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Non-Monogenity of an Infinite Family of Pure Octic Fields

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ABSTRACT

Let $m \neq 1$ be a square free integer. The aim of this paper is to prove that the infinite family of pure octic field $L = Q(\sqrt[8]{m})$ is non-monogenic if $m \equiv 1\ (mod\ 4)$, ultimately, to complete the classification of pure octic fields $L = Q(\sqrt[8]{m})$ with respect to monogenity. We prove our results by considering the relative norms of the partial differents $\xi - \xi^{\sigma^j}$ of an integer ξ from the Galois closure $\tilde{L}$ of L to Dirichlet optimum subfields of L, where σ is the isomorphism which maps $(\sqrt[8]{m})$ to $\zeta_8\sqrt[8]{m}$ of L with $\zeta_8 = e^{\frac{2\pi\iota}{8}}, \iota = \sqrt{-1}$, and j = 4, 2, 1

1. Introduction

Let L be a pure octic field $Q(\sqrt[8]{m})$ over the field Q of rational numbers with a square free integer $m \neq 1$ and Z_L the ring of integers in L. The purpose of this paper is to determine whether Z_L has a power integral basis over the ring Z of rational integers or does not in the case of $m \equiv 1\ (mod\ 4)$. For the case of $m \equiv 2, 3\ (mod\ 4)$ we have already proved the monogenity of L in [12]. With the proof of non-monogenity of L in the case of $m \equiv 1\ (mod\ 4)$ we will complete the classification of L with respect to monogenity, thus partially solving the problem 6 of [18] which states "Find a necessary and sufficient condition for a field to have index 1".

For a finite field extension F/E of degree n, an element $\eta \in Z_F$ is said to give a relative power integral basis $\{1, \eta, \eta^2, \dots, \eta^{n-1}\}$ for F over E if Z_F coincides with a Z_E-module $Z_E[\eta] = Z_E 1 + Z_E\eta + Z_E\eta^2 + \cdots + Z_E\eta^{n-1}$ of rank n. When a field F has a power integral basis over E, the field F is said to be relatively monogenic over E. In the case of $E = Q$, we say that Z_F has a power integral basis or equivalently F is monogenic.

On the characterization of monogenity for non-abelian extensions with degree not less than 4, there are a few works for the pure extensions [3], [5], [6], [10] and composites of polynomial orders of number fields [7]. If the fields K are abelian extensions over Q, the explicit integral bases of K have been determined by H. W. Loepoldt [13] and there exist infinitely many monogenic cyclic cubic and cyclic quartic extensions K of composite conductors over Q [2], [16] and non- monogenic characterizations [9], [16]. It is known that the fields K belonging to the family of cyclic quartic extensions of prime conductor not equal to 5, 2-elementary abelian extensions of degree $[K : Q] \geq 2^3$ not equal to the 24th cyclotomic field $Q(\zeta_{24}) = Q(\sqrt{-1}, \sqrt{2}, \sqrt{-3})$, which is a complex multiplication field over the maximal real subfield $Q(\zeta_{24} + \zeta_{24}^{-1})$ and the other types of abelian extensions are non-monogenic [17], [15], [14], [16], [20]. Recently, A. Pethö and M.E. Pohst obtained a generalization of [14] for multiquadratic fields F and precise classification of F according to the values of field indices Ind_F [19]. Here the index Ind_F is defined by $gcd_{\alpha \in Z_F}\{\mathrm{Ind}_F\alpha\}$ for the module index $\mathrm{Ind}_F(\alpha) = (Z_F : Z[\alpha])$ of a submodule $Z[\alpha]$ in Z_F. Modern expositions on this area are found in [10], [4], [8] and [6].

2. Notations and Terminologies

For a finite extension field F/Q of degree n, d_F and $d_F(\alpha_1, \dots, \alpha_n)$ with $\alpha_j \in Z_F$ $(1 \leq j \leq n)$ denote the field discriminant of F and the discriminant of numbers $\alpha_1, \dots, \alpha_n$ with respect to the extension F/Q, respectively. If $\alpha_j = \alpha^{j-1}$ for a number $\alpha \in F$, we denote $d_F(\alpha_1, \dots, \alpha_n)$ by $d_F(\alpha)$, which is the discriminant of α. Then $\mathrm{Ind}_F(\alpha)$ is equal to the value of $\sqrt{\frac{|d_F(\alpha)|}{d_f}}$ [1]. Let L be a pure octic field $Q(\theta)$ over Q with $\theta = \sqrt[8]{m}$, m a square free integer $\neq 1$, where $arg\theta = 0$ if $m > 0$ and $arg\theta = 2\pi/8$ if $m < 0$. We prove that for $m \equiv 2, 3\ (mod\ 4)$, the ring Z_L of integers in L have power integral basis over the ring Z and in Section 3 that for $m \equiv 1\ (mod\ 4)$, the ring Z_L does not have any power integral basis. For the proof of non-monogenity of L, we work in the relative extension $\tilde{L}/D$, where $\tilde{L}$ denotes the Galois closure of the algebraic number field L over Q and D denotes the biquadratic field $Q(\sqrt{m}, \zeta_8^2)$. Then we consider the relative norms $N_{\tilde{L}/D}(\eta - \eta^{\sigma^j})$ of the partial differents $\eta - \eta^{\sigma^j}$ $(j = 2, 1)$ of the different $\mathfrak{d}L(\eta)$ of an integer η in L, where σ denotes the automorphism of $\tilde{L}$ induced by $\sqrt[8]{m} \to \zeta_8\sqrt[8]{m}$, $\zeta_8 \to \zeta_8$.

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Let k_n be the nth cyclotomic field $Q(\zeta_n)$over Q with a primitive nth root $\zeta_n = e^{\frac{2\pi i}{n}}$ of unity.Then $\tilde{L} = L(\zeta_8) = Q(\sqrt[8]{m}, \zeta_8)$ has degree 32 over the field Q for a square free integer $m \neq \pm 1, \pm 2$. We denote the Galois group $G(\tilde{L}/Q)$ of $\tilde{L}$ over Q by G. The Group is generated by three automorphisms σ, τ and ρ whose actions on θ and ζ_8 are depicted in table 1:

Table 1. The Actions of Automorphisms of $\tilde{L}$ on θ and ζ_8.

