Available online at www.elixirpublishers.com (Elixir International Journal)

Applied Mathematics

Elixir Appl. Math. 113 (2017) 49269-49271

ISSN: 2229-712X

Generation of Compete Bipartite Graphs Using Normalized Hadamard Matrices

M.N.F.Sumaiya[1], W.V.Nishadi[1], K.D.E.Dhananjaya[1], A.A.I.Perera[1] and D.Uththamawadu[2]

1 Department of Mathematics, University of Peradeniya, Sri Lanka.

2. Department of Computer Science, University of Peradeniya, Sri Lanka.

ARTICLE INFO

Article history:

Received: 12 October 2017;

Received in revised form:

5 December 2017;

Accepted: 15 December 2017;

Keywords

Complete Bipartite Graph,

Hadamard Matrices,

Adjacency Matrix,

Sylvester's Method.

ABSTRACT

Complete Bipartite graph is the most important graph in design theory. There are several ways of constructing Complete Bipartite graph. In this paper we suggested an algorithm which can be used to construct Complete Bipartite graph using the Hadamard matrices. If the Order of Hadamard matrix is n then resultant Complete Bipartite graph is $\boldsymbol{K_{n,n}}$ where $\boldsymbol{n}$ n is even. The proposed method was tested manually $for\ \boldsymbol{n = 2, 4}$. Higher order Complete Bipartite graph were constructed using java program.

I. Introduction

In mathematics, a Hadamard matrix, named after the French mathematician Jacques Hadamard, is a square matrix whose entries are either +1 or -1 and whose rows are mutually orthogonal. The original four major families of Hadamard matrices discovered during the past century: the Sylvester, Paley, Hadamard design and Williamson families.[5] Examples of Hadamard matrices were first constructed by James Joseph Sylvester in 1867. Hadamard matrices play an important role in design theory. These matrices can be used to construct difference sets, designs, graphs etc. [10]

Normalized Hadamard matrices are special kind of Hadamard matrices whose first row and first column consist of all ones. [4] Here in this project, the ultimate purpose is to construct the complete bipartite graphs from normalized Hadamard matrices. These normalized Hadamard matrices are obtained from the Sylvester method.

If H is a Hadamard matrix of order $\boldsymbol{n}$, then the partition matrix $\begin{bmatrix} \boldsymbol{H} & \boldsymbol{H} \\ \boldsymbol{H} & -\boldsymbol{H} \end{bmatrix}$ is a Hadamard matrix of order $\boldsymbol{2n}$. This is known as Sylvester's method.[7]

Also note that multiplying a row or column by (-1) or permuting a row or column of a Hadamard matrix yields another Hadamard matrix. It is conjectured the order of Hadamard matrix is $\boldsymbol{1, 2}$ and $\boldsymbol{n \equiv 0 (mod\ 4)}$.[8]The most preliminary way of constructing Hadamard matrices is to use the Tensor (Kronecker) product. [1]

Tensor product

Let $A = \{a_{ij}\}$ be m x n matrix and B be a matrix then the Tensor product of A and B is defined as,

$$A \otimes B = \begin{bmatrix} a_{11}B & \cdots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{m1}B & \cdots & a_{mn}B \end{bmatrix}$$

Sylvester's construction can be applied repeatedly for Hadamard matrix H and it gives the result as following sequence of matrices

$$H_{2^k} = \begin{bmatrix} H_{2^{k-1}} & H_{2^{k-1}} \\ H_{2^{k-1}} & -H_{2^{k-1}} \end{bmatrix} = H_2 \otimes H_{2^{k-1}}$$ Where $\otimes$ denotes the Kronecker product. [6]

A complete bipartite graph is a special kind of bipartite graph where every vertex of the first set is connected to every vertex of the second set. Any complete bipartite graph that has m vertices in one of its bipartition subsets and n vertices in the other is denoted by $\boldsymbol{K_{m,n}}$. [2]

Examples,

$\boldsymbol{K_{2,3}}$

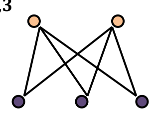

Fig. 1

$\boldsymbol{K_{3,3}}$

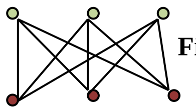

Fig. 2

Bipartite graphs are extensively used in modern coding theory. [3]In our work, complete bipartite graphs of $\boldsymbol{K_{n,n}}$ is considered, where $\boldsymbol{n}$ is even. [9]

Every Graph can be represented using a matrix. One such matrix is Adjacency matrix. A graph with $\boldsymbol{n}$ vertices is an $\boldsymbol{n}$ x $\boldsymbol{n}$ matrix if its $(\boldsymbol{i}, \boldsymbol{j})$entry is

1 if $\boldsymbol{i}^{th}$vertex and $\boldsymbol{j}^{th}$ vertex is connected and 0 otherwise.

