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Horizontal Dispersion of Solute in Semi-Infinite Porous Medium With Periodic Input Source and Velocity

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ABSTRACT

Analytical solution of advection-dispersion equation (ADE) is obtained for one-dimensional semi-infinite porous medium with periodic input boundary and periodic flow velocity. The dispersion coefficient is directly proportional to flow velocity. Laplace Transformation Technique is employing to get the analytical solution with exponentially increasing function of space variable initial condition and periodic input boundary condition. Analytical solution is illustrated graphically.

Introduction

The transport process, attenuation processes may cause movement of the pollutants to different from that of the bulk flowing groundwater, for example dispersion, sorption and chemical or biological degradation of the chemicals/contaminants. Transport and attenuation processes may cause movement of the pollutants to different from that of the bulk flowing groundwater, for example dispersion, sorption and chemical or biological degradation of the chemicals/contaminants. The natural hydrological conditions may also affect the behavior of some pollutants. Because, groundwater movement is typically slow and its residence time is long. There is potential for interaction between the water and the porous material through which it passes. Groundwater flow and transport models are applied to assess the present pollution situation and to predict future situations in order to prevent further contamination of subsurface water. Maximum theoretical models are focused on passive pollutants, which mean those**,** which have no influence on the shape of the flow velocity distribution. Solute concentrations can be obtained with a wide variety of techniques.

The models based on the advective-dispersion solute transport equation are developed analytically by [5] in one-dimension for point source. [8] demonstrated the influence of initial and boundary conditions on solute concentration in one-dimension. They considered the radioactive decay, simple chemical interaction of solute on hydrodynamic dispersion in finite and semi-infinite domain both. [7] and [15] studied solute transport in stratified aquifer of infinite thickness. They calculated dispersion under the assumption that the permeability of each layer is random and the flow is in a direction parallel to the layers. [14] obtained analytical solution for reactive solute transport in two-dimensional aquifers. [19] obtained an analytical solution against the flow in one-dimension. They considered porous domain of adsorbing nature and seepage velocity unsteady and dispersion coefficient proportional to seepage velocity. The advection-dispersion equation has served as the main theoretical framework for modeling the fate and transport of solutes in soil/porous media, and for addressing critical environmental issues stemming from agricultural practices or waste disposal operations during the last few decades [11]. [4] analyzed the transverse dispersion coefficient considering a spatially uniform flow field of a kinetically sorbing compound under periodic temporal fluctuations.[9], [12], [20] and [21] obtained analytical solutions for one and two-dimensional advection–diffusion equation with variable coefficients in a longitudinal semi-finite domain, for temporally and spatially dependent dispersion problems. [1] considered the inverse problem of identifying a moving source in a linear advection-dispersion-reaction equation. The main application, but not the only one, is the identification of an environmental pollution source in a river. [17-18] presented a large number of one- and multi-dimensional analytical solutions of the standard equilibrium advection-dispersion equation (ADE) with and without terms accounting for zero-order production and first-order decay in two-part series.

In this paper, we developed a theoretical model for the dispersion problem in porous media in which the flow is one-dimensional and periodic. The analytical solution is derived for semi-infinite porous medium with appropriate initial and boundary conditions involving periodic function. Laplace transform technique is employed to get the solution of the present problem.

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Mathematical Description of the Problem

Mass conservation of solutes transported through porous media is described by a second order partial differential equation known as advection-dispersion equation (ADE). The governing equation of solute concentration can be expressed by the one-dimensional ADE as [2],

$$\frac{\partial c}{\partial t}+\frac{1-n_p}{n_p}\frac{\partial (K_1 c+K_2)}{\partial t}=\frac{\partial}{\partial x}\left(D(x,t)\frac{\partial c}{\partial x}-u(x,t)c\right) \quad (1)$$

where *c* solute concentration is a function of time *t* and position x, *D* dispersion coefficient, presumably includes the effects of both molecular diffusion and mixing in the axial direction i.e., horizontal axis which is direction of flow velocity and vertical axis.

