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Elzaki Transform for Mixing Problems

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ABSTRACT

In this paper, we discuss the applicability of new integral transform called Elzaki transform method to solve few mixing problems of single tank which are comes under the category of applications of first order linear differential equations.

I.Introduction

In most of the areas of science and engineering, differential equations play an important role. In order to solve the complex differential equations, the integral transforms were extensively used. So the importance of an Integral Transforms is high, which provides powerful operational methods for solving initial value problems. Mostly we use Laplace Transform technique for solving differential equations. Here we discuss about ElzakiTransform is one of the new type integral transform method which is stronger than Sumudu and Laplace Transforms. Elzaki Transform was introduced by Tarig Elzaki [1] in 2011. Elzaki Transform defined for function of exponential order; we consider function on the set defined by

$$A=\left\{ f(t):\exists M,k_1,k_2>0,\left|f(t)\right|<Me^{\frac{|t|}{k_j}},\ if\ t\in(-1)^j\times[0,\infty)\right\}$$

For a given function in the set A the constant M must be a finite number, k_1 and k_2 may be finite or infinite.

The Elzaki transform is defined by $E[f(t)]=v\int_0^\infty f(t)e^{\frac{-t}{v}}dt=T(v), where\ \ t\geq 0\ and\ k_1\leq v\leq k_2.$

The sufficient conditions for the existence of the Elzaki transform are that f (t) for $t\geq 0$ be piecewise continuous and of the exponential order otherwise Elzaki transform may (or) may not exist.

II. Elzaki Transform of Some Functions and First Order Derivate

Here we consider some standard functions, the first order derivative which are mostly occurred in the problems; their Elzaki transforms are given below.

(i) If f(t)=1, Now $E\{f(t)\}=v\int_0^\infty e^{\frac{-t}{v}}f(t)dt=v\int_0^\infty e^{\frac{-t}{v}}dt=v\left(\frac{e^{\frac{-t}{v}}}{\frac{-1}{v}}\right)_0^\infty=v^2 \qquad \therefore E\{1\}=v^2$

(ii) If f(t)=t, then $E\{f(t)\}=v\int_0^\infty e^{\frac{-t}{v}}f(t)dt\Rightarrow E\{t\}=v\int_0^\infty t\,e^{\frac{-t}{v}}dt=v\left\{t(\frac{e^{-t/v}}{-1/v})-(1)\frac{e^{-t/v}}{1/v^2}\right\}_0^\infty=v(v^2)=v^3$

$$\therefore E(t)=v^3$$

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(iii) Similarly, we get $E(t^n)=n!\,v^{n+2}$

(iv) $E(e^{at})=v\int_0^\infty e^{at}e^{-\frac{t}{v}}dt \;=\dfrac{v^2}{1-av}$

Theorem: Let f(t) be the given function and $E\{f(t)\}=T(v)$ then $E\{f^1(t)\}=\dfrac{T(v)}{v}-vf(0)$

Proof: Given $E\{f(t)\}=T(v)$, By Definition of Elzaki Transform $E\{f(t)\}=v\int_0^\infty f(t)e^{-t/v}dt$

Now $E\{f^1(t)\}=v\int_0^\infty f^1(t)e^{\frac{-t}{v}}dt \;=-vf(0)+\int_0^\infty f(t)e^{-t/v}dt \;=-vf(0)+\dfrac{T(v)}{v}$

$$\therefore E\{f^1(t)\}=\frac{T(v)}{v}-vf(0)$$

Theorem: Let f(t) be the given function and $E\{f(t)\}=T(v)$ then $E\{t\,f(t)\}=v^2\dfrac{d}{dv}[T(v)]-vT(v)$

Proof: Given $E\{f(t)\}=T(v)$, By Definition of ElzakiTransform $E\{f(t)\}=v\int_0^\infty f(t)e^{-t/v}dt=T(v)$

