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Finite Population Loss and Delay Queue under No Passing Restriction and Discouragement

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ABSTRACT

This investigation deals with finite population loss and delay multi server queue with no passing, discouragement and additional servers. The concept of no passing has its own consideration as the customers are permitted to leave the system in the same chronological order in which they join the system. Due to no-passing restrictions, the customers can be categorized into two types (i) those who do not need service even waiting in the queue and (ii) the customers having exponentially distributed service time. Also customer is assumed to be discouraged on seeing long queue at the service station. This discouraging behaviour of the customer puts the system to loss and the customer decides not to join the queue. To overcome such situations the system organigers may facilitate the additional removable server to provide faster service. Keeping this in view the above-mentioned factors have been included in this study. By using birth death rate and product type solution, queue size distribution is established, which is further employed to derive various performance indices namely expected waiting time, average number of customers in the system and the difference between the expected waiting time of both types of customers in the system.

1. Introduction

Service station should offer some opportunities to the system designer to incorporate those factors, which are responsible for the loss and delay of the system. Due to undesirable waiting times, the customers either balk or renege from the system after impatience. There may happen some congestion situations in real life when the concept of no passing governs the policy of departure in the system. This leads to increase in the total waiting time. By providing removable additional servers we can avoid such problems. The management of service station needs to decide the number of servers based on the cost and queue theoretic information in this regard.

Some researchers have done a well-defined work in this direction of "no-passing" restriction according to which the customer depart from the system in the same chronological order of their arrival. Washburn (1974) did incredible work on multi-server queues taking the concept of no passing. Sharma et al. (1983) have considered the restriction of no passing in multi-server queueing models. Jain et al. (1989) analysed a no passing multi-server queueing model with two types of customers and discouragement. Jain and Premlata (1993) developed no passing M/M/Φ(.) time sharing queueing system. Jain and Singh (2001) extended no passing model for time-sharing queueing system with additional service position.

Discouraging behaviour of the customer is also incorporated in our study, due to which the customers may balk with certain probability or renege due to impatience. To cope up with such situations there may be a provision of removable additional servers along with the permanent servers. Abou-El-Ata and Hariri (1992) considered M/M/c/N queue with balking and reneging. Shawky (1997) developed the single server machine interference model with balking, reneging and an additional server for longer queues. Jain (1998) analysed M/M/m queue with discouragement and additional servers. Jain and Sharma (2002) analysed M/M/m/K queue with additional servers and discouragement. Jain and Sharma (2004) considered controllable multi server queue with balking and additional server. Controllable multi server queue with balking was investigated by Jain and Sharma (2005). Finite capacity queueing system with queue dependent servers and discouragement was analysed by Jain and Sharma (2008). Machine repair problem with spares, balking, reneging and N-Policy for vacation was developed by Sharma (2015). Yu et al. (2016) considered equilibrium strategies of the unobservable M/M/1 queue with balking and delayed repair. Sharma (2017) analysed transient analysis of interdependent M/M (a,d,b) /1 queue with discouragement and controllable arrival rate.

The loss and delay phenomenon of the customers in the system is likely to bring about the understanding that either the customers would like to wait in the queue to get service or may be lost when all the servers are busy in providing service to other customers. Multi server loss-delay queueing system with priority and no passing was studied by Jain et al. (1996). Jain et al. (2002) analysed loss-delay queueing model for time-sharing system with additional service position and no passing. Sharma (2013) considered a single unreliable server interdependent loss and delay queueing model with controllable arrival rate under N-policy. Transient analysis of loss and delay bulk service Markovian queue under N-policy was discussed by Sharma (2014). Gupta et al. (2017) developed optimal revenue management in two class pre-emptive delay dependent Markovian queues. Sharma (2017) developed unreliable server Mx /G/1 queue with loss-delay, balking and second optional service.

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In this paper, we consider finite population loss and delay queue with state dependent rates under no passing restriction by incorporating additional servers. The organization of the paper is as follows. The model descriptions and queue size distribution are given in sections 2 and 3, respectively. The expected waiting time is obtained in section 4. In section 5, special cases have been deduced by setting appropriate parameters. Discussions are outlined in the final section 6.