However molecular diffusion is negligible due to very low seepage velocity. In Eq. (1), *D* and *u* may be constants or functions of time/space and n_p is porosity, K_1 and K_2 are empirical constants of the medium. [13] considered two cases, namely equilibrium and non-equilibrium relationship between the concentrations in the two phases. Equilibrium relationship is adopted in the present paper. The use of equilibrium isotherms assumes that equilibrium exists at all times between the porous medium and the solute in solution. This assumption is generally valid when the adsorption process is fast in relative to the ground-water velocity [3].

The coefficient of dispersion is considered directly proportional to seepage velocity [22], i.e.

$$D(x,t)\propto u(x,t) \quad (2)$$

Let us write $u(x,t)=|u_0 cos(mt)|$, so that $D(x,t)=|D_0 cos(mt)|$, where m is angular frequency whose dimension is inverse of the time variable t and D_0 , u_0 are constant and dimension of D_0 , u_0 are L^2T^{-1} , LT^{-1} respectively. Hence these are initial diffusion and velocity coefficient since at $t=0$, $D(x,t)=D_0$ and $u(x,t)=u_0$. But $D(x,t)=0$ at $mt=\pi/2$ i.e., D_0 represents the maximum value of $D(x,t)$. Similarly $u(x,t)$ may be interpreted. Eq. (1) becomes,

$$R\frac{\partial c}{\partial t}=\frac{\partial}{\partial x}\left(|D_0 cos(mt)|\frac{\partial c}{\partial x}-|u_0 cos(mt)|\,c\right) \quad (3)$$

Or

$$\frac{R}{|cos(mt)|}\frac{\partial c}{\partial t}=\frac{\partial}{\partial x}\left(D_0\frac{\partial c}{\partial x}-u_0\,c\right) \quad (3a)$$

where $R=(1+\frac{1-n_p}{n_p}K_1)$ is a retardation factor describing solute sorption and it is dimension less quantity.

Let us introduce a new time variable T_n by using the following transformation [6],

$$\frac{\partial T_n}{\partial t}=|cos(mt)|$$

$$T_n=\int_0^t |cos(mt)|dt \quad (4)$$

Or

$$mT_n=|sin(mt)| \quad (5)$$

Now differential Eq. (3a) reduces into constant coefficients as

$$\frac{R}{|cos(mt)|}\frac{\partial c}{\partial T_n}\frac{\partial T_n}{\partial t}=\frac{\partial}{\partial x}\left(D_0\frac{\partial c}{\partial x}-u_0 c\right)$$

or

$$R\frac{\partial c}{\partial T_n}=\frac{\partial}{\partial x}\left(D_0\frac{\partial c}{\partial x}-u_0 c\right) \quad (6)$$

Analytical Solutions

The porous domain is semi-finite and initially not solute free initially. Let us assume that the source is exponentially increasing function of space variable. Periodic input source is considered at origin of the domain i.e. in the semi-infinite domain (0,∞), at *x* = 0 the input source concentration is $C_0\{1+cos(mt)\}$ which is periodic in nature. On other, end of the domain flux type boundary condition is assumed. Mathematically, initial and boundary conditions can be,

$$c(x,t)=C_i exp\,(\alpha x);0\le x<\infty,t=0 \quad (7)$$

$$c(x,t)=C_0\{1+cos\,(mt)\};x=0,t>0 \quad (8)$$

$$\frac{\partial c}{\partial x}=0;\,x\to\infty,t\ge 0 \quad (9)$$

where C_i and C_0 are the resident concentrations. To keep the initial concentration in feasible range α is taken less than one and its dimension is inverse of space variable. The initial and boundary conditions Eqs. (7 - 9) can be written in new time variable T_n by using Eq. (5), as

$$c(x,T_n)=C_i exp\,(\alpha x);0\le x<\infty,T_n=0 \quad (10)$$

$$c(x,T_n)=\frac{C_0}{2}\{4-m^2T_n^2\};x=0,T_n>0 \quad (11)$$

$$\frac{\partial c}{\partial x}=0;\,x\to\infty,T_n\ge 0 \quad (12)$$

Since $mT_n\ll 1$, so neglecting second and higher order term from binomial expansions of $(1-m^2T_n^2)^{1/2}$.