Consider

$$T^1(v)=\frac{d}{dv}[T(v)] \;=\frac{d}{dv}\left[\int_0^\infty v\,e^{\frac{-t}{v}}f(t)\,dt\right]$$

$$=\int_0^\infty \frac{\partial}{\partial v}\left[v\,e^{\frac{-t}{v}}f(t)\right]dt$$

$$=\int_0^\infty \frac{\partial}{\partial v}\left[v\,e^{\frac{-t}{v}}\right]f(t)dt$$

$$=\int_0^\infty\left\{v.\frac{\partial}{\partial v}\left[e^{\frac{-t}{v}}\right]+e^{\frac{-t}{v}}(1)\right\}f(t)\,dt$$

$$=\int_0^\infty\left\{v.e^{\frac{-t}{v}}.\frac{t}{v^2}+e^{\frac{-t}{v}}(1)\right\}f(t)\,dt$$

$$=\int_0^\infty\left\{e^{\frac{-t}{v}}.\frac{t}{v}.f(t)+e^{\frac{-t}{v}}\;f(t)\right\}dt$$

$$=\int_0^\infty\left\{e^{\frac{-t}{v}}.\frac{t}{v}.f(t))\right\}dt+\int_0^\infty\left\{e^{\frac{-t}{v}}\;f(t)\right\}dt$$

$$=\frac{1}{v^2}E\{t\,f(t)\}+\frac{1}{v}E\{f(t)\}$$

$$=\frac{E\{t\,f(t)\}+v\,E\{f(t)\}}{v^2}$$

$$v^2T^1(v)=E\{t\,f(t)\}+v\,E\{f(t)\}$$

$$\therefore E\{t\,f(t)\}=v^2T^1(v)-vT(v)$$

Using this theorem, we have $E\{t\,e^{at}\}=\dfrac{v^3}{(1-av)^2}$

III. Mixing Problems

All of the mixing problems that we will be dealing with will involve a tank into which a certain substance will be added at a certain input rate (measured in kg/hr or lb/min. or some such units) and the substance will be leave the system at a certain output rate (measured in the same units. The substance we are interested in these problems is generally contained in, and or being mixed with some other medium (water, salt etc...).Thus, the concentration of the substance in the medium when it enters the tank is generally not the same as the concentration of the substance in the medium when it leaves the tank. It is important to be clear on the fact that it is the input rate and output rate of the substance, not the volume of whole mixture, which is present in the tank at time t.

Often what we know is the inflow rate of a mixture containing the substance, and the outflow rate at which the mixture is leaving the tank. Knowing the inflow rate and the concentration of the substance in the mixture flowing in allows us to determine the input rate at which the substance itself is entering the tank.

We generally assume that mixing within the tank takes place instantaneously, so that the outflow concentration is simply the quantity of the substance present in the tank, divided by the total volume of the mixture in the tank. This together with the outflow rate allows us to determine the output rate.

The differential equation involved in this kind of problem arises from the following relationship –the rate at which the amount of the substance in the tank is changing is simply the difference between the rate at which it is coming into the tank and the rate at which it is leaving the tank.

$$\frac{dy}{dt} = \text{(Input rate) - (Output rate).}$$

By solving this model we will get the quantity of the substance present in the tank at any time t.

Problem.1: Initially 50 pounds of salt is dissolved in the large tank having 300 gallons of water. A brine solution is pumped into the tank at a rate of 3 gallons per minute, and a well-stirred solution is then pumped out at a same rate. If the concentration of the solution entering is 2 lb/gal. Find the amount of salt in the tank at any time. And also find how much salt is present after 50 minutes.

Solution:

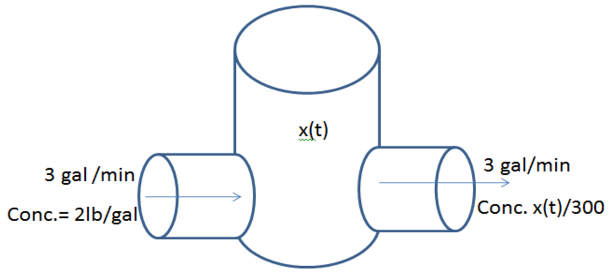

Figure-1

Let x (t) is the amount of salt in the tank at any time t. The differential equation involved in this kind of problem is

$$\frac{dx}{dt} = \text{(Rate of substance entering) - (Rate of substance leaving) i.e. } \frac{dx}{dt} = R_1 - R_2 \quad \text{\_\_\_\_\_\_(1)}$$

Where R_1 = Rate of substance entering = (concentration in) (rate of inflow water) = $\left(2\frac{lb}{gal}\right)\left(3\frac{gal}{\min}\right) = 6\,{}^{lb}\!/_{\min}$.

and R_2 = Rate of substance leaving = (concentration out) (rate of outflow water)

$$= \left(\frac{x(t)}{300}\,{}^{lb}\!/_{gal}\right)\left(3\,{}^{gal}\!/_{\min.}\right) = \frac{x(t)}{100}\,{}^{lb}\!/_{\min}.$$