2. Model Description

We consider the finite population loss and delay queueing model under no passing restriction and discouragement, in which the arrival and the service rates of the customer depend on the queue length. The discouragement behaviour of the customers can be reduced by incorporating the removable additional servers in the queue.

The following assumptions are made to formulate the system mathematically.

- The customers arrive in Poisson fashion and are classified as loss and delay customers depending whether they wish to wait on finding all permanent servers busy.
- The customers who have the patience to join the queue and wait for their service in case when all permanent servers busy, are called the delay customers.
- On the other hand the loss customers are those who would not like to wait in the queue and depart from the system on finding all servers busy on their arrival.
- The inter arrival times of loss and delay customers are exponentially distributed with mean rates λ_ℓ and λ_d, respectively.
- The arrival rate of delay customers is λ_{d_j} (j=0,1,2,….,r) when j additional servers are available.
- The service facility consists of c permanent and r additional removable servers.

The availability of additional servers depends upon the number of customers present in the system in the following manner:

- When there are less than or equals to K customers in the system only permanent servers are available.
- If there are greater than jK and less than or equal to (j+1)K customers in the system then j (j=1, 2, ….., r-1) additional servers along with all permanent servers provide service to the customers.
- When there are greater than rK customers in the system, all the permanent and additional servers will provide service to the customers.

The customer may balk with the probability b_n when there are n customers present in the system. When all c permanent servers and j $(0 \le j \le r)$ additional servers are busy then customers may renege exponentially with reneging parameter α.

The state dependent arrival rate for finite population model is given as follows.

$$\lambda(n)=\begin{cases}(M-n)(\lambda_d+\lambda_\ell)b_n; & 0\le n<c\\ (M-n)\lambda_{d_0}b_n; & c\le n<K\\ (M-n)\lambda_{d_j}b_n; & jK\le n<(j+1)K\\ (M-n)\lambda_{d_r}b_n; & rK\le n<M\end{cases} \qquad \text{…. (1)}$$

where $\lambda=(\lambda_d+\lambda_\ell)$, M is the population size and

$$b_n=\frac{\lambda}{n^p}\left(1-\frac{n}{M}\right)^q; \qquad 1\le n\le M$$

both p and q, $q \le p$ are scale parameters.

The state dependent service rate for finite population model is given by

$$\mu(n)=\begin{cases}n\mu_0; & 1\le n\le c\\ c\mu+(n-c)\alpha_0; & c<n\le K\\ c\mu+\sum_{i=1}^{j}(n-c+i)\alpha_j; & jK<n\le(j+1)K\\ c\mu+\sum_{i=1}^{r}(n-c+i)\alpha_r; & rK<n\le M\end{cases} \qquad \text{…. (2)}$$

Depending upon the requirement of the service, the customers are categorized into the following two types:

(i) Type A customers who are 1-p proportional of total customers, are those who do not need service but wait in the queue due to no-passing restriction.

(ii) The p proposition of total customers are of type B and have their service time exponential distributed.

3. Queue Size Distribution

In this section, the mathematical expressions for the queue size distribution and average queue length are determined by employing product type solution.

The traffic intensity is given by

$$\rho=\rho_d+\rho_\ell \qquad \text{…. (3)}$$

where $\rho_d = \frac{\lambda_d}{\mu_0}$ and $\rho_\ell = \frac{\lambda_\ell}{\mu_0}$ are the traffic intensities of delay and loss customers, respectively.

Using product type solution the steady state queue size distribution P_n is given by

