_g(P_{n-3}, i-1) \neq \Phi$ then $D_g(P_{n-2}, i-1) \neq \Phi$.

(iii) If $D_g(P_{n-1}, i-1) = \Phi$; $D_g(P_{n-2}, i-1) = \Phi$; $D_g(P_{n-3}, i-1) = \Phi$; $D_g(P_{n-4}, i-1) = \Phi$; $D_g(P_{n-5}, i-1) = \Phi$ then $D_g(P_n, i) = \Phi$.

Proof

(1) If $D_g(P_{n-1}, i-1) = \Phi$ and $D_g(P_{n-3}, i-1) = \Phi$

then $i - 1 < \left\lceil \frac{n+1}{5} \right\rceil$ or $i - 1 > n - 1$ and

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$i-1 < \left\lceil \frac{n-1}{5} \right\rceil$ or $i-1 > n-3$.

Therefore, $i-1 < \left\lceil \frac{n-1}{5} \right\rceil$ or $i-1 > n-1$

Therefore, $i-1 < \left\lceil \frac{n}{5} \right\rceil$ or $i-1 > n-2$ holds.

Therefore, $D_g(P_n-2, i-1) = \Phi$.

(ii) If $D_g(P_{n-1}, i-1) \neq \Phi$ and $D_g(P_{n-3}, i-1) \neq \Phi$,

then $\left\lceil \frac{n+1}{5} \right\rceil \leq i-1 \leq n-1$ and $\left\lceil \frac{n-1}{5} \right\rceil \leq i-1 \leq n-3$ Therefore, $\left\lceil \frac{n+1}{5} \right\rceil \leq i-1 \leq n-3$

Therefore, $\left\lceil \frac{n}{5} \right\rceil < \left\lceil \frac{n+1}{5} \right\rceil \leq i-1 \leq n-3 < n-2$

Therefore, $\left\lceil \frac{n}{5} \right\rceil < i-1 < n-2$

Hence, $D_g(P_{n-2}, i-1) \neq \Phi$.

(iii) If $D_g(P_{n-1}, i-1) = \Phi$; $D_g(P_{n-2}, i-1) = \Phi$; $D_g(P_{n-3}, i-1) = \Phi$; $D_g(P_{n-4}, i-1) = \Phi$ and $D_g(P_{n-5}, i-1) = \Phi$ then

$i-1 < \left\lceil \frac{n+1}{5} \right\rceil$ or $i-1 > n-1$;

$i-1 < \left\lceil \frac{n}{5} \right\rceil$ or $i-1 > n-2$;

$i-1 < \left\lceil \frac{n-1}{5} \right\rceil$ or $i-1 > n-3$;

$i-1 < \left\lceil \frac{n-2}{5} \right\rceil$ or $i-1 > n-4$;

$i-1 < \left\lceil \frac{n-3}{5} \right\rceil$ or $i-1 > n-5$

Therefore, $i-1 < \left\lceil \frac{n-3}{5} \right\rceil$ or $i-1 > n-1$.

Therefore, $i < \left\lceil \frac{n-3}{5} \right\rceil + 1$ or $i > n$

Therefore, $i < \left\lceil \frac{n+2}{5} \right\rceil$ or $i > n$.

Therefore, $D_g(P_n, i) = \Phi$

Hence, the theorem.

Lemma 2.4

Let P_n, $n \geq 2$ be the path with $|V(P_n)| = n$. Suppose that $D_g(P_n, i) \neq \Phi$, then we have,

(i) $D_g(P_{n-1}, i-1) = D_g(P_{n-2}, i-1) = D_g(P_{n-3}, i-1) = D_g(P_{n-4}, i-1) = \Phi$ and $D_g(P_{n-5}, i-1) \neq \Phi$ iff $n = 5k+3$, $i = k+1$, for some positive integer k.

(ii) If $D_g(P_{n-2}, i-1) = D_g(P_{n-3}, i-1) = D_g(P_{n-4}, i-1) = D_g(P_{n-5}, i-1) = \Phi$ then $D_g(P_{n-1}, i-1) \neq \Phi$ iff $i = n$.

