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Lagrangian Functions for Multi-Pendula Systems in Spatial Motion

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ABSTRACT

Functions of multi-Pendula systems in spatial motion in classical mechanics can be derived using the Lagrangian formalism. Limited research has been done for plane and spatial motion of linearly suspended mass units. This paper is therefore intended for multiple (many) masses linearly connected at varying lengths and for relatively small angular displacements in spatial motion. The objective is to develop governing relations of spatial motions in 3-dimensions for multi- pendula systems set to oscillate in space. The basis of the study is to formulate the Lagrangian for a multi-pendula system in spatial dynamics and determine the Lagangian functions and equations of the resultant motions.

1.0 Introduction

A few attempts have been made to obtain equations and solutions for a double pendulum system using Lagrange formalism. There is limited initiative to advance studies for motions of multi-pendula systems both in planner and spatial motions. The Lagrangian approach simplifies the complicated configurations that would be strenuous by any other formalism. The resultant equations so developed are the second order partial deferential equations where time is an explicit parameter. Given that the Hamiltonian functions derive their base from the Lagrange formulations, it is of paramount importance to note that Lagrange formulated a method that appears unpopular to mathematicians but serves as a bridge to the Hamiltonian principles. Spiegel [17] attempted to obtain equations of motion for plane double pendulum systems. In his book titled ``Theoretical mechanics'' on page 299 in problem 11.28, as an example, which states that, `A double pendulum vibrates in a vertical plane. (a) Write the Lagrangian of the system (b) obtain equations for the motion'. He derived the equations of motion for the double pendulum with even quantities as

$$2\ddot{\theta}_1 + \ddot{\theta}_2 = -2\frac{g}{l}\theta_1 \tag{1.1}$$

and

$$\ddot{\theta}_1 + \ddot{\theta}_2 = -\frac{g}{l}\theta_2 \tag{1.2}$$

This problem in plane motion is what triggered the research in spatial dynamics of even and uneven quantities of masses, lengths and angular displacements. A study had been done by Prichani et al [13]and Sakwa et al [15]about the planner motion of the masses under the same conditions.

Chow[3] used uneven quantities in plane motions to derive equations of motion for unequal mass units and unequal lengths of separation. He went further to obtain equations for spatial motion using a spherical pendulum and obtained the kinetic energy equation as

$$T = \frac{m}{2}l^2\left(\dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta\right) \tag{1.3}$$

and the potential energy equation as $V = mgl\cos\theta$ (1.4)

Subsequently the Lagrangian equation obtained was

$$L = \frac{m}{2}l^2\left(\dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta\right) - mgl\cos\theta \tag{1.5}$$

Braun [2] acknowledged that a multiple pendulum was considered for a plane motion in a vertical plane only and studied to establish the equations for n >1. The solutions were found by Prichani et al [13] for the n-tupple pendulum systems. They used the Lagrangian framework. It was observed that the angular acceleration for any mass was influenced by the immediate masses and angles. Joot [5] and Weiting et al [19] wrote about the multi- spherical pendula system with n components where energy was described and acknowledged that the description of the dynamics of the multi-pendula was of formidable difficulty. They therefore concentrated on the more accessible problem of small oscillations. Zhenglong and Nico [23] in their paper presented a framework for energy efficient dynamic human-like walk for a Nao humanoid robot. They used an inverted pendulum model to find an energy efficient stable walking gait. In this model they proposed a leg control policy which utilizes joint stiffness control. In order to identify the optimal parameters of the new gait for a Nao humanoid robot they used the policy gradient reinforcement. On testing this policy in a simulator and on a real Nao robot it was successful. It was shown that the new control policy had a dynamic walk that is more energy efficient than the standard walk of a Nao robot.

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Iyad and Kemalettin [4] in their paper present the best method for joint sensor faults detection and fault signal reconstruction. A virtual joint sensor is applied which consists of two interconnected models: The simple linear inverted pendulum model (LIPM) and the robot leg kinematic model (LKM).This proposed method was confirmed by simulations on 3D dynamics model of a humanoid robot SURALP while walking on a flat terrain and it was found to be valid.