Now introducing a transformation,

$$c(x,T_n)=K(x,T_n)exp\left[\frac{u_0}{2D_0}x-\frac{u_0^2}{4RD_0}T_n\right] \quad (13)$$

Eqs. (6) and Eqs.(10– 12) reduced into

$$R\frac{\partial K}{\partial T_n}=D_0\frac{\partial^2 K}{\partial x^2} \quad (14)$$

$$K(x,T_n)=C_i exp\,(\alpha x-\beta x);0\le x<\infty,T_n=0 \quad (15)$$

$$K(x,T_n)=\frac{C_0}{2}\{4-m^2{T_n}^2\}\exp\{\gamma^2T_n\};\,x=0,T_n>0 \quad (16)$$

$$\frac{\partial K}{\partial x} = -\frac{u_0}{2D_0}K;\ x \to \infty,\ \ T_n \geq 0 \qquad (17)$$

where $\gamma^2 = \frac{u_0^2}{4RD_0}$, $\beta = \frac{u_0}{2D_0}$.

Applying Laplace transformation on Eqs. (14 – 17), we may get

$$Rp\bar{K} - C_i R exp\ (\alpha x - \beta x) = D_0 \frac{d^2\bar{K}}{dx^2} \qquad (18)$$

$$\bar{K} = C_0\left[\frac{2}{(p-\gamma^2)} - \frac{m^2}{(p-\gamma^2)^3}\right]; x = 0, T_n > 0 \qquad (19)$$

$$\frac{d\bar{K}}{dx} = -\frac{u_0}{2D_0}\bar{K};\ x \to \infty,\ T_n \geq 0 \qquad (20)$$

Where $\bar{K}(x,p) = \int_0^{\omega} K(x,T_n)\, e^{-pT_n} dT_n$ (21)

in which p is the Laplace transformation parameter. Thus the solution of Eq. (18) on using condition Eqs.(19-20), may be written as,

$$\bar{K}(x,p) = \frac{C_i exp\{(\alpha-\beta)x\}}{\left\{p - \frac{(\alpha-\beta)^2 D_0}{R}\right\}} - \frac{C_i exp\left(-\sqrt{\frac{pR}{D_0}}x\right)}{\left\{p - \frac{D_0}{R}(\alpha-\beta)^2\right\}} + \frac{2C_0 exp\left(-\sqrt{\frac{pR}{D_0}}x\right)}{(p-\gamma^2)} - \frac{C_0 m^2 exp\left(-\sqrt{\frac{pR}{D_0}}x\right)}{(p-\gamma^2)^3} \qquad (22)$$

Taking inverse Laplace transform of Eq. (22), the solution of advection-dispersion solute transport for periodic input condition in terms of $c(x,T_n)$, as [10]