(iii) $D_g(P_{n-1}, i-1) \neq \Phi$; $D_g(P_{n-2}, i-1) \neq \Phi$; $D_g(P_{n-3}, i-1) \neq \Phi$; $D_g(P_{n-4}, i-1) \neq \Phi$ and $D_g(P_{n-5}, i-1) = \Phi$ iff $i = n-3$.

(iv) $D_g(P_{n-1}, i-1) = \Phi$; $D_g(P_{n-2}, i-1) \neq \Phi$; $D_g(P_{n-3}, i-1) \neq \Phi$; $D_g(P_{n-4}, i-1) \neq \Phi$; $D_g(P_{n-5}, i-1) \neq \Phi$ iff $n = 5k$ and $i = 2k$ for some $k \in N$.

(v) $D_g(P_{n-1}, i-1) \neq \Phi$; $D_g(P_{n-2}, i-1) \neq \Phi$; $D_g(P_{n-3}, i-1) \neq \Phi$; $D_g(P_{n-4}, i-1) \neq \Phi$; $D_g(P_{n-5}, i-1) \neq \Phi$

iff $\left\lceil \frac{n+1}{5} \right\rceil + 1 \leq i \leq n-4$.

Proof

(i) Since, $D_g(P_{n-1}, i-1) = D_g(P_{n-2}, i-1) = D_g(P_{n-3}, i-1) = D_g(P_{n-4}, i-1) = \Phi$, by Lemma 2.2,

$i-1 > n-1$ or $i-1 < \left\lceil \frac{n+1}{5} \right\rceil$,

$i-1 > n-2$ or $i-1 < \left\lceil \frac{n}{5} \right\rceil$,

$i - 1 > n - 3$ or $i - 1 < \left\lceil \frac{n-1}{5} \right\rceil$ and

$i - 1 > n - 4$ or $i - 1 < \left\lceil \frac{n-2}{5} \right\rceil$

Therefore, $i - 1 < \left\lceil \frac{n-2}{5} \right\rceil$ or $i - 1 > n - 1$

If $i - 1 > n - 1$ then $i > n$

Therefore, $D_g(P_n, i) = \Phi$ which is a contradiction.

Therefore, $i - 1 < \left\lceil \frac{n-2}{5} \right\rceil$

Therefore, $i < \left\lceil \frac{n-2}{5} \right\rceil + 1$, and since $D_g(P_n, i) \neq \Phi$,we have $\left\lceil \frac{n+2}{5} \right\rceil \leq i < \left\lceil \frac{n-2}{5} \right\rceil + 1$ which implies that $n = 5k + 3$ and $i = k + 1$ for some $k \in N$.

Conversely assume $n = 5k + 3$ and $i = k + 1$ for some $k \in N$.

By Lemma 2.2,

$$\gamma_g(P_n) = \left\lceil \frac{n+2}{5} \right\rceil$$

Therefore, $D_g(P_{n-1}, i -1) = D_g(P_{5k+3-1}, k) = \Phi$,

since $k < \left\lceil \frac{5k+3+2}{5} \right\rceil = \left\lceil \frac{5k+5}{5} \right\rceil$

Similarly, $D_g(P_{n-2}, i - 1) = \Phi$; $D_g(P_{n-3}, i - 1) = \Phi$

and $D_g(P_{n-4}, i -1) = \Phi$.

$D_g(P_{n-5}, i -1) = D_g(P_{5k+3-5}, k + 1-1) = D_g(P_{5k-2}, k)$,

since $k \geq \left\lceil \frac{5k-2+2}{5} \right\rceil = \left\lceil \frac{5k}{5} \right\rceil$

Therefore, $D_g(P_{n-5}, i -1) \neq \Phi$.

Hence, $D_g(P_{n-1}, i -1) = \Phi$; $D_g(P_{n-2}, i -1) = \Phi$;

$D_g(P_{n-3}, i -1) = \Phi$; $D_g(P_{n-4}, i -1) = \Phi$ and $D_g(P_{n-5}, i -1) \neq \Phi$.