Jung-Woo and Jan-Ho [8] in their paper a unit step pattern with a Center Of Mass (COM) position constrained in the supporting polygon at the end of each step, a robot can have a stationary time interval between steps to control and stabilize its posture. This is based on the linear inverted pendulum model. If there are zero moment point trajectories (ZMP), a simple solution form of the COM trajectories is formulated. Case studies based on different constraints on zero moment point (ZMP) references, COM and time differences are presented for an analytical solution. The unknown parameters of the COM trajectories in the solution form are formulated for the different cases. One of the cases was tested for the long stride walking with the DRC-HUBO robot developed at the Korea Advanced Institute of Science and Technology. Experiments on long-stride walking on bricks were successfully performed using the unit step pattern with several controllers of HUBO.Kaibing et al [9] proposed a whole-body control strategy for the walking of the the humanoid robots. This is based on the control of the centre of mass (COM) with the zero moment point (ZMP) regulation as well as the relative pose of the feet of the robot. A stable walking trajectory based on a 3D linear inverted pendulum model (3D-LIPM) is planned for study. In the proposed study it is shown that the control strategy perfectly tracks the planned trajectory of the COM and adjusts the ZMP back to the stability area when the robot is out of balance. Simulation results are presented to show the effectiveness of the proposed control scheme. Raudolph [14] clearly pointed out in one of his projections that "Lagrange discovered a method that is more generalized and simplifies solving the equations and positions with respect to two pendula for any given dimensions." He gave a method that applies for the difference between the kinetic and potential energy but he did not go far enough to apply in all situations. Therefore although some work had already been done, it did not cover the general areas especially the multi-pendula systems. The use of this method gives the energy functions and equations whose solutions are in tandem with the established classical work. According to Wells [21] and Arnold [1] about a double pendulum, coupled second-order ordinary differential equations were derived and could be solved numerically for $\boldsymbol{\theta_1(t)}$ and $\boldsymbol{\theta_2(t)}$. Plotting the resulting solutions quickly reveals the complicated motion. It could obviously be worse if the number of masses was increased. The equations of motion could then be written in the Hamiltonian formalism. Solving for $\dot{\boldsymbol{\theta}}_1$ and $\dot{\boldsymbol{\theta}}_2$ leads to the Hamiltonian equations.

According to a paper posted by Joot [6] it was indicated that although setting up the Lagrangian, for a double pendula system was difficult, it was worse solving it. From his research he did not know what to expect in future. He avoided doing more work because it proved to be complicated. In his second paper posted by the said Joot [7], it was noted that introducing any additional mass in system, for planer motion, the interaction coupling terms increase and thus complicate the kinetic energy specifications. This concept is captured by Prichani et al [12] where it was stated that if there are n masses suspended there will be $\frac{\boldsymbol{n(n+1)}}{\mathbf{2}}$ kinetic energy terms and for $(\boldsymbol{n+1})$ masses these terms will be $\frac{(\boldsymbol{n+1})(\boldsymbol{n+2})}{\mathbf{2}}$. It was observed that an increase by one mass would give an increase of $(\boldsymbol{n+1})$ kinetic energy terms. In this study the Lagrangian for n masses in planer motion was given by

$$\boldsymbol{L} = \sum_{k=f(i,j)}^{n} \boldsymbol{m_k} \left\{\frac{1}{2}\sum_{i=1}^{n} \boldsymbol{l_i^2 \dot{\theta}_i^2} + \sum_{j\neq i}^{n} \boldsymbol{l_i l_j \dot{\theta}_i \dot{\theta}_j} \cos\left(\boldsymbol{\theta_i - \theta_j}\right)\right\} - \sum_{k=i}^{n} \boldsymbol{m_k}\left(\sum_{i=1}^{n} \boldsymbol{l_i} - \sum_{i=1}^{k} \boldsymbol{l_i} \cos \boldsymbol{l_i}\right)\mathbf{g} \quad (1.6)$$

Before, it had been made clear by Joot [5] that, calculating the energy explicitly for a general multi pendula was likely thought to be too pedantic for even the most punishing instructor to inflict on students as a problem or an example.