$$c(x,T_n) = c_i exp\left\{(\alpha-\beta)x + \frac{D_0 T_n(\alpha-\beta)^2}{R} + \frac{u_0 x}{2D_0} - \frac{u_0{}^2 T_n}{4RD_0}\right\}$$
$$-\frac{c_i}{2} exp\left\{\frac{u_0 x}{2D_0} - \frac{u_0{}^2 T_n}{4RD_0} + \frac{D_0(\alpha-\beta)^2 T_n}{R}\right\} \times \left[exp\{-x(\alpha-\beta)\} erfc\left\{\frac{x\sqrt{R}}{2\sqrt{D_0 T_n}} - (\alpha-\beta)\sqrt{\frac{D_0 T_n}{R}}\right\}\right.$$
$$\left. + exp\{x(\alpha-\beta)\}\, erfc\left\{\frac{x\sqrt{R}}{2\sqrt{D_0 T_n}} + (\alpha-\beta)\sqrt{\frac{D_0 T_n}{R}}\right\}\right] + C_0\left[erfc\left\{\frac{x\sqrt{R}}{2\sqrt{D_0 T_n}} - \frac{u_0\sqrt{T_n}}{2\sqrt{D_0 R}}\right\} + exp\left\{\frac{x u_0}{D_0}\right\} erfc\left\{\frac{x\sqrt{R}}{2\sqrt{D_0 T_n}} + \frac{u_0\sqrt{T_n}}{2\sqrt{D_0 R}}\right\}\right]$$
$$-\frac{C_0 m^2}{4\gamma}\left[\left\{\gamma T_n{}^2 - \frac{x T_n\sqrt{R}}{\sqrt{D_0}} + \frac{x\sqrt{R}}{4\gamma^2\sqrt{D_0}} + \frac{x^2 R}{4\gamma D_0}\right\} erfc\left\{\frac{x\sqrt{R}}{2\sqrt{D_0 T_n}} - \gamma\sqrt{T_n}\right\}\right.$$
$$+\left\{\gamma T_n^2 + \frac{x T_n\sqrt{R}}{\sqrt{D_0}} - \frac{x\sqrt{R}}{4\gamma^2\sqrt{D_0}} + \frac{x^2 R}{4\gamma D_0}\right\} exp\left\{\frac{u_0 x}{D_0}\right\} erfc\left\{\frac{x\sqrt{R}}{2\sqrt{D_0 T_n}} + \gamma\sqrt{T_n}\right\}$$
$$\left. -\frac{x\sqrt{T_n}\sqrt{R}}{\gamma\sqrt{\pi}\sqrt{D_0}} \exp\left\{\frac{u_0 x}{2D_0} - \frac{u_0{}^2 T_n}{4RD_0} - \frac{x^2 R}{4T_n D_0}\right\}\right] \qquad (23)$$

Where $\gamma^2 = \frac{u_0^2}{4RD_0}$, $\beta = \frac{u_0}{2D_0}$ and $T_n = \frac{1}{m}|sin(mt)|$.

Results and Discussion

In the obtained analytical solutions given by Eq.(23) the input parameters, value and the ranges of these parameters within which they are varied taken either from published literature or empirical relationship. For example the range of seepage velocity, keeping in view the different types of soils, aquifer is lies between 2m/day to 2m/year [16]. The concentration values c/C_0 are evaluated assuming reference concentration as $C_0 = 1.0(kg\ m^{-3})$ and $C_i = 0.1(kg\ m^{-3})$, in a finite domain $0 \leq x(\text{km}) \leq 10$. The numerical values of other common parameters are taken as $D_0 = 1.33(m^2\, day^{-1})$, $u_0 = 1.12(m\ day^{-1})$ and $\alpha = 0.019(\ m^{-1})$.

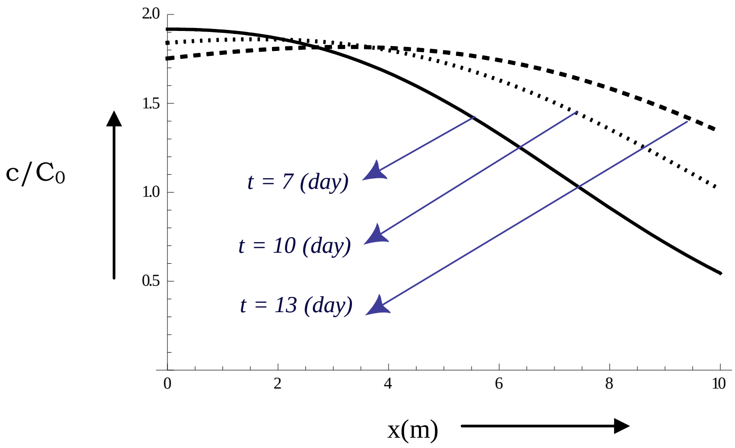

Figure 1. Distribution of dimensionless concentration for different time with fixed retardation $R = 1.2$ and angular frequency $m = 0.06(day^{-1})$.