(ii) Since $D_g(P_{n-2}, i- 1) = \Phi$; $D_g(P_{n-3}, i - 1) = \Phi$;

$D_g(P_{n-4}, i -1) = \Phi$ and $D_g(P_{n-5}, i -1) = \Phi$

By Lemma 2.2 ,

$i - 1 > n - 2$ or $i - 1 < \left\lceil \frac{n}{5} \right\rceil$

If $i - 1 < \left\lceil \frac{n}{5} \right\rceil$ then $i - 1 < \left\lceil \frac{n+1}{5} \right\rceil$

Therefore, by lemma 2.2,$D_g(P_{n-1}, i -1) = \Phi$, which is a contradiction.

So we have $i - 1 > n - 2$

i.e, $i > n - 1$

Therefore, $i \geq n$

Since, $D_g(P_{n-1}, i -1) \neq \Phi$, then $\left\lceil \frac{n+1}{5} \right\rceil \leq i -1 \leq n - 1$

Therefore, $i \leq n$.

Hence, $i = n$.

Conversely, if $i = n$, then $D_g(P_{n-2}, i -1) = D_g(P_{n-2}, n -1) = \Phi$,

$D_g(P_{n-3}, i - 1) = D_g(P_{n-3}, n - 1) = \Phi$,

$D_g(P_{n-4}, i - 1) = D_g(P_{n-4}, n - 1) = \Phi$,

$D_g(P_{n-5}, i - 1) = D_g(P_{n-5}, n - 1) = \Phi$ and

$D_g(P_{n-1}, i - 1) = D_g(P_{n-1}, n -1) \neq \Phi$.

Since, $D_g|P_{n-1}, n-1| = 1$.

(iii) Since, $D_g(P_{n-5}, i -1) = \Phi$, by Lemma 2.2,

$i - 1 > n - 5$ or $i - 1 < \left\lceil \frac{n-3}{5} \right\rceil$.

Since, $D_g(P_{n-2}, i-1) \neq \Phi$, $\left\lceil \frac{n}{5} \right\rceil < i - \leq n-2$

i.e, $i-1 < \left\lceil \frac{n-3}{5} \right\rceil$ is not possible.

Therefore, $i-1 > n-5$

Therefore, $i-1 \geq n-4$

But $i-1 \leq n-4$

Therefore, $i = n-3$

Conversely, suppose $i = n-3$, then

$D_g(P_{n-1}, i-1) = D_g(P_{n-1}, n-4) \neq \Phi$,

$D_g(P_{n-2}, i-1) = D_g(P_{n-2}, n-4) \neq \Phi$,

$D_g(P_{n-3}, i-1) = D_g(P_{n-3}, n-4) \neq \Phi$,

$D_g(P_{n-4}, i-1) = D_g(P_{n-4}, n-4) \neq \Phi$,

but $D_g(P_{n-5}, i-1) = D_g(P_{n-5}, n-4) = \Phi$.

By Lemma 2.2,

$D_g(P_{n-1}, i-1) \neq \Phi$, $D_g(P_{n-2}, i-1) \neq \Phi$, $D_g(P_{n-3}, i-1) \neq \Phi$,

$D_g(P_{n-4}, i-1) \neq \Phi$ and $D_g(P_{n-5}, i-1) = \Phi$.

(iv) Since $D_g(P_{n-1}, i-1) = \Phi$, by Lemma 2.2,

$i-1 > n-1$ or $i-1 < \left\lceil \frac{n+1}{5} \right\rceil$

If $i-1 > n-1$ then $i-1 > n-2$, by Lemma 2.2,

$D_g(P_{n-2}, i-1) = D_g(P_{n-3}, i-1) = D_g(P_{n-4}, i-1) = D_g(P_{n-5}, i-1) = \Phi$ which is a contradiction.

Therefore, $i-1 < \left\lceil \frac{n+1}{5} \right\rceil$

$$i < \left\lceil \frac{n+1}{5} \right\rceil + 1.$$

But $i-1 \geq \left\lceil \frac{n}{5} \right\rceil$, because $D_g(P_{n-2}, i-1) \neq \Phi$

Therefore, $i-1 \geq \left\lceil \frac{n}{5} \right\rceil + 1$

Hence, $\left\lceil \frac{n}{5} \right\rceil + 1 \leq i < \left\lceil \frac{n+1}{5} \right\rceil + \mathbf{1}$

This holds only if $n = 5k$ and $i = 2k$ for some $k \in N$.