$$P_n = \begin{cases} \frac{M!}{(M-n)!}\frac{\rho^n \beta_{n-1}}{n!}P_0; & 0<n\le c \\ \frac{M!}{(M-n)!}\frac{\rho^c \beta_{n-1}(\lambda_{d_0})^{n-c}}{c!\prod_{i=c+1}^{n}\{c\mu+(i-c)\alpha_0\}}P_0; & c<n\le K \\ \frac{M!}{(M-n)!}\frac{\rho^c \beta_{n-1}(\lambda_{d_0})^{K-c}\prod_{l=1}^{j-1}(\lambda_{d_l})^{K}(\lambda_{d_j})^{n-jK}}{c!\prod_{i=c+1}^{K}\{c\mu+(i-c)\alpha_0\}\prod_{l=1}^{j-1}\prod_{\theta=lK+1}^{(l+1)K}\left\{c\mu+\sum_{i=1}^{l}(\theta-\overline{c+i})\alpha_i\right\}\prod_{\theta=jK+1}^{n}\left\{c\mu+\sum_{i=1}^{j}(\theta-\overline{c+i})\alpha_i\right\}}P_0; & jK<n\le(j+1)K \\ \frac{M!}{(M-n)!}\frac{\rho^c \beta_{n-1}(\lambda_{d_0})^{K-c}\prod_{l=1}^{r-1}(\lambda_{d_l})^{K}(\lambda_{d_r})^{n-rK}}{c!\prod_{i=c+1}^{K}\{c\mu+(i-c)\alpha_0\}\prod_{l=1}^{r-1}\prod_{\theta=lK+1}^{(l+1)K}\left\{c\mu+\sum_{i=1}^{l}(\theta-\overline{c+i})\alpha_i\right\}\prod_{\theta=rK+1}^{n}\left\{c\mu+\sum_{i=1}^{r}(\theta-\overline{c+i})\alpha_i\right\}}P_0; & rK<n\le M \end{cases} \quad \text{.... (4)}$$

Where $\prod_{i=1}^{n-1} b_i = \beta_{n-1}$.

The value of P_0 can be obtained by using normalizing condition as follows

$$\begin{aligned} P_0 = \Bigg[& \sum_{n=0}^{c}\frac{M!}{(M-n)!}\frac{\rho^n\beta_{n-1}}{n!} + \sum_{n=c+1}^{K}\frac{M!}{(M-n)!}\frac{\rho^c\beta_{n-1}(\lambda_{d_0})^{n-c}}{c!\prod_{i=c+1}^{n}\{c\mu+(i-c)\alpha_0\}} \\ & + \sum_{j=1}^{r-1}\sum_{n=jK+1}^{(j+1)K}\frac{M!}{(M-n)!}\frac{\rho^c\beta_{n-1}(\lambda_{d_0})^{K-c}\prod_{l=1}^{j-1}(\lambda_{d_l})^{K}(\lambda_{d_j})^{n-jK}}{c!\prod_{i=c+1}^{K}\{c\mu+(i-c)\alpha_0\}\prod_{l=1}^{j-1}\prod_{\theta=lK+1}^{(l+1)K}\left\{c\mu+\sum_{i=1}^{l}(\theta-\overline{c+i})\alpha_i\right\}\prod_{\theta=jK+1}^{n}\left\{c\mu+\sum_{i=1}^{j}(\theta-\overline{c+i})\alpha_i\right\}} \\ & + \sum_{n=rK+1}^{M}\frac{M!}{(M-n)!}\frac{\rho^c\beta_{n-1}(\lambda_{d_0})^{K-c}\prod_{l=1}^{r-1}(\lambda_{d_l})^{K}(\lambda_{d_r})^{n-rK}}{c!\prod_{i=c+1}^{K}\{c\mu+(i-c)\alpha_0\}\prod_{l=1}^{r-1}\prod_{\theta=lK+1}^{(l+1)K}\left\{c\mu+\sum_{i=1}^{l}(\theta-\overline{c+i})\alpha_i\right\}\prod_{\theta=rK+1}^{n}\left\{c\mu+\sum_{i=1}^{r}(\theta-\overline{c+i})\alpha_i\right\}}\Bigg]^{-1} \end{aligned} \quad \text{.... (5)}$$