From (1) we have,

$$\frac{dx}{dt} = 6 - \frac{x}{100} \Rightarrow \frac{dx}{dt} + \frac{x}{100} = 6 \quad \Rightarrow x^1(t) + \frac{x}{100} = 6 \quad \text{\_\_\_\_\_\_\_\_\_\_(2)}$$

Taking Elzaki transform on both sides of equation (2), we get

$$E\left(x^1(t) + \frac{x}{100}\right) = E(6) \ \Rightarrow E\left(x^1(t)\right) + \frac{1}{100}E(x) = 6E(1) \Rightarrow \left[\frac{\bar{x}(v)}{v} - vx(0)\right] + \frac{1}{100}\bar{x}(v) = 6v^2 \quad \text{\_\_\_(3)}$$

Since, at the initial time tank having 50 pounds of salt. i.e. at t=0, x (0) =50

From (3), we can write

$$\left[\frac{\bar{x}(v)}{v}-50v\right]+\frac{1}{100}\bar{x}(v)=6v^2$$

$$\Rightarrow\left(\frac{1}{v}+\frac{1}{100}\right)\bar{x}(v)=6v^2+50v$$

$$\Rightarrow\left(1+\frac{1}{100}v\right)\bar{x}(v)=6v^3+50v^2$$

$$\Rightarrow \bar{x}(v)=6\left(\frac{v^3}{1+\frac{1}{100}v}\right)+50\left(\frac{v^2}{1+\frac{1}{100}v}\right) \qquad \text{\_\_\_\_\_\_\_\_\_\_(4)}$$

Taking both sides on inverse Elzaki transform of (4), we get $x(t)=6(100-100e^{-0.01t})+50e^{-0.01t}=600-550e^{-0.01t}$

It gives the amount of salt present in the tank at any time t. The amount of salt present in the tank after 50minutes is $x(50)=600-550e^{-0.01\times 50}=266.41lb$

The schematic diagram for the amount of salt in the tank is given bellow.

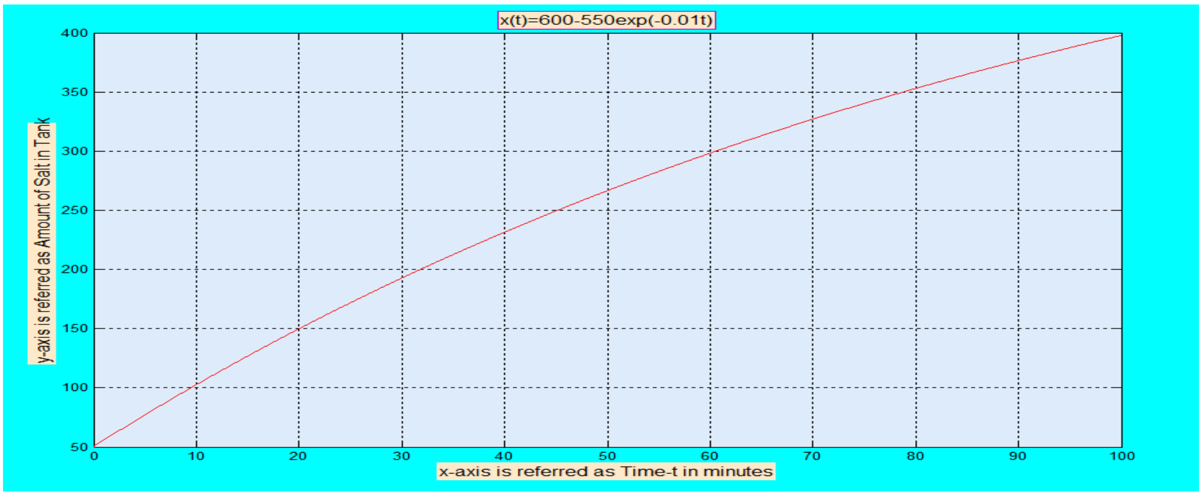

Figure-2

Problem.2: A tank contains 2000 ml of brine with 10 kg of dissolved salt. Pure water enters the tank at a rate of 5 liter per minute. The solution is kept thoroughly mixed and drains from the tank at the same rate. How much salt will be there in the tank after 20 minutes?