A paper by Shen et al [16] was to motivate continuing research on dynamics and control problems for many body 3-D rigid pendula systems. They suggested that many of the problems had not been previously studied and that there are many opportunities for creative research. During the proceedings of the 17th World Congress, a paper by the Ouyang et al [11] addressed a stability analysis and synthesis problem for pendulum systems with multinon-linearities. A method for analyzing the Lagrange stability of a pendulum-like system with multiple nonlinearities was proposed. Sakwa et al [15] worked on systems of up to five interlinked masses restricted to move vertically in a plane whose equations of motion motivated them to make further studies on energy equations. It was found that for the generalized angular acceleration $\ddot{\boldsymbol{\theta}}_i$:

$$\ddot{\boldsymbol{\theta}}_1 = \frac{1}{\boldsymbol{m_1}}\left\{\left(\sum_{i=2}^{n} \boldsymbol{m_i}\right)\boldsymbol{\theta_2} - \left(\sum_{i=1}^{n} \boldsymbol{m_i}\right)\boldsymbol{\theta_1}\right\}\frac{\mathbf{g}}{\boldsymbol{l_1}} \quad (1.7)$$

$$\ddot{\boldsymbol{\theta}}_{(n-1)} = \left\{\left(\frac{\sum_{i=(n-2)}^{n} m_i}{m_{(n-2)}}\right)\boldsymbol{\theta}_{(n-2)} - \left(\frac{m_{(n-2)}+m_{(n-1)}}{m_{(n-1)}}\right)\left(\frac{\sum_{i=(n-1)}^{n} m_i}{m_{(n-1)}}\right)\boldsymbol{\theta}_{(n-1)} + \left(\frac{m_n}{m_{(n-1)}}\right)\boldsymbol{\theta}_n\right\}\frac{\mathbf{g}}{l_{(n-1)}} \quad (1.8)$$

$$\ddot{\boldsymbol{\theta}}_n = \left\{\left(\frac{m_{(n-1)}+m_n}{m_{(n-1)}}\right)\left(\boldsymbol{\theta}_{(n-1)} - \boldsymbol{\theta}_n\right)\right\}\frac{\mathbf{g}}{l_n} \quad (1.9)$$

where m, l, θ are uneven quantities

Warren et al [18] showed that the Lagrange equations depend exclusively on the difference between the total kinetic and potential energies of the system. They presented the equations without using variation calculus, a tool we find useful in the current method of developing equations of motion for the multi-pendula settings.White et al [22] went further and introduced the Lagrange dynamical equations as a robotics course at the university. There was need to study more of the arm control in Robotics at that level. Other systems of compound pendula that have been studied are somewhat similar to that of Leaderich [10] and our current work in that it is just a chain suspended at one end in a gravitational field in which the dimension of the problem is reduced by eliminating the reference to an inertial frame.

The vast literature is now elegantly reviewed for applicability and could have some similarities, though not entirely, to Prichani et al [13] coupled system of masses. In this study, freely moving system whose mass displacements are very small but could have different mass sizes, varying inter-link lengths with masses at different angular displacements at different times is considered. Weiting [20] studied a double spherical pendula setting with assigned co-ordinates (u, v, w) and (x, y, z) for which of the two spherical pendulum where the co-ordinates satisfy the equations

$$u^2 + v^2 + w^2 = l^2 \tag{1.10}$$

$$\text{and}(x-u)^2 + (y-v)^2 + (z-w)^2 = l^2 \tag{1.11}$$

He also assigned $(\dot{u}, \dot{v}, \dot{w})$ *and* $(\dot{x}, \dot{y}, \dot{z})$ as the velocity of the bobs for the first and second pendulum that must satisfy the constraint equations given by