The average number of customers in the system is given by

$$L = \sum_{n=0}^{M} nP_n$$

$$\begin{aligned} = \Bigg[& \sum_{n=0}^{c}\frac{M!}{(M-n)!}\frac{\rho^n\beta_{n-1}}{(n-1)!} + \sum_{n=c+1}^{K}\frac{M!}{(M-n)!}\frac{c\rho^c\beta_{n-1}(\lambda_{d_0})^{n-c}}{c!\prod_{i=c+1}^{n}\{c\mu+(i-c)\alpha_0\}} \\ & + \sum_{j=1}^{r-1}\sum_{n=jK+1}^{(j+1)K}\frac{M!}{(M-n)!}(c+j)\frac{\rho^c\beta_{n-1}(\lambda_{d_0})^{K-c}\prod_{l=1}^{j-1}(\lambda_{d_l})^{K}(\lambda_{d_j})^{n-jK}}{c!\prod_{i=c+1}^{K}\{c\mu+(i-c)\alpha_0\}\prod_{l=1}^{j-1}\prod_{\theta=lK+1}^{(l+1)K}\left\{c\mu+\sum_{i=1}^{l}(\theta-\overline{c+i})\alpha_i\right\}\prod_{\theta=jK+1}^{n}\left\{c\mu+\sum_{i=1}^{j}(\theta-\overline{c+i})\alpha_i\right\}} \\ & + \sum_{n=rK+1}^{M}\frac{M!}{(M-n)!}(c+r)\frac{\rho^c\beta_{n-1}(\lambda_{d_0})^{K-c}\prod_{l=1}^{r-1}(\lambda_{d_l})^{K}(\lambda_{d_r})^{n-rK}}{c!\prod_{i=c+1}^{K}\{c\mu+(i-c)\alpha_0\}\prod_{l=1}^{j-1}\prod_{\theta=lK+1}^{(l+1)K}\left\{c\mu+\sum_{i=1}^{j}(\theta-\overline{c+i})\alpha_i\right\}\prod_{\theta=rK+1}^{n}\left\{c\mu+\sum_{i=1}^{r}(\theta-\overline{c+i})\alpha_i\right\}}\Bigg]P_0 \end{aligned} \quad \text{.... (6)}$$

4. The Expected Waiting Time

For type A and B customers who need zero and exponentially distributed service times respectively, the expected waiting times are denoted by $E(W_A)$ and $E(W_B)$. Then the expected waiting time $E(W)$ for the tagged customer in the system is given by

$E(W)=(1-p)\,E(W_A) + pE(W_B)$ …. (7)

The tagged customer has a cumulative distribution function (c.d.f) of service times and is expressed as follows:

$$E(W)=(1-p)+p\{1-\exp(\mu_n x)\}\quad for\ x\geq 0,\ 0<p\leq 1 \qquad \text{.... (8)}$$

Now the expression for the mean waiting time of both types of customers are obtained as (cf. Jain and Singh, 2003)

$$E\{W_A\}=\frac{1}{\mu}\left[\sum_{n=0}^{c}a_n+\sum_{n=c+1}^{K}\left\{\frac{n-c+1}{c}+a_{c-1}\right\}+\sum_{j=1}^{r-1}\sum_{n=jK+1}^{(j+1)K}\left\{\frac{n-(c+j)+1}{(c+j)}+a_{c+j-1}\right\}+\sum_{n=rK+1}^{M}\left\{\frac{n-(c+r)+1}{(c+r)}a_{c+r-1}\right\}\right]P_n \qquad \text{.... (9)}$$

and

$$E\{W_B\}=\frac{1}{\mu}\left[\sum_{n=0}^{c}a_{n+1}+\sum_{n=c+1}^{K}\left\{\frac{n-c+1}{c}+a_c\right\}+\sum_{j=1}^{r-1}\sum_{n=jK+1}^{(j+1)K}\left\{\frac{n-(c+j)+1}{(c+j)}+a_{c+j}\right\}+\sum_{n=rK+1}^{M}\left\{\frac{n-(c+r)+1}{(c+r)}a_{c+r}\right\}\right]P_n \qquad \text{.... (10)}$$

where

$$a_n=\begin{cases}0; & n=0\\ \sum_{i=1}^{n}1/i; & n=1,2,....,n.\end{cases}$$

The expected waiting times of type A and B customers are determined using equation (4) in equations (9) and (10) respectively as