	θ	ζ_8
σ	$\theta\zeta_8$	ζ_8
ρ	θ	ζ_8^{-1}
τ	θ	ζ_8^3

Thus $G =< \sigma, \tau, \rho;\ \sigma^8 = \tau^2 = \rho^2 = (\tau\,\sigma)^2 = (\sigma\,\rho)^2 = (\tau\,\rho)^2 = \iota >$ is the Galois group with the identity map ι of $\tilde{L}$.

In the following Hasse diagram, we identify an isomorphism $\upsilon \in G$ and its restriction map $\in\upsilon_F$ to any subfield F of $\tilde{L}$. Then we have the subfield structure of $\tilde{L}$ and the corresponding subgroup structure of Galois group G for a square free integer $m \neq \pm 1, \pm 2$ in Fig. 1.

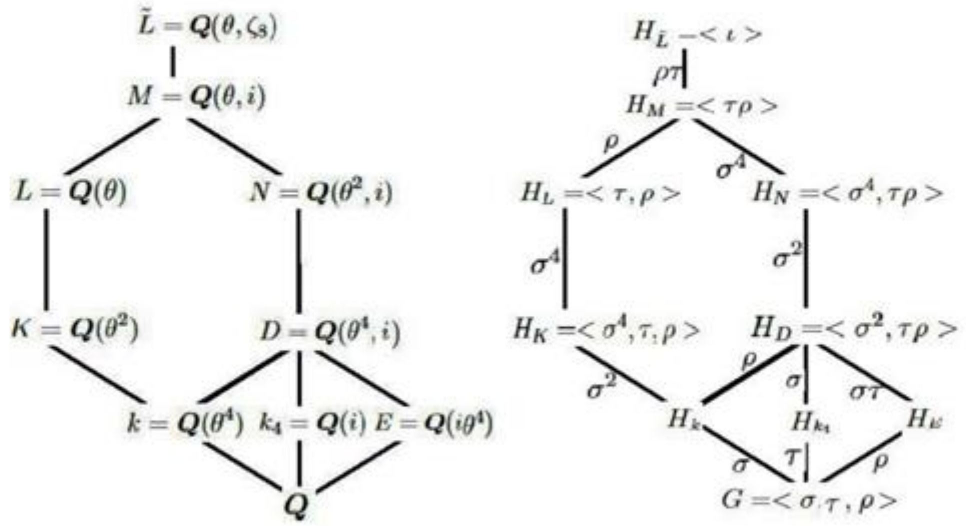

Fig. 1. The Subfield Structure of Galois Closure of $L = Q(\sqrt[8]{m})$ for $m \neq \pm 1, \pm 2$.

Here we denote $k = Q(\theta^4) = Q(\sqrt{m})$, $E = Q(i\theta^4) = Q(\sqrt{-m})$, $k_4 = Q(i)$, $K = Q(\theta^2) = Q(\sqrt[4]{m})$, $L = Q(\theta) = Q(\sqrt[8]{m})$, $D = Q(\theta^4, i) = Q(\sqrt{m}, \sqrt{-1})$, $N = Q(\theta^2, i) = Q(\sqrt[4]{m}, \sqrt{-1})$, $M = Q(\theta, i) = Q(\sqrt[8]{m}, \sqrt{-1})$ and $\tilde{L} = L(\zeta_8) = Q(\theta, \zeta_8)$. The corresponding Galois groups are $H_Q = G =< \sigma, \tau, \rho >$, $H_k =< \sigma^2, \tau, \rho >$, $H_{k_4} =< \sigma, \tau, \rho >$, $H_E =< \sigma^2, \sigma\rho, \sigma\tau >$,
$H_k =< \sigma^4, \tau, \rho >$, $H_D =< \sigma^2, \tau\rho >$, $H_L =< \tau, \rho >$, $H_N =< \sigma^4, \tau\rho >$, $H_M =< \tau\rho >$ and $H_L =< \tau >$.Here $< \rho_1, \ldots, \rho_s >$ for $\rho_j \in G\ (1 \leq j \leq s)$ denotes the subgroup of G generated by $\rho_1, \ldots, \rho_s$.

Let $\eta \in Z_L$ be the generator of power integral basis for L, then there exist $\alpha, \beta \in K$ such that $\eta = \alpha + \beta\theta$. This is typical throughout the paper unless stated otherwise. Elements of quadratic subfield k of L are denoted by α_j, β_j and the integers in Z are denoted by a_{ij}, b_{ij} with $i, j = 0, 1$.

3. Monogenity of Pure Octic Fields $Q(\sqrt[8]{m})$ with Square Free Integers $m \neq 1$

For an eighth root $\theta = \sqrt[8]{m}$ of a square free integer $m \neq 1$, let $L = Q(\theta)$ be a pure octic field, $K = Q(\theta^2), k = Q(\theta^4)$ its quartic and quadratic subfields respectively. Basing on the integral bases of pure quartic fields determined by T. Funakara [3], we obtained the integral basis of the pure octic field L and ascertained relative monogenity over its subfields in [11] as stated below.

3.1 Theorem. [11]. *For an eighth root $\theta = \sqrt[8]{m}$ of a square free integer $m \neq 1$, let L be a pure octic field $Q(\theta)$ and Z_L be the ring of integers in L. Then for $\omega = \frac{1+\theta^4}{2}$ integral bases for Z_L and field discriminants of L for different classes of m are as follows*

$$Z_L = \begin{cases} Z[\theta] = Z_K[\theta] = Z_k[\theta^2][\theta] & \text{if } m \equiv 2,3 (mod\ 4) \\ Z[1, \omega, \theta^2, \omega\theta^2, \theta, \omega\theta, \theta^3, \omega\theta^3] & \text{if } m \equiv 5, 13 (mod\ 16) \\ \quad = Z_K[\theta] = Z_k[\theta^2][\theta] & \\ Z\left[1, \omega, \theta^2, \omega\frac{1+\theta^2}{2}, \theta, \omega\theta, \theta^3, \omega\frac{\theta+\theta^3}{2}\right] & \text{if } m \equiv 9 (mod\ 16) \\ Z\left[1, \omega, \theta^2, \omega\frac{1+\theta^2}{2}, \theta, \omega\theta, \theta^3, \omega\frac{\theta+\theta^3}{2}\frac{1+\theta}{2}\right] & \text{if } m \equiv 1 (mod\ 16) \end{cases}$$

and hence

$$d_L = \begin{cases} -2^{24}m^7 = -2^8 \cdot d_k \cdot d_K^2 & \text{if } m \equiv 2,3(mod\ 4) \\ -2^{16}m^7 = -2^8 \cdot d_k \cdot d_K^2 & \text{if } m \equiv 5,13(mod\ 16) \\ -2^{12}m^7 = -2^8 \cdot d_k \cdot d_K^2 & \text{if } m \equiv 9(mod\ 16) \\ -2^{10}m^7 = -2^8 \cdot d_k \cdot d_K^2 & \text{if } m \equiv 1(mod\ 16) \end{cases}$$

We also determined the monogenity of the pure octic fields $L = Q(\theta)$ for $m \equiv 2,3\ (mod\ 4)$ in [12]. Thus, for the complete classification on monogenity of pure octic fields we are left with the proof of monogenity or non-monogenity of L for $m \equiv 1\ (mod\ 4)$. for which the following lemma is fundamental.