$$u\dot{u} + v\dot{v} + w\dot{w} = 0 \tag{1.12}$$

$$and\ \ x\dot{x} + y\dot{y} + z\dot{z} = u\dot{x} + v\dot{y} + w\dot{z} + x\dot{u} + y\dot{v} + z\dot{w} \tag{1.13}$$

subject to the constraints (u, v, w, x, y, z) that define the configuration of the system and $(u, v, w, x, y, z, \dot{u}, \dot{v}, \dot{w}, \dot{x}, \dot{y}, \dot{z})$ that define the state of the system.

According to him one need not attempt to describe the general dynamics for a multi-spherical pendula setting because the problem is of formidable difficulty. Calculation of the energy explicitly for a general n-pendula system was derived by Prichani et al [13] and Sakwa et al [15]. There were unique possibilities of advancing studies in n-tupple pendulum systems by Lagrange method which we eagerly embraced. This work was restricted to a series of n-pendula systems in spatial motion. The glaring gap in the increased number of uneven mass units, other contemporary uneven quantities of lengths and angles for the same pendula setting and the complicated methods of solving the resultant equations provide the motivation to research more in this area. The fact that researchers admit the strenuous and difficult stages to go through before obtaining the required equations make it even more real to do same. Furthermore, the energy matrix operator method to obtain the equation lays a direct avenue to do this research. In this exercise, it should be clear that the determination of the energy eigen functions and coefficients makes it easier to obtain the solutions for the equations of motion.

2.0 LAGRANGIAN FUNCTIONS OF MULTI-PENDULA SYSTEMS

When a general co-ordinate origin **is** 0_k and the respective axes *are* X_k, Y_k and Z_k, then the position of the k^{th} mass m_k in Cartesian coordinate $(x_k, y_k\ and\ z_k)$ is expressed in polar co-ordinates as

$$\left(\sum_{i=1}^{k} l_i \sin\theta_i \cos\emptyset_i, \sum_{i=1}^{k} \ell_i \sin\theta_i \sin\emptyset_i, \sum_{i=1}^{k} \ell_i \cos\theta_i\right) \tag{2.1}$$

The translated origin 0_k takes on the position of the origin *O* if $k = 0$ and therefore it is the sum of the resolved components of l_i in the respective axes, where θ_k^0 is the zenith angle, $\emptyset_k^0$ is the azimuth angle and l_i is the distance of the k[th] mass from translated origin 0_k.

Adding the squares of the velocities gives

$$\dot{x}_k^2 + \dot{y}_k^2 + \dot{z}_k^2 = \sum_{i=1}^{k} \ell_i^2\left(\dot{\theta}_i^2 + \dot{\emptyset}_i^2 \sin^2\theta_i\right) + 2\sum_{j\neq i}^{k} l_i l_j\left\{\dot{\theta}_i\dot{\theta}_j \cos(\theta_i - \theta_j) + \dot{\emptyset}_i\dot{\emptyset}_j \sin\theta_i \sin\theta_j\right\} \tag{2.2}$$

$\therefore$ the Lagrangian formulation for the n **masses** is

$$L = \sum_{k=f(i,j)}^{n} \frac{1}{2} m_k \left[\sum_{i=1}^{n} l_i^2\left(\dot{\theta}_i^2 + \dot{\emptyset}_i^2 \sin^2\theta_i\right) + 2\sum_{j\neq i}^{n} l_i l_j\left\{\dot{\theta}_i\dot{\theta}_j \cos(\theta_i - \theta_j) + \dot{\emptyset}_i\dot{\emptyset}_j \sin\theta_i \sin\theta_j\right\}\right] - \sum_{k=i}^{n} m_k\left(\sum_{i=1}^{n} l_i - \sum_{i=1}^{k} l_i \cos\theta_i\right)g \tag{2.3}$$