Solution:

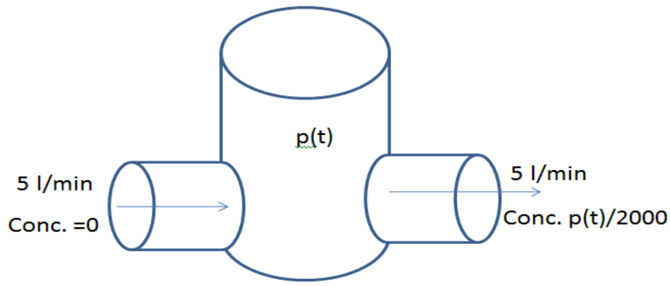

Figure-3

Let p (t) is the amount of salt in the tank at any time t. In this case we use the differential equation.

$$\frac{dp}{dt} = \text{(Rate of substance entering) - (Rate of substance leaving)} \qquad \text{i.e } \frac{dp}{dt} = R_1 - R_2 \qquad \_\_\_\_\_\_\_\_(5)$$

where R_1 = Rate of substance entering = (concentration in) (rate of inflow water) = $\left(0\,{}^{kg}/_{l}\right)\left(5\,{}^{l}/_{\min.}\right) = 0\,{}^{kg}/_{\min.}$

and R_2 = Rate of substance leaving = (concentration out) (rate of outflow water)

$$= \left(\frac{p(t)}{2000}\,{}^{kg}/_{l}\right)\left(5\,{}^{l}/_{\min.}\right) = \frac{p(t)}{400}\,{}^{kg}/_{\min.}$$

From (5) we have,

$$\frac{dp}{dt} = 0 - \frac{p}{400} \quad \Rightarrow \frac{dp}{dt} + \frac{p}{400} = 0 \quad \Rightarrow p^1(t) + \frac{p(t)}{400} = 0 \qquad \_\_\_\_\_\_\_\_(6)$$

Taking Elzaki transform on both sides of equation (6)

$$E\left(p^1(t) + \frac{p}{400}\right) = E(0) \quad \Rightarrow E\left(p^1(t)\right) + \frac{1}{400}E(p) = 0 \Rightarrow \left[\frac{\overline{p}(v)}{v} - vp(0)\right] + \frac{1}{400}\overline{p}(v) = 0 \qquad \_\_\_\_\_\_\_\_(7)$$

Since at the initial time tank having 10 kg of salt. i.e. at t=0, p (0) =10. From (7), we can write

$$\left[\frac{\overline{p}(v)}{v} - 10v\right] + \frac{1}{400}\overline{p}(v) = 0$$

$$\Rightarrow \left(\frac{1}{v} + \frac{1}{400}\right)\overline{p}(v) = 10v$$

$$\Rightarrow \left(1 + \frac{1}{400}v\right)\overline{p}(v) = 10v^2$$

$$\Rightarrow \overline{p}(v) = 10\left(\frac{v^2}{1 + \frac{1}{400}v}\right) \qquad \_\_\_\_\_\_\_\_\_\_\_(8)$$

Taking both sides on inverse Elzaki transform of (8), we get $p(t) = 10e^{-\frac{1}{400}t} = 10e^{-0.0025t}$

The amount of salt present in the tank after 20 minutes is $p(20) = 10e^{-0.0025\times 20} = 9.5kg$ and schematic diagram for the amount of salt in the tank is shown in the graph.