$$E\{W_A\}=\frac{1}{\mu}\Bigg[\sum_{n=0}^{c}a_n\frac{M}{(M-n)!}\frac{\rho^n\beta_{n-1}}{n!}+\sum_{n=c+1}^{K}\left\{\frac{n-c+1}{c}+a_{c-1}\right\}\frac{M}{(M-n)!}\frac{\rho^c\beta_{n-1}(\lambda_{\theta_0})^{n-c}}{c!\prod_{i=c+1}^{n}\{c\mu+(i-c)\alpha_0\}}$$
$$+\sum_{j=1}^{r-1}\sum_{n=jK+1}^{(j+1)K}\left\{\frac{n-(c+j)+1}{(c+j)}+a_{c+j-1}\right\}\frac{M}{(M-n)!}\frac{\rho^c\beta_{n-1}(\lambda_{\theta_0})^{K-c}\prod_{l=1}^{j-1}(\lambda_{\theta_l})^K(\lambda_{\theta_j})^{n-jK}}{c!\prod_{i=c+1}^{K}\{c\mu+(i-c)\alpha_0\}\prod_{l=1}^{j-1}\prod_{\theta=lK+1}^{(l+1)K}\left\{c\mu+\sum_{i=1}^{l}(\theta-\overline{c+i})\alpha_i\right\}\prod_{\theta=jK+1}^{n}\left\{c\mu+\sum_{i=1}^{j}(\theta-\overline{c+i})\alpha_i\right\}}$$
$$+\sum_{n=rK+1}^{M}\left\{\frac{n-(c+r)+1}{(c+r)}a_{c+r-1}\right\}\frac{M}{(M-n)!}\frac{\rho^c\beta_{n-1}(\lambda_{\theta_0})^{K-c}\prod_{l=1}^{r-1}(\lambda_{\theta_l})^K(\lambda_{\theta_r})^{n-rK}}{c!\prod_{i=c+1}^{K}\{c\mu+(i-c)\alpha_0\}\prod_{l=1}^{r-1}\prod_{\theta=lK+1}^{(l+1)K}\left\{c\mu+\sum_{i=1}^{l}(\theta-\overline{c+i})\alpha_i\right\}\prod_{\theta=rK+1}^{n}\left\{c\mu+\sum_{i=1}^{r}(\theta-\overline{c+i})\alpha_i\right\}}$$
$$\text{.... (11)}$$

and

$$E\{W_B\}=\frac{1}{\mu}\Bigg[\sum_{n=0}^{c}a_{n+1}\frac{M!}{(M-n)!}\frac{\rho^n\beta_{n-1}}{n!}+\sum_{n=c+1}^{K}\left\{\frac{n-c+1}{c}+a_c\right\}\frac{M!}{(M-n)!}\frac{\rho^c\beta_{n-1}(\lambda_{\theta_0})^{n-c}}{c!\prod_{i=c+1}^{n}\{c\mu+(i-c)\alpha_0\}}$$
$$+\sum_{j=1}^{r-1}\sum_{n=jK+1}^{(j+1)K}\left\{\frac{n-(c+j)+1}{(c+j)}+a_{c+j}\right\}\frac{M!}{(M-n)!}\frac{\rho^c\beta_{n-1}(\lambda_{\theta_0})^{K-c}\prod_{l=1}^{j-1}(\lambda_{\theta_l})^K(\lambda_{\theta_j})^{n-jK}}{c!\prod_{i=c+1}^{K}\{c\mu+(i-c)\alpha_0\}\prod_{l=1}^{j-1}\prod_{\theta=lK+1}^{(l+1)K}\left\{c\mu+\sum_{i=1}^{l}(\theta-\overline{c+i})\alpha_i\right\}\prod_{\theta=jK+1}^{n}\left\{c\mu+\sum_{i=1}^{j}(\theta-\overline{c+i})\alpha_i\right\}}$$
$$+\sum_{n=rK+1}^{M}\left\{\frac{n-(c+r)+1}{(c+r)}a_{c+r}\right\}\frac{M!}{(M-n)!}\frac{\rho^c\beta_{n-1}(\lambda_{\theta_0})^{K-c}\prod_{l=1}^{r-1}(\lambda_{\theta_l})^K(\lambda_{\theta_r})^{n-rK}}{c!\prod_{i=c+1}^{K}\{c\mu+(i-c)\alpha_0\}\prod_{l=1}^{r-1}\prod_{\theta=lK+1}^{(l+1)K}\left\{c\mu+\sum_{i=1}^{l}(\theta-\overline{c+i})\alpha_i\right\}\prod_{\theta=rK+1}^{n}\left\{c\mu+\sum_{i=1}^{r}(\theta-\overline{c+i})\alpha_i\right\}}\Bigg]P_0 \qquad \text{.... (12)}$$