Then $\sum_{k=i}^{n} m_k l_i^2\left\{\ddot{\emptyset}_i\theta_i^2 + 2\dot{\emptyset}_i\dot{\theta}_i\theta_i\right\} + \sum_{k=f(i,j)}^{n} m_k\left\{\sum_{j\neq i=1}^{n} l_i l_j\left(\ddot{\emptyset}_i\theta_i\theta_j + \dot{\emptyset}_i\left[\dot{\theta}_i\theta_j + \dot{\theta}_j\theta_i\right]\right)\right\}$ (2.4)

and

$$\sum_{k=f(i,j)}^{n} m_k l_j \sum_{j\neq i=1}^{n} l_i\left\{\ddot{\emptyset}_i\theta_i\theta_j + \dot{\emptyset}_i\left(\dot{\theta}_i\theta_j + \dot{\theta}_j\theta_i\right)\right\} = 0 \tag{2.5}$$

In matrix form obtains

$$\begin{pmatrix} \sum_{k=1}^{n} m_k\left(1 - \dot{\emptyset}_1^2 + \lambda\right) & \sum_{k=2}^{n} m_k\left(1 - \dot{\emptyset}_1\dot{\emptyset}_2\right) & \cdots \sum_{k=(n-2)}^{n} m_k\left(1 - \dot{\emptyset}_1\dot{\emptyset}_{(n-2)}\right) & \sum_{k=(n-1)}^{n} m_k\left(1 - \dot{\emptyset}_1\dot{\emptyset}_{(n-1)}\right) & m_n\left(1 - \dot{\emptyset}_1\dot{\emptyset}_n\right) \\ \sum_{k=2}^{n} m_k\left(1 - \dot{\emptyset}_1\dot{\emptyset}_2\right) & \sum_{k=2}^{n} m_k\left(1 - \dot{\emptyset}_2^2 + \lambda\right) & \cdots \sum_{k=(n-2)}^{n} m\left(1 - \dot{\emptyset}_2\dot{\emptyset}_{(n-2)}\right) & \sum_{k=(n-1)}^{n} m_k\left(1 - \dot{\emptyset}_2\dot{\emptyset}_{(n-1)}\right) & m_n\left(1 - \dot{\emptyset}_2\dot{\emptyset}_n\right) \\ \sum_{k=3}^{n} m_k\left(1 - \dot{\emptyset}_1\dot{\emptyset}_3\right) & \sum_{k=3}^{n} m_k\left(1 - \dot{\emptyset}_2\dot{\emptyset}_3\right) & \cdots \sum_{k=(n-2)}^{n} m_k\left(1 - \dot{\emptyset}_3\dot{\emptyset}_{(n-2)}\right) & \sum_{k=(n-1)}^{n} m_k\left(1 - \dot{\emptyset}_3\dot{\emptyset}_{(n-1)}\right) & m_n\left(1 - \dot{\emptyset}_3\dot{\emptyset}_n\right) \end{pmatrix} \begin{pmatrix} \theta_1 \\ \theta_2 \\ \vdots \\ \theta_n \end{pmatrix} = 0$$

⋮⋮⋮⋮⋮⋮

$$\left\{ \begin{matrix} \sum_{k=(n-2)}^{n} m_k\left(1 - \dot{\emptyset}_1\dot{\emptyset}_{(n-2)}\right) & \sum_{k=(n-2)}^{n} m_k\left(1 - \dot{\emptyset}_2\dot{\emptyset}_{(n-2)}\right) & \sum_{k=(n-2)}^{n} m_k\left(1 - \dot{\emptyset}_3\dot{\emptyset}_{(n-2)}\right) \cdots\cdots \\ \sum_{k=(n-1)}^{n} m_k\left(1 - \dot{\emptyset}_1\dot{\emptyset}_{(n-1)}\right) & \sum_{k=(n-1)}^{n} m_k\left(1 - \dot{\emptyset}_2\dot{\emptyset}_{(n-1)}\right) & \sum_{k=(n-1)}^{n} m_k\left(1 - \dot{\emptyset}_3\dot{\emptyset}_{(n-1)}\right) \\ m_n\left(1 - \dot{\emptyset}_1\dot{\emptyset}_n\right) & m_n\left(1 - \dot{\emptyset}_2\dot{\emptyset}_n\right) & m_n\left(1 - \dot{\emptyset}_3\dot{\emptyset}_n\right) \end{matrix} \right.$$