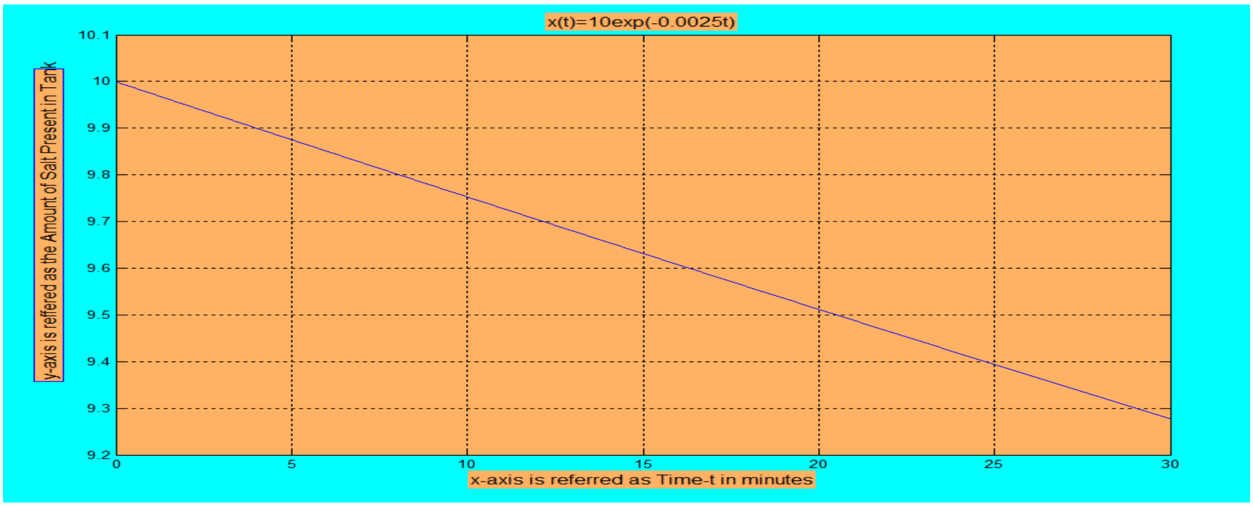

Figure-4

Problem.3: Initially, a water tank contains 100000 liters of pure water. Two valves are there. One valve allows a solution of water and fluoride, with a concentration of 0.5 kg of fluoride per liter of water, to flow into the tank at a rate of 50 liters per minute. The other valve allows the solution in the tank to the drained at 50 liter per minute. Assume that the solution is mixed constantly. Find

(i) An expression for the amount in kg of fluoride in the water tank at any time t.
(ii) Determine how it will take for the concentration of fluoride in the water to reach 0.25 kg per liter.

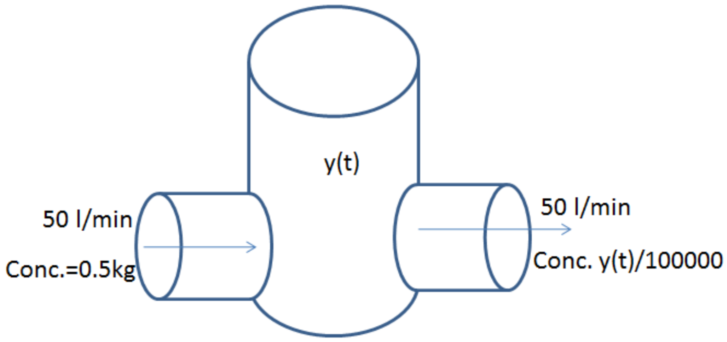

Figure-5

Let y (t) is the amount of salt in the tank at any time t.

We have $\frac{dy}{dt}=$(Rate of substance entering) - (Rate of substance leaving) i.e. $\frac{dy}{dt}=R_1-R_2$ ____________(9)

Where $R_1=$Rate of substance entering = (concentration inflow) (rate of inflow water)

$$= \left(0.5\, {}^{kg}\!/\!_{l}\right)\left(50\, {}^{l}\!/\!_{\min.}\right) = 25\, {}^{kg}\!/\!_{\min.}$$

and $R_2=$ Rate of substance leaving = (concentration outflow) (rate of outflow water)

$$= \left(\frac{y(t)}{100000}\, {}^{kg}\!/\!_{l}\right)\left(50\, {}^{l}\!/\!_{\min.}\right) = \frac{y(t)}{2000}\, {}^{kg}\!/\!_{\min.} \quad = 0.0005\, y(t)\ \text{kg/min},$$

From (9), we have

$$\frac{dy}{dt}=25-0.0005y(t) \quad \Rightarrow \frac{dy}{dt}+0.0005y(t)=25 \ \Rightarrow y^1(t)+0.0005y(t)=25 \qquad \text{\_\_\_\_\_(10)}$$

Taking both sides on Elzaki transform of equation (10), we get $E\left(y^1(t)+0.0005y(t)\right)=E(25)$

$$\Rightarrow E\left(y^1(t)\right)+0.0005E\left(y(t)\right)=E(25) \Rightarrow \left[\frac{\bar{y}(v)}{v}-vy(0)\right]+0.0005\bar{y}(v)=25E(1) \qquad \text{\_\_\_\_\_\_\_(11)}$$