Conversely, assume $n = 5k$ and $i = 2k$ for some $k \in N$, then by Lemma 2.2,

$D_g(P_{n-1}, i-1) = \Phi$; $D_g(P_{n-2}\, i-1) \neq \Phi$; $D_g(P_{n-3}\, i-1) \neq \Phi$,

$D_g(P_{n-4}\, i-1) \neq \Phi$ and $D_g(P_{n-5,} i-1) \neq \Phi$.

(v) Since $D_g(P_{n-1}, i-1) \neq \Phi$; $D_g(P_{n-2}, i-1) \neq \Phi$; $D_g(P_{n-3}, i-1) \neq \Phi$, $D_g(P_{n-4}, i-1) \neq \Phi$ and $D_g(P_{n-5}, i-1) \neq \Phi$ then we have

$\left\lceil \frac{n+1}{5} \right\rceil \leq i-1 \leq n-1$; $\left\lceil \frac{n}{5} \right\rceil \leq i-1 \leq n-2$;

$\left\lceil \frac{n-1}{5} \right\rceil \leq i-1 \leq n-3$; $\left\lceil \frac{n-2}{5} \right\rceil \leq i-1 \leq n-4$ and

$\left\lceil \frac{n-3}{5} \right\rceil \leq i-1 \leq n-5$.

$\left\lceil \frac{n+1}{5} \right\rceil \leq i-1 \leq n-5$ and hence $\left\lceil \frac{n+1}{5} \right\rceil + 1 \leq i \leq n-4$

Therefore, $\left\lceil \frac{n-1}{5} \right\rceil \leq i-1 \leq n-1$ and $\left\lceil \frac{n}{5} \right\rceil \leq i-1 \leq n-2$,

$\left\lceil \frac{n-1}{5} \right\rceil \leq i-1 \leq n-3$; $\left\lceil \frac{n-2}{5} \right\rceil \leq i-1 \leq n-4$ and $\left\lceil \frac{n-3}{5} \right\rceil \leq i-1 \leq n-5$.

From these, we obtain that $D_g(P_{n-1}, i-1) \neq \Phi$; $D_g(P_{n-2}, i-1) \neq \Phi$; $D_g(P_{n-3}, i-1) \neq \Phi$; $D_g(P_{n-4}, i-1) \neq \Phi$ and $D_g(P_{n-5}, i-1) \neq \Phi$.

Lemma 2.5

For every $n \geq 4$ and $i > \left\lceil \frac{n+2}{5} \right\rceil$;

(i) If $D_g(P_{n-2}, i-1) = D_g(P_{n-3}, i-1) = D_g(P_{n-4}, i-1) = D_g(P_{n-5}, i-1) = \Phi$ and $D_g(P_{n-1}, i-1) \neq \Phi$ then $D_g(P_n, i) = D_g(P_n, n) = \{\{1,2,3,..., n\}\}$.

(ii) If $D_g(P_{n-3}, i-1) = \Phi$; $D_g(P_{n-2}, i-1) \neq \Phi$, $D_g(P_{n-1}, i-1) \neq \Phi$ then $D_g(P_n, i) = \{[n] - \{x\}, x \in [n]\}$.

(iii) If $D_g(P_{n-1}, i-1) \neq \Phi$; $D_g(P_{n-2}, i-1) \neq \Phi$; $D_g(P_{n-3}, i-1) \neq \Phi$, $D_g(P_{n-4}, i-1) \neq \Phi$; $D_g(P_{n-5}, i-1) \neq \Phi$ then $D_g(P_n, i)=\{x_1 \cup \{n\} / x_1 \in D_g(P_{n-1}, i-1)\} \cup$
$\{x_2 \cup \{n-1\} / x_2 \in D_g(P_{n-2}, i-1)\} \cup$
$\{x_3 \cup \{n-2\} / x_3 \in D_g(P_{n-3}, i-1)\} \cup$
$\{\{n-2\} \cup (x_4 - x_3) / x_4 \in D_g(P_{n-4}, i-1)\} \cup$
$\{\{n-1\} \cup (x_5 - x_4) /, x_5 \in D_g(P_{n-5}, i-1)\}$.