$$\begin{bmatrix} \sum_{k=(n-2)}^{n} m_k\left(1-\dot\phi_{(n-2)}^2+\lambda\right) & \sum_{k=(n-1)}^{n} m_k\left(1-\dot\phi_{(n-1)}\dot\phi_{(n-2)}\right) & m_n\left(1-\dot\phi_n\dot\phi_{(n-2)}\right) \\ \sum_{k=(n-1)}^{n} m_k\left(1-\dot\phi_{(n-2)}\dot\phi_{(n-1)}\right) & \sum_{k=(n-1)}^{n} m_k\left(1-\dot\phi_{(n-1)}^2+\lambda\right) & m_n\left(1-\dot\phi_n\dot\phi_{(n-1)}\right) \\ m_n\left(1-\dot\phi_{(n-1)}\dot\phi_n\right) & m_n\left(1-\dot\phi_{(n-1)}\dot\phi_n\right) & m_n\left(1-\dot\phi_n^2+\lambda\right) \end{bmatrix} \begin{pmatrix}\theta_1\\ \theta_2\\ \vdots \\ \theta_n\end{pmatrix} = 0 \quad (2.6)$$

When the mass units are equal i.e. $m_i = m_j = m$ and $\dot\phi_i, \dot\phi_j \cong 0$ then the above matrix equation has a determinant as hereunder

$$\begin{vmatrix} n(1+\lambda) & (n-1) & (n-2) & \cdots & \cdots & 4 & 3 & 2 & 1 \\ (n-1) & (n-1)(1+\lambda) & (n-2) & \cdots & \cdots & 4 & 3 & 2 & 1 \\ (n-2) & (n-2) & (n-2)(1+\lambda)\cdots & & \cdots & 4 & 3 & 2 & 1 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 3 & 3 & 3 \ldots\ldots & & & 3 & 3(1+\lambda) & 2 & 1 \\ 2 & 2 & 2 \ldots\ldots & & & 2 & 2 & 2(1+\lambda) & 1 \\ 1 & 1 & 1 \ldots\ldots & & & 1 & 1 & 1 & (1+\lambda) \end{vmatrix} = 0 \quad (2.7)$$

This determinant has an equation of the form

$$n!\left(\lambda^n + n\lambda^{(n-1)} + \frac{n(n-1)\lambda^{(n-2)}}{2^2} + \frac{n(n-1)(n-2)\lambda^{(n-3)}}{2^2.3^2} + \frac{n(n-1)(n-2)(n-3)\lambda^{(n-4)}}{2^2.3^2.4^2} + \cdots\right) = 0 \quad (2.8)$$

Applying this formula to various mass units suspended with the conditions as above give the respective determinant equations.

Table 1.

No of mass units	Determinant equation=0
1	$\lambda + 1$
2	$2\lambda^2 + 4\lambda + 1$
3	$6\lambda^3 + 18\lambda^2 + 9\lambda + 1$
4	$24\lambda^4 + 96\lambda^3 + 72\lambda^2 + 16\lambda + 1$
5	$120\lambda^5 + 600\lambda^4 + 600\lambda^3 + 200\lambda^2 + 25\lambda + 1$
n	$n!\left(\lambda^n + n\lambda^{(n-1)} + \frac{n(n-1)\lambda^{(n-2)}}{2^2} + \frac{n(n-1)(n-2)\lambda^{(n-3)}}{2^2.3^2} + \frac{n(n-1)(n-2)(n-3)\lambda^{(n-4)}}{2^2.3^2.4^2} + \cdots\right)$

Note that when $\dot\phi_i, \dot\phi_j \cong 0$, it means that the motion is planner depending only on the zenith angle. The values in the above table were obtained by Prichani et al (2010) in an M.sc. thesis titled Planner Motion of n-tupple Pendulum systems.