3.2 Lemma. Let $L = Q(\theta)$ be a pure octic field with $\theta = \sqrt[8]{m}$ and $m \equiv 1\ (mod\ 4)$. Let $K = Q(\theta^2)$ and $k = Q(\theta^4)$ be quartic and quadratic subfields of L respectively. Let $\eta = \alpha + \beta\theta \in L$ with $\alpha, \beta \in K$. If L is monogenic with η a generator of power integral bases of Z_L, then

a. $N_K(\beta) = \pm 1$ for $m \equiv 5, 9, 13(mod\ 16)$.

b. $\alpha_1 = 0$ for $m \equiv 5, 13(mod\ 16)$, with $\alpha = \alpha_0 + \alpha_1\theta^2$ and $\alpha_0, \alpha_1 \in Z_k$.

$b_3 = 1$ for $m \equiv 1(mod\ 16)$ with $\eta = \alpha + \beta\theta + b_3\frac{1+\theta^2}{2}\frac{1+\theta}{2}b_3 \in Z$.

Proof. The case a.

Since there is no relative integral basis of Z_L over Z_k for the case of $m \equiv 9(mod\ 16)$ by [11], we deal with the following two cases separately;

(i) $m \equiv 5, 13(mod\ 16)$ and (ii) $m \equiv 9(mod\ 16)$.

The case (i) $m \equiv 5, 13(mod\ 16)$.

By theorem 3.1 it holds that $Z_L = Z_K[\theta] = Z[1, \omega, \theta^2, \omega\theta^2, \theta, \omega\theta, \theta^3, \omega\theta^3]$, then for $\eta = \alpha + \beta\theta$ with $\alpha, \beta \in Z_K$, we have $\eta - \eta^{\sigma^4} = \alpha + \beta\theta - (\alpha + \beta(-\theta)) = 2\beta\theta$.

Moreover for $\alpha, \beta \in Z_K = Z_k[\theta^2]$ we take $\alpha = \alpha_0 + \alpha_1\theta^2$ and $\beta = \beta_0 + \beta_1\theta^2$ with $\alpha_j, \beta_j \in Z_k (0 \le j \le 1)$, such that $\eta - \eta^{\sigma^2} = (\alpha_0 + \alpha_1\theta^2) + (\beta_0 + \beta_1\theta^2)\theta - \left((\alpha - \alpha\theta^2) + (\beta - \beta\theta^2)i\theta\right) = 2\alpha_1\theta^2 + \beta_0(1-i)\theta + \beta_1(1+i)\theta^3 \equiv 0(mod(1-i)\theta Z_M)$.

For $\alpha_0 = a_0 + a_1\omega$ with $a_j \in Z (0 \le j \le 1)$ and $\omega = \frac{1+\theta^4}{2}$, the next partial different becomes

$$\eta - \eta^{\sigma} = \alpha_0 - \alpha_0^{\sigma} + (\alpha_1 - \alpha_1^{\sigma}i)\theta^2 + (\beta - \beta^{\sigma}\zeta_8)\theta$$

$= a_1\theta^4 + (\alpha_1 - \alpha_1^{\sigma}i)\theta^2 + (\beta - \beta^{\sigma}\zeta_8)\theta \equiv 0(mod\ \theta Z_{\tilde{L}})$ by $\alpha_0 - \alpha_0^{\sigma} = a_1(\omega - \omega^{\sigma}) = a_1\theta^4$.

We use $\eta - \eta^{\sigma^6} = -\left(\eta - \eta^{\sigma^2}\right)^{\sigma^6}$, $\eta - \eta^{\sigma^3} = (\eta - \eta^{\sigma^2}) + (\eta - \eta^{\sigma})^{\sigma^2}$ and $\eta - \eta^{\sigma^5} = (\eta - \eta^{\sigma^4}) + (\eta - \eta^{\sigma})^{\sigma^4}$ so that $d_L(\eta) = N_L(d_L(\eta)) = N_L((\eta - \eta^{\sigma})\left(\eta - \eta^{\sigma^3}\right)\left(\eta - \eta^{\sigma^5}\right) \cdot \left(\eta - \eta^{\sigma^2}\right)\left(\eta - \eta^{\sigma^6}\right) \cdot (\eta - \eta^{\sigma^4})) \equiv 0(mod\ (\theta^4)^8) \cdot ((1-i)^8\theta^8)^2 \cdot 2^8\theta^8 N_{L/K}(N_K(\beta)) \equiv 0(mod\ N_K(\beta)^2 \cdot (\theta^8)^7 \cdot (2^4)^2 \cdot 2^8) \equiv 0(mod\ N_K(\beta)^2 \cdot m^7 \cdot 2^{16}) \equiv 0\ (mod\ N_K(\beta)^2 \cdot d_L)$.

By $d_L = 2^{16}m^7$, we conclude that β should be a unit in K.

The case (ii) $m \equiv 9(mod\ 16)$.

In this case $Z_L = Z_K[\theta] = Z[1, \omega, \theta^2, \omega\frac{1+\theta^2}{2}, \theta, \omega\theta, \theta^3, \omega\ \frac{\theta+\theta^3}{2}]$.

For $\eta = \alpha + \beta\theta$ with $\alpha, \beta \in Z_K$, we take $\alpha = a_0 + a_1\omega + a_2\theta^2 + a_3\omega\frac{1+\theta^2}{2}$ and $\beta = b_0 + b_1\omega + b_2\theta^2 + b_3\frac{1+\theta^2}{2}$ with $a_j, b_j \in Z (0 \le j \le 3)$, so that $\eta - \eta^{\sigma^4} = 2\beta\theta$ and $(\eta - \eta^{\sigma^2})/\theta = 2a_2\theta + a_3\omega\theta + (\beta - \beta^{\sigma^2}i)$.