II. Methodology

The proposed algorithm constructs the complete bipartite graph $\boldsymbol{K_{n,n}}$ using Normalized Hadamard matrices.

2. Proposed algorithm

2.1 $K_{2,2}$ Construction

Consider Normalized Hadamard matrix of order 2.

$\boldsymbol{H_2} = \begin{bmatrix} \boldsymbol{1} & \boldsymbol{1} \\ \boldsymbol{1} & -\boldsymbol{1} \end{bmatrix}$. In the next step label column vectors $\boldsymbol{c_1}$ and $\boldsymbol{c_2}$ such that alternate sign pattern column as $\boldsymbol{c_2}$ and other column as $\boldsymbol{c_1}$. That is,

$$\boldsymbol{H_2} = \begin{bmatrix} \boldsymbol{1} & \boldsymbol{1} \\ \boldsymbol{1} & -\boldsymbol{1} \end{bmatrix} = [\boldsymbol{c_1 c_2}]$$

Then, find the$\boldsymbol{C_i}$where $\boldsymbol{i = 1, 2}$ matrices such that

Tele:

E-mail address: athula103@yahoo.com

$\boldsymbol{C_i = c_i . c_i^T}$,where $\boldsymbol{i = 1, 2}$. Thus,

$$\boldsymbol{C_1} = \begin{bmatrix} \mathbf{1} \\ \mathbf{1} \end{bmatrix} [\mathbf{1} \quad \mathbf{1}] = \begin{bmatrix} \mathbf{1} & \mathbf{1} \\ \mathbf{1} & \mathbf{1} \end{bmatrix}$$

$$\boldsymbol{C_2} = \begin{bmatrix} \mathbf{1} \\ \mathbf{-1} \end{bmatrix} . [\mathbf{1} \quad \mathbf{-1}] = \begin{bmatrix} \mathbf{1} & \mathbf{-1} \\ \mathbf{-1} & \mathbf{1} \end{bmatrix}$$

Then, $\boldsymbol{C_1 \otimes C_2} = \begin{bmatrix} \boldsymbol{C_2} & \boldsymbol{C_2} \\ \boldsymbol{C_2} & \boldsymbol{C_2} \end{bmatrix}$.Therefore resultant matrix is

$$\begin{bmatrix} \mathbf{1} & \mathbf{-1} & \mathbf{1} & \mathbf{-1} \\ \mathbf{-1} & \mathbf{1} & \mathbf{-1} & \mathbf{1} \\ \mathbf{1} & \mathbf{-1} & \mathbf{1} & \mathbf{-1} \\ \mathbf{-1} & \mathbf{1} & \mathbf{-1} & \mathbf{1} \end{bmatrix}$$

Then replacing 1 by 0 and -1 by 1 yields to the adjacency matrix of the complete bipartite graph obtained from the normalized Hadamard matrix of order 2.

Hence the adjacency matrix of the complete bipartite graph $\boldsymbol{K_{2,2}}$ obtained from $\boldsymbol{H_2}$ is

$$\begin{bmatrix} \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{1} \\ \mathbf{1} & \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{1} \\ \mathbf{1} & \mathbf{0} & \mathbf{1} & \mathbf{0} \end{bmatrix}$$

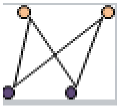

Fig. 3

This method is applied to construct $\boldsymbol{K_{n,n}}$ from normalized Hadamard matrices of order $\boldsymbol{2^n}$where $\boldsymbol{n} \in \mathbb{N}$.

2.2 $\boldsymbol{K_{4,4}}$ Construction

The normalized Hadamard matrix of order 4 $(\boldsymbol{H_4})$ can be constructed using $\boldsymbol{H_2}$ as follows by Sylvester's construction.

$$\boldsymbol{H_4} = \begin{bmatrix} \boldsymbol{H_2} & \boldsymbol{H_2} \\ \boldsymbol{H_2} & \boldsymbol{H_2} \end{bmatrix} = \begin{bmatrix} \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} \\ \mathbf{1} & \mathbf{-1} & \mathbf{1} & \mathbf{-1} \\ \mathbf{1} & \mathbf{1} & \mathbf{-1} & \mathbf{-1} \\ \mathbf{1} & \mathbf{-1} & \mathbf{-1} & \mathbf{1} \end{bmatrix}$$

In the next step label column vectors $\boldsymbol{c_1, c_2, c_3}$ and $\boldsymbol{c_4}$ such that alternate sign pattern column as $\boldsymbol{c_2}$.That is,

$$\begin{bmatrix} \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} \\ \mathbf{1} & \mathbf{-1} & \mathbf{1} & \mathbf{-1} \\ \mathbf{1} & \mathbf{1} & \mathbf{-1} & \mathbf{-1} \\ \mathbf{1} & \mathbf{-1} & \mathbf{-1} & \mathbf{1} \end{bmatrix} = [\boldsymbol{c_1} \quad \boldsymbol{c_2} \quad \boldsymbol{c_3} \quad \boldsymbol{c_4}]$$