By denoting the difference between mean waiting times for both types of customers by D, we obtain it as

$$D=\mu\left[E\{W_A\}-E\{W_B\}\right]$$
$$=\mu\Bigg[\sum_{n=0}^{c}(a_{n-1}-a_n)\frac{M!}{(M-n)!}\frac{\rho^n\beta_{n-1}}{n!}+\sum_{n=c+1}^{K}(a_c-a_{c-1})\frac{M!}{(M-n)!}\frac{\rho^c\beta_{n-1}(\lambda_{\theta_0})^{n-c}}{c!\prod_{i=c+1}^{n}\{c\mu+(i-c)\alpha_0\}}$$
$$+\sum_{j=1}^{r-1}\sum_{n=jK+1}^{(j+1)K}(a_{c+j}-a_{c+j-1})\frac{M!}{(M-n)!}\frac{\rho^c\beta_{n-1}(\lambda_{\theta_0})^{K-c}\prod_{l=1}^{j-1}(\lambda_{\theta_l})^K(\lambda_{\theta_j})^{n-jK}}{c!\prod_{i=c+1}^{K}\{c\mu+(i-c)\alpha_0\}\prod_{l=1}^{j-1}\prod_{\theta=lK+1}^{(l+1)K}\left\{c\mu+\sum_{i=1}^{l}(\theta-\overline{c+i})\alpha_i\right\}\prod_{\theta=jK+1}^{n}\left\{c\mu+\sum_{i=1}^{j}(\theta-\overline{c+i})\alpha_i\right\}}\Bigg]P_0 \qquad \text{.... (13)}$$

5. Special cases

Now we are in the position to deduce earlier existing results by setting appropriate parameters in the different performance indices derived in previous section as follows:

Case1: Queue with discouragement, no passing and additional server.

In this case there are only delay customers. Hence on putting $\lambda_\ell = 0$, equations (4) and (11) reduce to

$$P_n = \begin{cases} \dfrac{M!}{(M-n)!}\dfrac{\rho_d^{\,n}\beta_{n-1}}{n!}P_0; & 0<n\le c \\[2ex] \dfrac{M!}{(M-n)!}\dfrac{\rho_d^{\,c}\beta_{n-1}(\lambda_{d_0})^{n-c}}{c!\prod\limits_{i=c+1}^{n}\{c\mu+(i-c)\alpha_0\}}P_0; & c<n\le K \\[2ex] \dfrac{M!}{(M-n)!}\dfrac{\rho_d^{\,c}\beta_{n-1}(\lambda_{d_0})^{K-c}\prod\limits_{i=1}^{j-1}(\lambda_{d_i})^{K}(\lambda_{d_j})^{n-jK}}{c!\prod\limits_{i=c+1}^{K}\{c\mu+(i-c)\alpha_0\}\prod\limits_{i=1}^{j-1}\prod\limits_{\theta=iK+1}^{(i+1)K}\left\{c\mu+\sum\limits_{i=1}^{j}(\theta-\overline{c+i})\alpha_i\right\}\prod\limits_{\theta=jK+1}^{n}\left\{c\mu+\sum\limits_{i=1}^{j}(\theta-\overline{c+i})\alpha_i\right\}}P_0; & jK<n\le(j+1)K \\[2ex] \dfrac{M!}{(M-n)!}\dfrac{\rho_d^{\,c}\beta_{n-1}(\lambda_{d_0})^{K-c}\prod\limits_{i=1}^{r-1}(\lambda_{d_i})^{K}(\lambda_{d_r})^{n-rK}}{c!\prod\limits_{i=c+1}^{K}\{c\mu+(i-c)\alpha_0\}\prod\limits_{i=1}^{j-1}\prod\limits_{\theta=iK+1}^{(i+1)K}\left\{c\mu+\sum\limits_{i=1}^{j}(\theta-\overline{c+i})\alpha_i\right\}\prod\limits_{\theta=rK+1}^{n}\left\{c\mu+\sum\limits_{i=1}^{r}(\theta-\overline{c+i})\alpha_i\right\}}P_0; & rK<n\le M \end{cases} \quad \text{.... (14)}$$