From $\beta - \beta^{\sigma^2}i = (b_0 + b_1\omega)(1-i) + b_2\theta^2(1+i) + b3\omega\left(\frac{1+\theta^2}{2} - \frac{1+\theta^2}{2}i\right)$

$= (1-i)(b_0 + b_1\ \omega + b_2\theta^2 i) + b_3\omega\frac{1+i\theta^2}{2}(1-i)$, it follows that

$(\eta - \eta^{\sigma^2})/\theta \equiv 0 + a_3\omega\theta + b_3\omega\frac{1+i\theta^2}{2}(1-i)\ (mod(1-i)Z_M)$.

Put $\xi = \frac{1+i\theta^2}{2}(1-i) \in N$. Then ξ is an integer of the field N, because the relative norm $N_{N/D}(\xi) = \xi\xi^{\sigma^2} = \frac{1+\theta^4}{4}(-2i) = \omega(-i)$ and the relative trace $T_{N/D}(\xi) = 1 - i$ are integers in D. Thus, we may put $\eta_M = \frac{\eta - \eta^{\sigma^2}}{\theta} = \omega\lambda_M + (1-i)\mu_M$ with suitable integers $\lambda_M, \mu_M \in Z_M$.With this substitution,we proceed further to have $N_{M/N}(\eta) = \eta_M \cdot (\eta_M)^{\sigma^4} = (\omega\lambda_M + (1-i)\mu_M) \cdot \left(\omega\lambda_M^{\sigma^4} + (1-i)\mu_M^{\sigma^4}\right) = \omega^2 N_{M/N}(\lambda_M) + \omega(1-i)\ T_{M/N}\left(\lambda_M\mu_M^{\sigma^4}\right) - 2iN_{M/N}(\mu M) = \omega^2\lambda_N + \omega(1-i)\mu_N + 2\nu_N$, which is denoted by η_N with $\lambda_N, \mu_N, \nu_N \in Z_N$.

The next relative norm of η_N until the biquadratic field $D = Q(\sqrt{m}, \zeta_8^2)$ gives $N_{N/D}(\eta_N) = \eta_N \cdot \eta_N^{\sigma^2} = \omega^4\lambda_{D_1} + \omega^2 \cdot 2\lambda_{D_2} + 2^2\lambda_{D_3} + \omega^3(1-i)\lambda_{D_4} + 2\omega^2\lambda_{D_5} + \omega(1-i) \cdot 2\lambda_{D_6} = \omega^4\lambda_{D_1} + \omega^2(1-i)\lambda_{D_7} + 2^2\lambda_{D_3} + \omega(1-i) \cdot 2\lambda_{D_6}$, which is denoted by η_D with $\lambda_{D_j} \in Z_D (1 \le j \le 7)$.

Next we have $N_{D/k_4}(\eta_D) = \eta_D \cdot \eta_D^{\sigma} = (\omega \cdot \omega^{\sigma})^4\lambda_1 + (\omega \cdot \omega^{\sigma})^2(-2)\lambda_2 + 2^4\lambda_3 + (\omega \cdot \omega^{\sigma})(-2) \cdot 2^2\lambda_4 + (\omega \cdot \omega^{\sigma})^2(1-i)\lambda_5 + 2^2\lambda_6 + \omega \cdot \omega^{\sigma}(1-i) \cdot 2\lambda_7 + (1-i) \cdot 2^2\lambda_8 + (\omega \cdot \omega^{\sigma})(-2)\lambda_9 + 2^3(1-i)\lambda_9 \equiv 0(mod\ 2^2)$ with $\lambda_j \in Z_{k_4}$, $(1 \le j \le 9)$, because of $\omega \cdot \omega^{\sigma} = \frac{1-m}{4} \equiv 0\ (mod\ 2)$. By $N_{N/k_4}\left(\frac{\eta - \eta^{\sigma^2}}{\theta}\right) \equiv 0\ (mod\ 2^2)$, it holds that $N_L\left(\frac{\eta - \eta^{\sigma^2}}{\theta}\right) \equiv$

$0\ (mod\ 2^2)$. Thus $N_L\prod_{j=1}^{3}\left(\frac{\eta-\eta^{\sigma^{2j}}}{\theta}\right)\equiv 0(mod\ (N_K(\beta))^2\cdot 2^{2+8+2}$, which gives $d_L=-2^{12}\cdot m^7\cdot (N_K(\beta))^2$. But $d_L=-2^{12}\cdot m^7$ by theorem 3.1. Thus, it holds that $N_K(\beta)=\pm 1$.

The case b.

From the case a(i), since β is a unit of K, we put $\eta=\alpha+\theta=\alpha_0+\alpha_1\theta^2+\theta$ with $\alpha_0,\alpha_1\in Z_k$. Thus, we have

$$\eta-\eta^{\sigma^2}=(\alpha_0+\alpha_1\theta^2+\theta)-(\alpha_0+\alpha_1(-\theta^2)+i\theta)=2\alpha_1\theta^2+(1-i)\theta=[(1+i)\alpha_1\theta+1](1-i)\theta.$$

Here $N_L[(1-i)\theta]=2^4m$ which is a factor of $d_L(\eta)$. Put $(1+i)\alpha_1\theta+1=\varepsilon$, then ε must be a unit of $Z_{\tilde{L}}$. Here $N_{\tilde{L}/M}(\varepsilon)=\varepsilon\cdot\varepsilon^{\rho\tau}=[(1+i)\alpha_1\theta+1]^2$,which we denote by ε_M.Then $N_{\tilde{L}/N}(\varepsilon)=N_{\tilde{L}/M}(\varepsilon_M)=\varepsilon_M\cdot\varepsilon_M^{\sigma^4}=[(1+i)\alpha_1\theta+1]^2\cdot[(1+i)\alpha_1(-\theta)+1]^2=\left[-(1+i)^2\alpha_1^2\theta^2+1\right]^2=\left[-2i\alpha_1^2\theta^2+1\right]^2$ holds, whose value is denoted by ε_N.