Proof

(i) We have $D_g(P_{n-2}, i-1) = D_g(P_{n-3}, i-1) = D_g(P_{4\square n}, i-1) = D_g(P_{n-5}, i-1) = \Phi$ and $D_g(P_{n-1}, i-1) \neq \Phi$ by Lemma 2.4 (ii) we have i = n Therefore, $D_g(P_n, i) = D_g(P_n, n) = \{\{1,2,3, \ldots, n\}\}$.

(ii) If $D_g(P_{n-3}, i-1) = \Phi$; $D_g(P_{n-2}, i-1) = \Phi$;$D_g(P_{n-1}, i-1) \neq \Phi$,
by Lemma 2.4 , i = n-1.
Therefore, $D_g(P_n, i) = \{[n] - \{x\}, x \in [n]\}$.

(iii)Suppose $D_g(P_{n-1}, i-1) \neq \Phi$, $D_g(P_{n-2}, i-1) \neq \Phi$,
$D_g(P_{n-3}, i-1) \neq \Phi$, $D_g(P_{n-4}, i-1) \neq \Phi$; $D_g(P_{n-5}, i-1) \neq \Phi$.
Let $x_1 \in D_g(P_{n-}, i-1)$, then n- 2 or n- 3 is in x_1.
If $n-2$ or $n-3 \in x_1$ then $x_1 \cup \{n\} \in D_g(P_n, i)$.
Let $x_2 \in D_g(P_{n-2}, i-1)$, then n- 3 or n- 4 is in x_2.
If $n-3$ or $n-4 \in x_2$ then $x_2 \cup \{n-1\} \in D_g(P_n, i)$.
Now let $x_3 \in D_g(P_{n-3}, i-1)$, then n- 4 or n- 5 is in x_3.
If $n-4$ or $n-5 \in x_3$ then $x_3 \cup \{n-2\} \in D_g(P_n, i)$.
Now let $x_4 \in D_g(P_{n-4}, i-1)$, then n- 5 or n- 6 is in x_4.
If $n-5 \in x_4$ then $x_5 \cup \{n\} \in D_g(P_n, i)$. If $n-6 \in x_4$ then $x_5 \cup \{n-1\} \in D_g(P_n, i)$.
Thus,we have $\{x_1 \cup \{n\} / x_1 \in D_g(P_{n-1}, i-1)\} \cup$
$\{x_2 \cup \{n-1\} / x_2 \in D_g(P_{n-2}, i-1)\} \cup$
$x_3 \cup \{n-2\} / x_3 \in D_g(P_{n-3}, i-1)\} \cup \{\{n-2\} \cup$
$(x_4 - x_3) / x_4 \in D_g(P_{n-4}, i-1)\} \cup \{(n-1) \cup$
$(x_5 - x_4) / x_5 \in D_g(P_{n-5}, i-1)\} \subseteq D_g(P_n, i)$ ......(2.5)
If $n \in Y$ then $Y = x_1 \cup \{n\}$ for some $x_1 \in D_g(P_{n-1}, i-1)$.
If $n \notin Y$ and $n-1 \in Y$ then $Y = x_2 \cup \{n-1\}$, for some $x_2 \in D_g(P_{n-2}, i-1)$.
If $n \notin Y$, $n-1 \notin Y$ and $n-2 \in Y$ then $Y = x_3 \cup \{n-2\}$,for some $x_3 \in D_g(P_{n-3}, i-1)$.
If $n \notin Y$, $n-1 \notin Y$, $n-2 \notin Y$ and $n-3 \in Y$, then $Y = x_4 \cup \{n\}$ for some $x_4 \in D_g(P_{n-4}, i-)$.
If $n \notin Y$, $n-1 \notin Y$, $n-2 \notin Y$, $n-3 \notin Y$, $n-4 \in Y$ and $n-5 \in Y$ then $Y = (x_5 - x_4) \cup \{n-1\}$, for some $x_4 \in D_g(P_{n-4}, i-1)$, $x_5 \in D_g(P_{n-5}, i-1)$.
So, $D_g(P_n, i) \subseteq \{x_1 \cup \{n\} / x_1 \in D_g(P_{n-1}, i-1)\} \cup$
$\{x_2 \cup \{n-1\} / x_2 \in D_g(P_{n-2}, i-1)\} \cup$
$\{x_3 \cup \{n-2\} / x_3 \in D_g(P_{n-3}, i-1)\} \cup$
$\{\{n-2\} \cup (x_4 - x_3)\} / x_4 \in D_g(P_{n-4}, i-1)\} \cup$
$\{\{n-1\} \cup (x_5 - x_4) / x_5 \in D_g(P_{n-5}, i-1)\} \subseteq D_g(P_n, i)$.......(2.6) From (2.5) and (2.6), we have,
$D_g(P_n, i) = \{x_1 \cup \{n\} / x_1 \in D_g(P_{n-1}, i-1)\} \cup$
$\{x_2 \cup \{n-1\} / x_2 \in D_g(P_{n-2}, i-1)\} \cup$
$\{x_3 \cup \{n-2\} / x_3 \in D_g(P_{n-3}, i-1)\}$
$\{\{n-2\} \cup (x_4 - x_3) / x_4 \in D_g(P_{n-4}, i-1)\} \cup$
$\{\{n1\} \cup (x_5\, x_4) / x_5 \in D_g(P_{n-5}, i-1)\}$.