and the expected waiting time of type A customer is as follows

$$E\{W_A\} = \frac{1}{\mu}\Bigg[\sum_{n=0}^{c}q_n\frac{M}{(M-n)!}\frac{\rho_d^{\,n}\beta_{n-1}}{n!} + \sum_{n=c+1}^{K}\left\{\frac{n-c+1}{c}+q_{n-1}\right\}\frac{M!}{(M-n)!}\frac{\rho_d^{\,c}\beta_{n-1}(\lambda_{d_0})^{n-c}}{c!\prod\limits_{i=c+1}^{n}\{c\mu+(i-c)\alpha_0\}}$$

$$+\sum_{j=1}^{r-1}\sum_{n=jK+1}^{(j+1)K}\left\{\frac{n-(c+j)+1}{(c+j)}+q_{n-j-1}\right\}\frac{M!}{(M-n)!}\frac{\rho_d^{\,c}\beta_{n-1}(\lambda_{d_0})^{K-c}\prod\limits_{i=1}^{j-1}(\lambda_{d_i})^{K}(\lambda_{d_j})^{n-jK}}{c!\prod\limits_{i=c+1}^{K}\{c\mu+(i-c)\alpha_0\}\prod\limits_{i=1}^{j-1}\prod\limits_{\theta=iK+1}^{(i+1)K}\left\{c\mu+\sum\limits_{i=1}^{j}(\theta-\overline{c+i})\alpha_i\right\}\prod\limits_{\theta=jK+1}^{n}\left\{c\mu+\sum\limits_{i=1}^{j}(\theta-\overline{c+i})\alpha_i\right\}}$$

$$+\sum_{n=rK+1}^{M}\left\{\frac{n-(c+r)+1}{(c+r)}q_{n-r-1}\right\}\frac{M}{(M-n)!}\frac{\rho_d^{\,c}\beta_{n-1}(\lambda_{d_0})^{K-c}\prod\limits_{i=1}^{r-1}(\lambda_{d_i})^{K}(\lambda_{d_r})^{n-rK}}{c!\prod\limits_{i=c+1}^{K}\{c\mu+(i-c)\alpha_0\}\prod\limits_{i=1}^{j-1}\prod\limits_{\theta=iK+1}^{(i+1)K}\left\{c\mu+\sum\limits_{i=1}^{j}(\theta-\overline{c+i})\alpha_i\right\}\prod\limits_{\theta=rK+1}^{n}\left\{c\mu+\sum\limits_{i=1}^{r}(\theta-\overline{c+i})\alpha_i\right\}}\Bigg]; \quad \text{.... (15)}$$

Case II: Loss and delay multi server queue with discouragement, no passing and additional server.

For this case we set balking probability b_n=1 so that using (4) and (11), we find

$$P_n = \begin{cases} \dfrac{M!}{(M-n)!}\dfrac{\rho^{n}}{n!}P_0; & 0<n\le c \\[2ex] \dfrac{M!}{(M-n)!}\dfrac{\rho^{c}(\lambda_{d_0})^{n-c}}{c!\prod\limits_{i=c+1}^{n}\{c\mu+(i-c)\alpha_0\}}P_0; & c<n\le K \\[2ex] \dfrac{M!}{(M-n)!}\dfrac{\rho^{c}(\lambda_{d_0})^{K-c}\prod\limits_{i=1}^{j-1}(\lambda_{d_i})^{K}(\lambda_{d_j})^{n-jK}}{c!\prod\limits_{i=c+1}^{K}\{c\mu+(i-c)\alpha_0\}\prod\limits_{i=1}^{j-1}\prod\limits_{\theta=iK+1}^{(i+1)K}\left\{c\mu+\sum\limits_{i=1}^{j}(\theta-\overline{c+i})\alpha_i\right\}\prod\limits_{\theta=jK+1}^{n}\left\{c\mu+\sum\limits_{i=1}^{j}(\theta-\overline{c+i})\alpha_i\right\}}P_0; & jK<n\le(j+1)K \\[2ex] \dfrac{M!}{(M-n)!}\dfrac{\rho^{c}(\lambda_{d_0})^{K-c}\prod\limits_{i=1}^{r-1}(\lambda_{d_i})^{K}(\lambda_{d_r})^{n-rK}}{c!\prod\limits_{i=c+1}^{K}\{c\mu+(i-c)\alpha_0\}\prod\limits_{i=1}^{j-1}\prod\limits_{\theta=iK+1}^{(i+1)K}\left\{c\mu+\sum\limits_{i=1}^{j}(\theta-\overline{c+i})\alpha_i\right\}\prod\limits_{\theta=rK+1}^{n}\left\{c\mu+\sum\limits_{i=1}^{r}(\theta-\overline{c+i})\alpha_i\right\}}P_0; & rK<n\le M \end{cases} \quad \text{.... (16)}$$