Similarly, $N_{N/D}(\varepsilon_N)=\varepsilon_N\cdot(\varepsilon_N)^{\sigma^2}=\left[4{\alpha_1}^4\theta^4+1\right]^2$Holds. Put $4{\alpha_1}^4\theta^4+1=\varepsilon_D$. For $m<0$, as ${\alpha_1}^4\theta^4\in Z_k=Z[1,\omega]$, we put ${\alpha_1}^4\theta^4=s+t\omega$ with $s,t\in Z$, Such that $N_k(\varepsilon_D)=(4s+1+4t\omega)(4s+1+4t\omega^\sigma)=(4s+1)^2+(4s+1)4t+(4t)^2\frac{1-m}{4}$. Here $\frac{1-m}{4}=1+2m_1>0$ with $m_1\in Z^+$.Thus $N_k(\varepsilon_D)=(4s+1)^2+(4s+1)4t+(4t)^2.(1+2m_1)=(4s+1+2t)^2+(2t)^2(3+8m_1)=+1$ holds if and only if $t=0$ and $s=0$, namely $\alpha_1=0$ follows. If $m>0$, then $N_{D/K}(\varepsilon_D)=(\varepsilon_D)(\varepsilon_D)^2=[4(s+4\omega)+1][4(s+4\omega^\sigma)+1]=16\left(s^2+st+t^2\frac{1-m}{4}\right)+4(2s+t)+1=\pm 1$ holds. Therefore, by $c=s^2+st+t^2\frac{1-m}{4}\in Z$ and $d=2s+t\in Z$, we have $16c+4d+1=\pm 1$. For $16c+4d+1=-1$ we have $8c+2d=-1$ which is impossible. For the case of +1, consider again $N_{D/k}(\varepsilon_D)=\left[4\alpha_1^4\theta^4+1\right]\left[4\alpha_1^{4^\sigma}(-\theta^4)+1\right]=1$, namely

$4^2[N_k(\alpha_1)]^4m+4\left(\alpha_1^4+\alpha_1^{4^\sigma}\right)\theta^4+1=1$.This implies

$0=4\alpha_1^4\alpha_1^{4^\sigma}\theta^4+\left(\alpha_1^4+\alpha_1^{4^\sigma}\right)\geq 4\alpha_1^4\alpha_1^{4^\sigma}\theta^4+2\alpha_1^2\alpha_1^{2^\sigma}=2\alpha_1^2\alpha_1^{2^\sigma}\left(2\alpha_1^2\alpha_1^{2^\sigma}\theta^4+1\right)\geq 2[N_k(\alpha 1)]^2$, and hence $N_k(\alpha_1)=0$. Since $\{1,\omega\}$ is an integral basis of Z_k then $N_k(\alpha_1)=0$ if and only if $\alpha_1=0$.

The case c.

For $m\equiv 1\ (mod\ 16)$, we have $Z_L=Z\left[1,\omega,\theta^2,\omega\frac{1+\theta^2}{2},\theta,\omega\theta,\theta^3,\omega\frac{1+\theta^2}{2}\frac{1+\theta}{2}\right]$.For $\eta\in Z_L$, we use $\eta=\alpha+\beta\theta+b_3\omega\frac{1+\theta^2}{2}\frac{1+\theta}{2}$ with $\alpha,\beta\in Z_K$ and $b_3\in Z$, and hence $\eta-\eta^{\sigma^4}=2\beta\theta+b_3\omega\frac{\theta+\theta^3}{2}$, and $\eta-\eta^{\sigma^2}=\alpha-\alpha^{\sigma^2}+\left(\beta\theta-\beta^{\sigma^2}i\theta\right)+b3\left(\omega\frac{1+\theta^2}{2}\frac{1+\theta}{2}-\omega\frac{1+\theta^2}{2}\frac{1+i\theta^2}{2}\right)$. Put $\alpha=a_0+a_1\omega+a_2\theta^2+a_3\omega\frac{1+\theta^2}{2}$ and $\beta=b_0+b_1\omega+b_2\theta^2,a_l,b_n\in Z$ with $0\leq l\leq 3$ and $0\leq n\leq 2$. Then $\alpha-\alpha^{\sigma^2}=2a_2\theta^2+a_3\omega\theta^2\equiv a_3\omega\theta^2(mod\ (1-i)Z_M)$, $\beta-\beta^{\sigma^2}i=b_0(1-i)+b_1\omega(1-i)+b_2\theta^2(1+i)\equiv 0\ (mod(1-i)Z_M)$. If b_3 is even, then $N_L\left(\eta-\eta^{\sigma^4}\right)\equiv 0\ (mod\ 2^8)$ and $\eta-\eta^{\sigma^2}\equiv a_3\omega\theta^2(mod\ (1-i)Z_M)$ hold. Thus for $\lambda_M\in Z_M$ we write $\eta-\eta^{\sigma^2}=a_3\omega\theta^2+(1-i)\lambda_M$. Then $N_{M/N}(\eta-\eta^{\sigma^2})=(a_3\omega\theta^2+(1-i)\lambda_M)(a_3\omega\theta^2+(1-i)\lambda_M^{\sigma^4})=\omega^2\nu_N+(1-i)\omega\lambda_N+2\mu_N$ holds, which is denoted by η_N for $\lambda_N,\mu_N,\nu_N\in Z_N$. Proceeding in the same way, we have $N_{N/D}(\eta_N)=\eta_N\cdot\eta_N^{\sigma^2}=(\omega^4\lambda_1+2\omega^2\lambda_2+2^2\lambda_3+2(1-i)\omega^3\lambda_4+2^2\omega^2\lambda_5+2(1-i)\omega\lambda_6$, which is denoted by $\eta_D\in Z_D$ with $\lambda_j\in Z_D,(1\leq j\leq 6)$.Then we obtain $N_{D/k_4}(\eta_D)=\eta_D\cdot\eta_D^\sigma=(\omega\omega^\sigma)^4\mu_1+2^2(\omega\omega^\sigma)^2\mu_2+2^4\mu_3+2^2\cdot(-2)(\omega\omega^\sigma)^3\mu_4+2^4(\omega\omega^\sigma)^2\mu_5+2^2(-2)\omega\omega^\sigma\mu_6+0+\cdots+0\equiv 0(mod\ 2^2Z_E)$ for $\mu_j\in Z_E(1\leq j\leq 6)$.

Then we have $N_{N/k_4}(\eta-\eta^{\sigma^2})\equiv 0\equiv N_L(\eta-\eta^{\sigma^2})(mod\ 2^2)$.