III. **Geodetic Domination Polynomial of Path (P_n)**

Let $D_g(P_n, x) = \sum_{i=\lceil \frac{n+2}{5} \rceil}^{n} d_g(P_n, i)\, x^i$ be the geodetic domination polynomial of a Path P_n. In this section, we derive an expression for $D_g(P_n, x)$.

Theorem 3.1

a) If $D_g(P_n, i)$ is the family of geodetic dominating sets with cardinality i of P_n, then
$d_g(P_n, i) = d_g(P_{n-1}, i-1) + d_g(P_{n-2}, i-1) + d_g(P_{n-3}, i-1) +$
$d_g(P_{n-4}, i-1) + d_g(P_{n-5}, i-1)$, where
$d_g(P_n, i) = |D_g(P_n, i)|$.

b) For every $n \geq 5$,

$D_g(P_n, x) = x\,[D_g(P_{n-1}, x) + D_g(P_{n-2}, x) + D_g(P_{n-3}, x) + D_g(P_{n-4}, x) + D_g(P_{n-5}, x)]$ with initial values
$D_g(P_2, x) = x^2$
$D_g(P_3, x) = x^3 + x^2$
$D_g(P_4, x) = x^4 + 2x^3 + x^2$.

Proof

a) Suppose (iv) of theorem 2.5 holds, from (iv),
we have $D_g(P_n, i) = \{x_1 \cup \{n\} / x_1 \in D_g(P_{n-1}, i-1)\} \cup$
$\{x_2 \cup \{n-1\} / x_2 \in D_g(P_{n-2}, i-1)\} \cup$
$\{x_3 \cup \{n-2\} / x_3 \in D_g(P_{n-3}, i-1)\} \cup$
$\{(x_{4-} x_3) \cup \{n-2\} / x_4 \in D_g(P_{n-4}, i-1)\} \cup$
$\{(x_5 - x_4) \cup \{n-1\} / x_5 \in D_g(P_{n-5}, i-1)\}$
Therefore, $D_g|(P_n, i)| = |D_g(P_{n-1}, i-1)| \cup$
$|D_g(P_{n-2}, i-1)| \cup |D_g(P_{n-3}, i-1)| \cup$
$|D_g(P_{n-4}, i-1)| \cup |D_g(P_{n-5}, i-1)|$.
Therefore, $\sum d_g(P_n, i) = d_g(P_{n-1}, i-1) + d_g(P_{n-2}, i-1) + d_g(P_{n-3}, i-1) + d_g(P_{n-4}, i-1) + d_g(P_{n-5}, i-1)$
Therefore we have the theorem.
(ii) $d_g(P_n, i)x^i = d_g(P_{n-1}, i-1)\,x^i + d_g(P_{n-2}, i-1)\,x^i +$
$d_g(P_{n-3}, i-1)\,x^i + d_g(P_{n-4}, i-1)\,x^i + d_g(P_{n-5}, i-1)\,x^i$
$\sum d_g(P_n, i)x^i = \sum d_g(P_{n-1}, i-1)\,x^i + \sum d_g(P_{n-2}, i-1)\,x^i +$
$\sum d_g(P_{n-3}, i-1)\,x^i + \sum d_g(P_{n-4}, i-1)\,x^i + \sum d_g(P_{n-5}, i-1)\,x^i$.
$\sum d_g(P_n, i)x^i = x\sum d_g(P_{n-1}, i-1)\,x^{i-1} + x\sum d_g(P_{n-2}, i-1)\,x^{i-1} +$
$x\sum d_g(P_{n-3}, i-1)\,x^{i-1} + x\sum d_g(P_{n-4}, i-1)\,x^{i-1} + x\sum d_g(P_{n-5}, i-1)\,x^{i-1}$.
$\sum d_g(C_n, i) = x\,[D_g(P_{n-}, x) + D_g(P_{n-}, x) + D_g(P_{n-}, x) +$
$D_g(P_{n-}, x) + D_g(P_{n-}, x)]$
$\therefore D_g(P_n, i) = x\,[D_g(P_{n-1}, x) + D_g(P_{n-2}, x) + D_g(P_{n-3}, x) +$
$D_g(P_{n-4}, x) + D_g(P_{n-5}, x)]$ with initial values.
$D_g(P_2, x) = x^2$
$D_g(P_3, x) = x^3 + x^2$
$D_g(P_4, x) = x^4 + 2x^3 + x^2$
We obtain $d_g(P_n, i)$ $1 \le n \le 12$ as shown in the table