3.0 DISCUSSION AND CONCLUSION

The main objective of this research was to develop the governing relations of spatial motion for multiple pendula system set to oscillate in space for very small angular displacements. This objective has been achieved by the determination of the relationship between all the n mass units suspended linearly and oscillating as mentioned above. The major achievement was the Lagrangian equation for the kth. given by

$$L = \frac{1}{2}m_k\left[\sum_{i=1}^{k} l_i^2\left(\dot\theta_i^2 + \dot\phi_i^2\sin^2\theta_i\right) + 2\sum_{\substack{i=1\\ i\neq j}}^{k} l_i l_j\left\{\left(\dot\theta_i\dot\theta_j\cos(\theta_i-\theta_j) + \dot\phi_i\dot\phi_j\sin\theta_i\sin\theta_j\right)\right\}\right] - m_k\left(\sum_{i=1}^{k} l_i - \sum_{i=1}^{k} l_i\cos\theta_i\right)g \quad (3.1)$$

The kinetic energy for the k[th] mass particle, as an example, is found to be

$$K.E = \frac{1}{2}m_k\left[\sum_{i=1}^{k} l_i^2\left(\dot\theta_i^2 + \dot\phi_i^2\sin^2\theta_i\right) + 2\sum_{j\neq i}^{k} l_i l_j\left\{\left(\dot\theta_i\dot\theta_j\cos(\theta_i-\theta_j) + \dot\phi_i\dot\phi_j\sin\theta_i\sin\theta_j\right)\right\}\right] \quad (3.2)$$

And the respective potential energy is $P.E = m_k\left(\sum_{i=1}^{k} l_i - \sum_{i=1}^{k} l_i\cos\theta_i\right)g$ (3.3)

The sum of these two equations gives the total energy for the k[th] mass particle in the system. The integration of the energy for all the mass units gave the total energy in system. Complicated equations of motion from the dynamics were obtained. Then matrices that were positive definite and symmetric about the leading diagonal were found. The angular accelerations varied directly as the product of the zenith angular velocity and the azimuth angular displacement. According to the second paper posted by Peeter Joot[2] it was noted that introducing any additional mass in the system, for a planer motion, the interacting coupling terms increase and thus complicate the kinetic energy specifications. The results from this thesis show clearly that for n linearly suspended masses in spatial motion there are $2n + \frac{2n!}{2(n-1)!2!}$ kinetic energy terms. This means that for 1,2,3,4……… suspended mass units the expected kinetic energy terms will $3, 10, 21, 36 \cdots\cdots$ respectively. If there is an increase by one mass unit the kinetic energy terms for the $(n+1)$ **mass units** will be $2(n+1) + \frac{2(n+1)!}{2n!2!}$. The increase in the number of kinetic energy terms is $4n+3$. From this calculation the additional terms will be $7, 11, 15, 19 \cdots\cdots\cdots$ **respectively.**

All the primary matrix operators are $n \times n$ **square and** symmetric matrices e.g.

$\begin{pmatrix}\sum_1^2 m_k & m_2\\ m_2 & m_2\end{pmatrix}$, $\begin{pmatrix}\sum_1^3 m_k & \sum_2^3 m_k & m_3\\ \sum_2^3 m_k & \sum_2^3 m_k & m_3\\ m_3 & m_3 & m_3\end{pmatrix}$ for unequal masses and $\begin{pmatrix}10 & 5 & 2\\ 5 & 5 & 2\\ 2 & 2 & 2\end{pmatrix}$ for equal mass units.A few equations analyzed give the various symmetric forms mentioned above.

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