Thus if b_3 is even, then by $\eta-\eta^{\sigma^6}=\left(\eta^{\sigma^2}-\eta\right)^{\sigma^6}$, it follows that $N_L(\prod_{j=1}^{3}\left(\eta-\eta^{\sigma^{2j}}\right))\equiv 0(mod\ 2^{2+8+2})$, which contradicts the fact that 2^{10} is a maximal even divisor of d_L. Then b_3 is an odd number, say $1+2c_3$; namely $\eta=\alpha+\beta'\theta+(1+2c_3)\omega\frac{1+\theta^2}{2}\frac{1+\theta}{2}=\alpha+\beta\theta+\omega\frac{1+\theta^2}{2}\frac{1+\theta}{2}$ for some integers $\beta',\beta\in Z_K$. Therefore Lemma 3.2 has been proved.

We are now in a position to prove the non-monogenity of a family of pure octic fields.

4. Non-monogenity of Pure Octic Fields $Q(\sqrt[8]{m})$ with Square Free Integers $m\neq 1$

4.1. Theorem. The ring Z_L of integers in $L=Q(\sqrt[8]{m})$ with a square free integer $m\equiv 1(mod\ 4)$ and $m\neq 1$ has no power integral basis. Proof. First we consider the case of $m\equiv 5,13\ (mod\ 16)$.

Assume that $Z_L=Z[\eta]$ holds for some integer $\eta\in Z_L$. Then for the different $d_L(\eta)$ and the field discriminant d_L it should hold that $d_L(\eta)\cong d_L$.

For $\eta=\alpha+\beta\theta=\alpha_0+\alpha_1\theta^2+\beta\theta$, by Lemma 3.2a and 3.2b we have $\beta\cong 1$ and $\alpha_1=0$. Hence we put $\eta=\alpha_0+\theta=a_0+a_1\omega+\theta$ with $a_s\in Z,s=0,1$. Then $\eta-\eta^\sigma=a_1\theta^4+(1-\zeta_8)\theta$ holds. By the proof of Lemma 3.2a, $(\eta-\eta^\sigma)/\theta$ should be equal to a unit. If $a_1=0$, then $\eta-\eta^\sigma\equiv 0(mod\ 1-\zeta_8)$ and since $(1-\zeta_8)$ is a prime ideal in k_8, therefore $(1-\zeta_8)$ is not a unit in $\tilde{L}$. Thus $a_1\neq 0$.

Put $\varepsilon_{\tilde{L}}=\frac{\eta-\eta^\sigma}{\theta}=a_1\theta^3+(1-\zeta_8)$,then $N_{\tilde{L}/M}(\theta\varepsilon_{\tilde{L}})=[a_1\theta^4+(1-\zeta_8)\theta][a_1\theta^4+(1-\zeta_8)\theta]^{\rho\tau}=\theta^2N_{\tilde{L}/M}(\varepsilon_{\tilde{L}})$, $N_{\tilde{L}/M}(\varepsilon_{\tilde{L}})\in U_M$.

Put $\varepsilon_M = N_{\tilde{L}/M}(\varepsilon_{\tilde{L}}) = (a_1\theta^3+1)^2 - i$,so that $N_{M/N}(\varepsilon_M) = \varepsilon_M \cdot \varepsilon_M^{\sigma^4} = ((a_1\theta^3+1)^2 - i)((-a_1\theta^3+1)^2 - i)$ $= a_1^4\theta^{12} - 2a_1^2\theta^6 - 2ia_1^2\theta^6 - 2i = a_1^4 m\sqrt{m} - 2i - 2(1+i)a_1^2\sqrt{m}\,\theta^2$, which is denoted by ε_N.Then we have $N_{N/D}(\varepsilon_N) = (\varepsilon_N)(\varepsilon_N)^{\sigma^2} = \left(a_1^4 m\sqrt{m} - 2i\right)^2 - 8ia_1^4 m\sqrt{m} = a_1^4 m^3 - 4 - 12i\,a_1^4 m\sqrt{m}$, which we denote by ε_D.

Finally, we have $N_{D/k_4}(\varepsilon_D) = a_1^{16}m^6 + 136a_1^8m^3 + 16$ which should be a unit in $U_{k_4} \cap Q = \{\pm 1\}$. Namely it holds that $a_1^8m^3(a_1^8m^3 + 136) = -15$ or -17. Since $|m|^3 \geq |-3|^3$ therefore $\left|a_1^8m^3(a_1^8m^3+136)\right| = 0 \text{ or } > 27$ which is a contradiction. Thus, for the case of $m \equiv 5, 13 (mod\ 16)$, Z_L has no power integral basis.

Next, we consider the case of $m \equiv 9 (mod\ 16)$.

In this case $Z_L = Z_K[\theta] = Z[1, \omega, \theta^2, \omega\frac{1+\theta^2}{2}, \theta, \omega\theta, \theta^3, \omega\frac{\theta+\theta^3}{2}]$ and $d = -2^{12}m^7$ hold by Theorem 3.1.

Assume $Z_L = Z[\eta]$ for some integer $\eta \in Z_L$. Then $\eta = \alpha + \theta$ with $\alpha = \alpha_0 + a_2\theta^2 + a_3\,\omega\frac{1+\theta^2}{2}$, $\alpha_0 \in Z_k, a_s \in Z, s = 2,3$ by Lemma 3.2. In this case $\eta - \eta^{\sigma^4} = 2\theta$ and hence $N_L\left(\eta - \eta^{\sigma^4}\right) \equiv 0\ (mod\ 2^8)$ holds. By $\eta - \eta^{\sigma^2} = 2a_2\theta^2 + a_3\omega\theta^2$ $+(1-i)\theta = \theta\left(2a_2\theta + a_3\omega\theta + (1-i)\right)$, we put $\mu_M = 2a_2\theta + a_3\omega\theta + (1-i)$. Then we have $N_{M/N}(\mu_M) = \mu_M \cdot \mu_M^{\sigma^4}$ $= -(2a_2\theta + a_3\omega\theta)^2 - 2i$ which is denoted by $-\mu_N$. Then $N_{N/D}(\mu_N) = \mu_N \cdot \mu_N^{\sigma^2} = -((2a_2 + a_3\omega)^4\theta^4 + 4)$, which is denoted by μ_D. The relative norm $N_{D/E}(\mu_D) = \mu_D \cdot \mu_D^{\sigma\tau}$ of μ_D then gives $\{(2a_2 + a_3\omega)(2a_2 + a_3\bar{\omega})\}^4(-m) - \{(2a_2 + a_3\omega)^4$ $-(2a_2 + a_3\bar{\omega})^4\}4\theta^4 + 16$. If $2a_2 + a_3\omega = 0$, then $a_2 = a_3 = 0$, so that $\eta - \eta^{\sigma^2} = (1-i)\theta \equiv 0 (mod\ 1-i)$ and hence $N_L(\prod_{j=1}^3\left(\eta - \eta^{\sigma^{2j}}\right) \equiv 0\ (mod\ 2^{4+8+4})$, which is impossible as $2^{16} \nmid dL$. Thus $2a_2 + a_3\omega \neq 0$, and the relative norm becomes $N_{D/E}(\mu_D) = \left(4a_2^2 + 2a_2a_3 + a_3^2\frac{1-m}{4}\right)^2(-m) + 16 - \{s + t\omega - (s + t\omega)\}^4\theta^4$. Therefore by $\frac{1-m}{4} \equiv 0 (\text{mod } 2)$, we get $N_{D/E}(\mu_D) = -4\left(2a_2^2 + a_2a_3 + a_3^2\,\frac{1-m}{8}\right)^2 m + 16 - 4tm \equiv 0 (mod\ 4Z_E)$ and hence not in $U_E \cap Z = \{\pm 1\}$, Which is a contradiction. Thus for $m \equiv 9\ (mod\ 16)$, Z_L has no power integral basis.