Fig. 1., demonstrates dimensionless concentration profiles at various time $t(day) = 7, 10, 13$ with retardation factor $R = 1.20$ and angular frequency $m = 0.06\ (day^{-1})$. It reveals that at $x = 0$, the concentration values are higher for lower time and lower for higher time. Concentration at particular time varies with position.

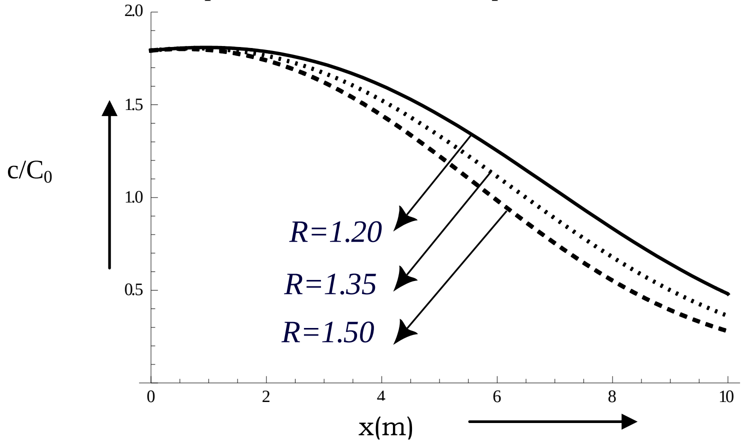

Figure 2. Distribution of dimensionless concentration for different retardation and for fixed angular frequency $m = 0.1(day^{-1})$ and time $t = 7(day)$

Fig. 2., demonstrates the effect of retardation factor on concentration profiles computed at various the retardation factor $R = 1.20, 1.35$ and 1.50 for angular frequency $m = 0.1(day^{-1})$. It reveals that concentration value decreases sharply as the retardation factor increases.

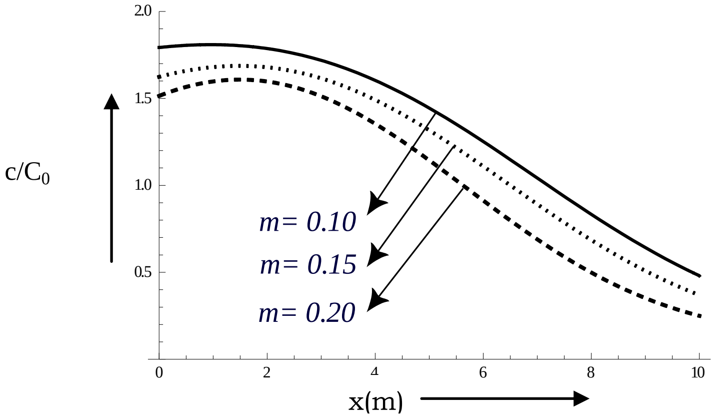

Figure 3. Distribution of dimensionless concentration for different angular frequency with fixed retardation $R = 1.2$ and time $t = 7(day)$

Fig.3., Illustrates the effect of angular frequency on concentration behavior computed at various $m(day^{-1}) = 0.10, 0.15, 0.20$ for retardation factor $R = 1.2$. It may be observed that concentration values at $x = 0(m)$ are higher for lower angular frequency and lower for higher. At particular position the concentration values are lower for higher angular frequency.

Many practical applications of ADE with initial and boundary condition in a semi-infinite/finite domain need to be considered and numerical/analytical solutions are required. Groundwater/surface water interactions in hydro-environments are influenced by a number of processes forming complex spatially and temporally variable systems.

Conclusion

A solute transport model with time dependent periodic boundary conditions and periodic flow is formulated to predict contaminant concentration along transient groundwater flow in a homogeneous semi-infinite aquifer is solved analytically using Laplace Transformation Technique. Effect of parameter, periodic boundary conditions and periodic flow on concentration profiles are illustrated with numerical solution. The boundary condition (8) seems to have relatively more effect on the results. The obtained result may be useful in examining the degradation levels of the surface as well as subsurface, particularly in assessing the rehabilitation time period.

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