n	2	3	4	5	6	7	8	9	10	11	12
1	0										
2	1										
3	1	1									
4	1	2	1								
5		3	3	1							
6		2	6	4	1						
7		1	8	10	5	1					
8			9	18	15	6	1				
9			8	27	33	21	7	1			
10			6	34	60	54	28	8	1		
11			3	37	93	114	82	36	9	1	
12			1	34	126	206	196	118	45	10	1

In the following theorem we obtain some properties of $d_g(P_n, i)$

Theorem 3. 2

The following properties hold for the co efficient of $D_g(P_n, x)$.
(i) $d_g(P_n, n) = 1$ for every $n \ge 2$.
(ii) $d_g(P_n, n-1) = n-2$ for every $n \ge 3$.
(iii) $d_g(P_n, n-2) = \frac{1}{2}[n^2 - 5n + 6]$ for every $n \ge 4$.
(iv) $d_g(P_n, n-3) = \frac{1}{6}(n^3 - 9n^2 + 26n - 36)$ for every $n \ge 5$.
(v) $d_g(P_n, n-4) = \frac{1}{24}(n^4 - 14n^3 + 71n^2 - 202n + 360)$.

Proof

(i) $d_g(P_n, n) = \{[n]\}$, we have the result.

(ii) $d_g(P_n, n-1) = n-2$, for every $n \geq 3$.

Since $D_g(P_n, n-1) = \{[n] - \{x\}, x \in [x]$,

we have $d_g(P_n, n-1) = n- 2$.

(iii) By induction on n, the result is true for n = 4

L.H.S = $d_g(4, 2) = 1$ (from table).

R.H.S = $\frac{1}{2}(4^2 - 5(4) + 6) = 1$

Therefore, the result is true for n = 4 .

Now, suppose that the result is true for all numbers less than n, and we prove it for 'n'.

By theorem 3.1, we have

$d_g(P_n, n-2) = d_g(P_{n-1}, n-3) + d_g(P_{n-2}, n-3) + d_g(P_{n-3}, n-3) + d_g(P_{n-4}, n-3) + d_g(P_{n-5}, n-3)$

$= \frac{1}{2}[(n-1)^2 - 5(n-1) + 6] + (n-2) - 2 + 1$

$= \frac{1}{2}[n^2 - 2n + 1 - 5n + 5 + 6 + 2n - 6]$.

$= \frac{1}{2}[n^2 - 5n + 6]$.

Hence, the result is true for all n.

(v) By induction on n, the result is true for n = 6.

L.H.S = $d_g(P_{6,3}) = 2$ (from table)

R.H.S = $\frac{1}{6}(6^3 - 9 \times 6^2 + 26 \times 6 - 36)$

$= 2$

Therefore, the result is true for all natural numbers less than n.