Finally, we consider the case of $m \equiv 1 (mod\ 16)$.

In this case, $Z_L = Z[1, \omega, \theta^2, \omega\frac{1+\theta^2}{2}, \theta, \omega\theta, \theta^3, \omega\frac{1+\theta^2}{2}\frac{1+\theta}{2}]$ and $d_L = -2^{10}\,m^7$

Let $Z_L = Z[\eta]$. Then by Lemma 3.2c we may put $\eta = \alpha + \beta\theta + \eta_7$ with $\alpha = \alpha_0 + \alpha_1\omega + \alpha_2\theta^2 + \alpha_3\,\eta_3$, $\beta = \beta_0 + b_2\,\theta^2$,

$\eta_3 = \omega\frac{1+\theta^2}{2}$ and $\eta_7 = \omega\frac{1+\theta^2}{2}\frac{1+\theta}{2}$. Thus $\eta - \eta^{\sigma^4} = (2\beta + \eta_3)\theta$ holds. Put $(\eta - \eta^{\sigma^4})/\theta = 2\beta + \eta_3 = \xi_M$, then $N_{M/N}(\xi_M)$ $= \xi_M{\xi_M}^{\sigma^4} = {\xi_M}^2$,Which is denoted by ξ_N.Then it follows that $N_{M/D}(\xi_M) = N_{N/D}(\xi_N) = \xi_N{\xi_N}^{\sigma^2} = [(2\beta_0 + 2b_2\theta^2 + \eta_3)(2\beta_0 + 2b_2\theta^2 + {\eta_3}^{\sigma^2})]^2 = [4\beta_0^2 - 4b_2^2\theta^4 + {\eta_3}^{\sigma^2}) + 2b_2\theta^2\left({\eta_3}^{\sigma^2} - \eta_3\right) + \eta_3{\eta_3}^{\sigma^2}]^2$.

Here ${\eta_3}^{\sigma^2}$ is equal to $\omega\frac{1-\theta^2}{2}$, so that $\eta_3 + {\eta_3}^{\sigma^2} = \omega, {\eta_3}^{\sigma^2} - \eta_3 = -\omega\theta^2$ and $\eta_3{\eta_3}^{\sigma^2} = \frac{1}{2}\omega^2\omega^\sigma = -2m_1\,\omega$ with $m = 1 + 16m_1$. Moreover using $\theta^4 = 2\omega - 1$, $\omega^2 = \omega + 4m_1, \frac{1}{2}\omega\omega^\sigma = -2m_1$, $\gamma = c + d\omega \in Z[1, \omega]$ and the above relations, we have

$N_{M/D}(\xi_M) = 4(2\gamma + b\omega)^2$ (4.1)

In the case of $m < 0$, we take the process from the biquadratic field D to the quadratic subfield K as shown in fig 1, such that $N_{D/k}(2\gamma + b\omega) = (2c)^2 + 2c(2d + b) + (2d + b)^2\,\frac{1-m}{4}$. If $2d + b = 0$, then $N_{D/k_4}(\xi_M) \equiv 0 (mod\ 2^8)$. If $2d + b \neq 0$, then by $\frac{1-m}{4} = 4\,m_1$ with $m_1 > 0$, we have $|N_{M/D}(2\gamma + b\omega)| = (2c(2d + b)/2)^2 + (2d + b)^2, 4m_1 \geqq$ ¼ $+4m_1 \geqq$ 4. Thus, it is deduced that

$N_{M/D}(\xi_M) \geqq (2^2.2^2)^2$ (4.2)

For $m > 0$ we evaluate the $N_M(\xi_M)$ as follows:

$N_{D/K}(2\gamma + b\omega) = (2\gamma + b\omega)(2\gamma + b\omega)^\sigma = 4\gamma\gamma^\sigma + b^2\omega\omega^\sigma + 2b(\gamma\omega^\sigma + \gamma^\sigma\omega)$. Here $\omega\omega^\sigma = -4m_1, m_1 > 0$, therefore $N_{D/K}(2\gamma + b\omega) \equiv 0 (mod\ 2)$.

Thus $N_M(\xi_M) \equiv 0 (mod\ 4^4.2^4)$ and $N_L(\xi_M) \equiv 0\ (mod\ \sqrt{4^4.2^4} = 2^6)$. Thus, in both the cases

$N_L(\xi_M) \geqq 2^6$ (4.3)

Next, we evaluate norm of $\xi - \xi^{\sigma^2} = \eta_M \in L$ along the field tower $M \supset L \supset K \supset k \supset Q$. Consider $\eta_M = \alpha_M - \frac{1}{2}\eta_3^{\sigma^2}$ $-\left(\beta_K\theta i + \frac{1}{2}\eta_3^{\sigma^2}\theta i\right)$ with $\alpha_M = 2a_2\theta^2 + 2a_3\omega\theta^2 + \beta_0\theta + b_2\theta^3 + \eta_7$ and $\beta_K = \beta_0 - b_2\theta^2$.Thus on the integer $2\eta_M = 2\alpha_M - \eta_3^{\sigma^2} - \left(2\beta_K + \eta_3^{\sigma^2}\right)\theta i$, for $\eta_L = N_{M/L}(\eta_M)$, we have