and the expected waiting time of type A customers

$$E(W_A)=\frac{1}{\mu}\Bigg[\sum_{n=0}^{c} q_n\frac{M}{(M-n)!}\frac{\rho^n}{n!}+\sum_{n=c+1}^{K}\left\{\frac{n-c+1}{c}+q_{n-1}\right\}\frac{M}{(M-n)!}\frac{\rho^c(\lambda_{d_0})^{n-c}}{c!\prod_{i=c+1}^{n}\{c\mu+(i-c)\alpha_0\}}$$

$$+\sum_{j=1}^{r-1}\sum_{n=jK+1}^{(j+1)K}\left\{\frac{n-(c+j)+1}{(c+j)}+q_{n-j-1}\right\}\frac{M!}{(M-n)!}\frac{\rho^c(\lambda_{d_0})^{K-c}\prod_{l=1}^{j-1}(\lambda_{d_l})^K(\lambda_{d_j})^{n-jK}}{c!\prod_{i=c+1}^{K}\{c\mu+(i-c)\alpha_0\}\prod_{l=1}^{j-1}\prod_{\theta=lK+1}^{(l+1)K}\left\{c\mu+\sum_{i=1}^{l}(\theta-\overline{c+i})\alpha_i\right\}\prod_{\theta=jK+1}^{n}\left\{c\mu+\sum_{i=1}^{j}(\theta-\overline{c+i})\alpha_i\right\}}$$

$$+\sum_{n=rK+1}^{M}\left\{\frac{n-(c+r)+1}{(c+r)}q_{n-r-1}\right\}\frac{M}{(M-n)!}\frac{\rho^c(\lambda_{d_0})^{K-c}\prod_{l=1}^{r-1}(\lambda_{d_l})^K(\lambda_{d_r})^{n-rK}}{c!\prod_{i=c+1}^{K}\{c\mu+(i-c)\alpha_0\}\prod_{l=1}^{j-1}\prod_{\theta=lK+1}^{(l+1)K}\left\{c\mu+\sum_{i=1}^{l}(\theta-\overline{c+i})\alpha_i\right\}\prod_{\theta=rK+1}^{n}\left\{c\mu+\sum_{i=1}^{r}(\theta-\overline{c+i})\alpha_i\right\}}\Bigg]P_0 \qquad \ldots.(17)$$

Case III: Finite Population Loss and delay multi server queue with balking and without reneging, no-passing and additional servers.

In this case b_n=1 and □ $_1$=0 so that the arrival rate and the service rate are given by:

$$\lambda(n)=\begin{cases}(M-n)(\lambda_d+\lambda_l); & 0<n\le c\\ (M-n)\lambda_{d_0}; & c<n\le K\\ (M-n)\lambda_{dj}; & jK<n\le (j+1)K\\ (M-n)\lambda_{dr}; & rK<n\le M\end{cases} \qquad \ldots(18)$$

and

$$\mu(n)=\begin{cases}n\mu_0; & 1\le n\le c\\ c\mu+(n-c)\alpha_1; & c<n\le M\end{cases} \qquad \ldots.(19)$$

Various performance indices in this case are obtained by substituting appropriate rate in (4), (6), (9) and (10).

Case IV: When $\lambda_d=\lambda_{d_0}=\lambda_{d_j}=\lambda_{d_r}=\lambda;\ \alpha_1=0; r=0$, b_n=1, and (M-n) = 1 i.e. infinite capacity then our model coincides with the model of Washburn (1974).

6. Discussion

In this study, a loss and delay queueing model with state dependent rates and no passing is considered. The incorporation of additional servers has also done to reduce the discouraging behaviour of the customers, which may be a cost effective alternative in particular when no passing constraint is imposed. The explicit formulae for queue size distribution, average number of customers in the system, expected waiting times of both types of customers and the difference between waiting times of both types of customers are established, which provide insight to system managers and decision makers to determine optimal combination of permanent and additional servers in a gainful manner.

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