By Theorem 3.1, we have,

$d_g(P_n, n-3) = d_g(P_{n-1}, n-4) + d_g(P_{n-2}, n-4) + d_g(P_{n-3}, n-4) + d_g(P_{n-4}, n-4) + d_g(P_{n-5}, n-4) =$

$\frac{(n-1)^3 - 9(n-1)^2 + 26(n-1) - 36}{6} + \frac{(n-2)^2 - 5(n-2) + 6}{2} + (n-3) - 2 + 1$

$= \frac{1}{6}[(n^3 - 3n^2 + 3n - 1) - 9(n^2 - 2n + 1) + 26n - 26 - 36) +$

$\frac{1}{2}(n^2 - 4n + 4 - 5n + 10 + 6) + n - 4)]$

$= \frac{1}{6}[(n^3 - 3n^2 + 3n - 1 - 9n^2 - 18n - 9 + 26n - 26 - 36) +$

$\frac{1}{2}(n^2 - 9n + 20) + n - 4$

$= \frac{1}{6}[n^3 - 12n^2 + 47n - 72 - 3n^2 - 72n + 60 + 6n - 24]$

$= \frac{1}{6}[(n^3 - 9n^2 + 26n - 36]$

Hence, the result is true for all n.

(vi) By induction on n. Let n = 7.

L.H.S = $d_g(P_{7,3}) = 1$ (from table)

R.H.S = $\frac{1}{24}(74 - 14 \times 7^3 + 71 \times 7^2 - 202 \times 7 + 360)$

$= 1$

Therefore, the result is true for n = 1.

Now, suppose that the result is true for all natural numbers less than n.

$d_g(P_n, n-4) = d_g(P_{n-1}, n-5) + d_g(P_{n-2}, n-5) + d_g(P_{n-3}, n-5) +$

$d_g(P_{n-4}, n-5) + d_g(P_{n-5}, n-5)$

$= \frac{(n-1)^4 - 14(n-1)^3 + 71(n-1)^2 - 202(n-1) + 360}{24} +$

$\frac{1}{6}[(n-2)^2 - 9(n-2)^2 + 26(n-2) - 36)] +$

$\frac{1}{2}((n-2)^3 - 5(n-3) + 6)) + (n-4) - 2 + 1$

$$= \frac{n^4 - 18n^3 + 119n^2 - 390n + 648}{24} + \frac{n^3 - 15n^2 + 74n - 132}{6} + \frac{n^2 - 11n + 30}{6} + n - 5$$

$$= \frac{1}{2}[n^2 - 18n^3 + 119n^2 - 390n + 648 + 4n^3 - 60n^2 + 296n - 528 +$$

$$12n^2 - 132n + 460 + 24n - 120]$$

$$= \frac{n^4 - 14n^3 + 7n^2 - 202n + 360}{24}$$

Hence the result is true for all n.

Theorem 3.3

$$\sum_{i=n}^{3n} d_g(P_i, n) = 5\sum_{i=4}^{3n-5} d_g(P_i, n-1) \text{ for every } n \geq 4.$$

Proof

Proof by induction on n.

First suppose that n = 4. Then

$$\sum_{i=4}^{12} d_g(P_i, 4) = 45 = 5\sum_{i=4}^{7} d_g(4, 3).$$

$$\sum_{i=k}^{3k} d_g(P_i, k) = \sum_{i=k}^{3k} d_g(P_{i-1}, k-1) + \sum_{i=k}^{3k} d_g(P_{i-2}, k-1) + \sum_{i=k}^{3k} d_g(P_{i-3}, k-1) + \sum_{i=k}^{3k} d_g(P_{i-4}, k-1) +$$

$$\sum_{i=k}^{3k} d_g(P_{i-5}, k-1).$$

$$= 5\sum_{i=k-1}^{3(k-1)} d_g(P_{i-1}, k-2) + 5\sum_{i=k-1}^{3(k-1)} d_g(P_{i-2}, k-2) + 5\sum_{i=k-1}^{3(k-1)} d_g(P_{i-3}, k-2) + 5\sum_{i=k-1}^{3(k-1)} d_g(P_{i-4}, k-2) +$$

$$5\sum_{i=k-1}^{3(k-1)} d_g(P_{i-5}, k-2).$$

$$= 5\sum_{i=k-1}^{3(k-5)} d_g(P_i, k-1).$$

We have the result.

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