$2^2\eta_L = N_{M/L}(2\eta_M) = \left(2\alpha_M - \eta_3^{\sigma^2}\right) + (2\beta_K + \eta_3^{\sigma^2})^2\theta^2 \geqq 2\left|2\alpha_M - \eta_3^{\sigma^2}\right|\left|2\beta_K + \eta_3^{\sigma^2}\right|\theta$. Thus we have the inequality

$N_{M/L}(\eta_M) \geqq 2\left|2\alpha_M - \eta_3^{\sigma^2}\right|\left|2\beta_K + \eta_3^{\sigma^2}\right|\theta$ (4.4)

Here by $2\alpha_M - \eta_3^{\sigma^2} = 2a_2\theta^2 + a_3\omega\theta^2 + \beta\theta + \omega\theta^2 + \eta_3\theta$ we notice that $2\alpha_M - \eta_3^{\sigma^2} \equiv 0 (mod\ \theta)$. Then it follows that $2^4\eta_K = N_{L/K}\left(2^2N_{L/K}(\eta_M)\right) \geqq 2.2\left|(2\alpha_M - \eta_3^{\sigma^2})(2\alpha_M - \eta_3^{\sigma^2})^{\sigma^4}((2\beta_K - \eta_3^{\sigma^2})^2\right|\theta^2$ for $\eta_K = N_{L/K}(\eta_L)$,

$2^8\eta_K = N_{K/k}(2^4 N_{M/K}(\eta_M)) \geqq 2^4 \left|(2\alpha_M - \eta_3^{\sigma^2})^{i+\sigma^4+\sigma^2+\sigma^6}((2\beta_K + \eta_3^{\sigma^2})^2)^{i+\sigma^2})^2)\right| \theta^4$ for $\eta_K = N_{K/k}(\eta_K)$ and hence $2^{16}N_k(\eta_K) \geqq 2^8 \left|(2\alpha_M - \eta_3^{\sigma^2})^{i+\sigma^4+\sigma^2+\sigma^6+\sigma+\sigma^5+\sigma^3+\sigma^7}\right| \left|((2\beta_K + \eta_3^{\sigma^2})^2)^{i+\sigma^2+\sigma+\sigma^3}\right| \theta^8$. Here we denote $\xi^{\rho_1}\xi^{\rho_2}$ by $\xi^{\rho_1+\rho_2}$ for any $\xi \in \tilde{L}$ and $\rho_1, \rho_2 \in G(\tilde{L}/Q)$.

Since 2 is completely decomposed in the quadratic subfield k, we put $2 \cong \mathfrak{B}\mathfrak{B}^\sigma$ for a prime ideal $\mathfrak{B}$, Which devides ω. Using $\omega\omega^\sigma = \pm 4m_1 \equiv 0(mod\, \mathfrak{B}^2\mathfrak{B}^\sigma)^2)$, and hence $\omega \equiv 0(mod\, \mathfrak{B}^2)$ by $\omega + \omega^\sigma = 1, \eta_3\eta_3^{\sigma^2}\omega\omega\frac{1}{2}\omega^\sigma = \omega(\pm 2m_1) \equiv 0(mod\, 2\mathfrak{B})$ for $m = 1 + 16m_1$ and $\beta_K^2\eta_3 = a\omega + a_1\omega + a_2\theta^2 + a_3\omega\frac{1+\theta^2}{2}$, $4\beta_K^2\eta_3^2 + 4(4)^{\sigma^2} = 8a_0\omega + 8a_1\omega\omega + 4\omega\theta^4 + 4_{aq3}\omega \equiv 0(mod\, 4\omega)$, we deduce that

$((2\beta_k + \eta_3^{\sigma^2})^2)^{1+\sigma^2} = (4\beta_K^2 + 4\beta_k\eta_3^{\sigma^2} + (\eta_3^{\sigma^2})^2)(4\beta_K^{\sigma^2})^2 + (v\eta_3 + \eta_3^2) \equiv$
$4\beta_K^2\eta_3^2 + 4\left(\beta_k\left(\eta_3^{\sigma^2}\eta_3\right)\eta_3 + \right)4\left(\beta_K^{\sigma^2})^2\left(e_3^{\sigma^2}\right)^2 + \right)4\beta_K^{\sigma^2}\left(\eta_3\eta_3^{\sigma^2}\right)\eta_3^{\sigma^2}\right) + (\eta_3^{\sigma^2}\eta_3)^2 \equiv 0(mod\, 4\omega)$, namely $\equiv 0(mod\, 4\mathfrak{B}^2)$.

Thus, we obtain

$N_K(2^8\eta_K) = 2^{16}N_K(\eta_K) \geqq 2^8 N_K(2\beta_k + \eta_3^{\sigma^2})^2)^{i+\sigma^2+\sigma+\sigma^3} \equiv 0(mod\, 2^8\, (4\mathfrak{B}^2)^{\sigma^2})^2) \equiv 0\, (mod\, 2^8\, (4^2.2^2)^2 \equiv 0(mod\, 2^{8+8+4})$. Therefore $N_K(\eta_K) \equiv 0(mod\, 2^4)$, thus

$$N_L(\eta_M) \geqq 2^2 \quad (4.5)$$

By $N_L\left(\eta - \eta^{\sigma^6}\right) = N_L\left(-\eta - \eta^{\sigma^2}\right))^{\sigma^6}, N_L\left(-\eta - \eta^{\sigma^i}\right), i = 1, 3, 5\ 7$ and from inequalities (4.3), (4.4) and (4.5) we conclude that $N_L(\partial_L(\eta)) \geqq 1.2^2.1.2^6.1.2^2.1.N_L(\theta^7) = 2^{10}m^7$.

From inequality (4.4), the equality holds if and if $2\alpha_M - \eta_3\sigma^2 = 2\beta_k + \eta_3\sigma^2$. However, $2\alpha_M - \eta_3\sigma^2 \neq 2\beta_k + \eta_3\sigma^2$ because $2\alpha_M - \eta_3\sigma^2 \notin K$ and $2\beta_k + \eta_3\sigma^2 \in K$. Thus $N_L(\partial_L(\eta)) > 2^{10}m^7$, which is a contrary to $|d_L/N_L(\theta)^7| = 2^{10}$.

Thus partial solution to the problem 6 of [18] follows:

4.2. Theorem.

Let $m \neq 1$ be a square free integer. The pure octic field $L = Q(\sqrt[8]{m})$ is monogenic if and only if $m \equiv 2, 3, (mod\, 4)$.

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