Since, at the initial time tank having pure water. i.e. at t=0, y (0) =0. From (11), we can write

$$\left[\frac{\bar{y}(v)}{v}-0\right]+0.0005\bar{y}(v)=25v^2$$

$$\Rightarrow \left(\frac{1}{v}+0.0005\right)\bar{y}(v)=25v^2$$

$$\Rightarrow (1+0.0005v)\,\bar{y}(v)=25v^3$$

$$\Rightarrow \bar{y}(v)=25\left(\frac{v^3}{1+0.0005v}\right) \qquad \text{\_\_\_\_\_\_\_\_\_\_\_(12)}$$

Taking both sides on inverse Elzaki transform of (12), we get

$$y(t)=25\left(\frac{1}{0.0005}-\frac{1}{0.0005}e^{-0.0005t}\right) = 50000(1-e^{-0.0005t})$$

It gives the amount of fluoride in the water tank at any time t.

(b) we need determine how long it will take for the concentration to reach a level of 0.25 kg/l.
This concentration of fluoride in the 100000 liters in the tank corresponding to having (100000)(0.25) = 25000 kg of fluoride in the tank.
Thus, we see that we need to find the value of t that satisfies y (t) =25000. From above we have,

$$25000=50000(1-e^{-0.0005t})$$

$$\Rightarrow 1-e^{-0.0005t}=\frac{1}{2} \Rightarrow e^{-0.0005t}=\frac{1}{2}$$

Taking logarithms on both sides, then we have, $0.0005t=\ln(2) \Rightarrow t=\frac{\ln(2)}{0.0005}\approx 1386.29$

We see that it will take approximately 1386.29 minutes or 23.10 hours for the concentration of fluoride to reach 0.25 kg/l. The schematic diagram for the amount of fluoride in the tank is shown bellow.

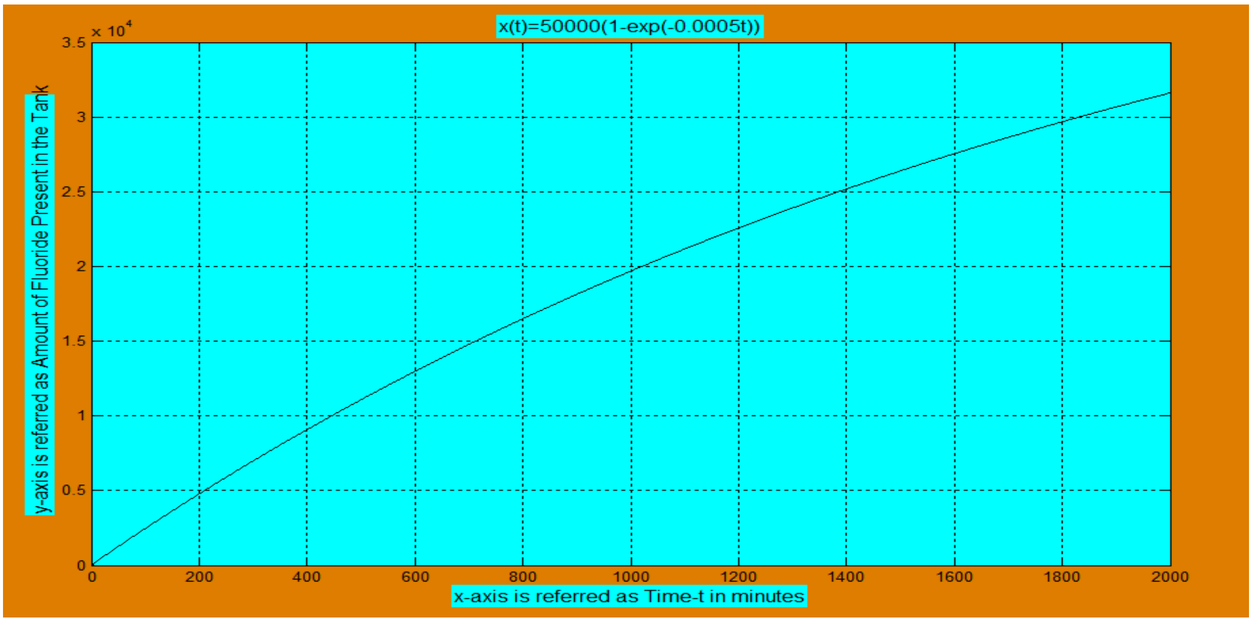

Figure-6

IV. Conclusion:

We can use Elzaki Transform Method to solve mixing problems of single tank which is an application of first order linear differential equations.

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