Then, multiply that $\boldsymbol{c_2}$ by its transpose and label the resulting matrix as $\boldsymbol{C_2}$ and it can be constructed as follows:

$$\boldsymbol{C_2 = c_2 . c_2^T}$$

$$\begin{bmatrix} \mathbf{1} \\ \mathbf{-1} \\ \mathbf{1} \\ \mathbf{-1} \end{bmatrix} [\mathbf{1} \quad \mathbf{-1} \quad \mathbf{1} \quad \mathbf{-1}] = \begin{bmatrix} \mathbf{1} & \mathbf{-1} & \mathbf{1} & \mathbf{-1} \\ \mathbf{-1} & \mathbf{1} & \mathbf{-1} & \mathbf{1} \\ \mathbf{1} & \mathbf{-1} & \mathbf{1} & \mathbf{-1} \\ \mathbf{-1} & \mathbf{1} & \mathbf{-1} & \mathbf{1} \end{bmatrix}$$

Now,as before obtain $\boldsymbol{C_1 \otimes C_2} = \begin{bmatrix} \boldsymbol{C_2} & \boldsymbol{C_2} \\ \boldsymbol{C_2} & \boldsymbol{C_2} \end{bmatrix}$ where $\boldsymbol{C_1} = \begin{bmatrix} \mathbf{1} & \mathbf{1} \\ \mathbf{1} & \mathbf{1} \end{bmatrix}$ is obtained from Hadamard matrix of order 2, $(\boldsymbol{H_2})$.

$$\begin{bmatrix} \mathbf{1} & \mathbf{-1} & \mathbf{1} & \mathbf{-1} & \mathbf{1} & \mathbf{-1} & \mathbf{1} & \mathbf{-1} \\ \mathbf{-1} & \mathbf{1} & \mathbf{-1} & \mathbf{1} & \mathbf{-1} & \mathbf{1} & \mathbf{-1} & \mathbf{1} \\ \mathbf{1} & \mathbf{-1} & \mathbf{1} & \mathbf{-1} & \mathbf{1} & \mathbf{-1} & \mathbf{1} & \mathbf{-1} \\ \mathbf{-1} & \mathbf{1} & \mathbf{-1} & \mathbf{1} & \mathbf{-1} & \mathbf{1} & \mathbf{-1} & \mathbf{1} \\ \mathbf{1} & \mathbf{-1} & \mathbf{1} & \mathbf{-1} & \mathbf{1} & \mathbf{-1} & \mathbf{1} & \mathbf{-1} \\ \mathbf{-1} & \mathbf{1} & \mathbf{-1} & \mathbf{1} & \mathbf{-1} & \mathbf{1} & \mathbf{-1} & \mathbf{1} \\ \mathbf{1} & \mathbf{-1} & \mathbf{1} & \mathbf{-1} & \mathbf{1} & \mathbf{-1} & \mathbf{1} & \mathbf{-1} \\ \mathbf{-1} & \mathbf{1} & \mathbf{-1} & \mathbf{1} & \mathbf{-1} & \mathbf{1} & \mathbf{-1} & \mathbf{1} \end{bmatrix}$$

III. Results and Discussion

This method constructs complete bipartite graphs of the form $\boldsymbol{K_{n,n}}$ where $\boldsymbol{n = 2^k, k \geq 1}$, from the normalized Hadamard matrices of order $\boldsymbol{n}$. This method is developed to begin with the normalized Hadamard matrix of order 2. It is easy to observe that, in each $\boldsymbol{K_{n,n}}$ construction, each block $\boldsymbol{C_n}$ that are constructed by the operation $\boldsymbol{C_n = c_n c_n^T}$ for $\boldsymbol{n = 2^k}$. Note that each block $\boldsymbol{C_n}$ are symmetric that is $\boldsymbol{C_n = C_n^T}$. Since the normalized Hadamard matrices have been used, in every construction the second column will be a column matrix with the entries having alternate sign pattern. Then the block $\boldsymbol{C_n}$ in each $\boldsymbol{K_{n,n}}$ construction is a matrix of order $\boldsymbol{n}$. Hence for every construction we take the matrix $\boldsymbol{C_1}$ where $\boldsymbol{C_1}$ is a order 2 matrix that is obtain using $\boldsymbol{H_2}$.

Further, it can be seen that the corresponding complete bipartite graph of order $\boldsymbol{n}$ can be obtained easily for all $\boldsymbol{= 2, 4, 8}$ ... . For each $\boldsymbol{K_{n,n}}$ construction, $\boldsymbol{H_n}$ the normalized Hadamard matrix of order $\boldsymbol{n}$ has been taken to construct block matrix $\boldsymbol{C_n}$.

The above table summarizes the relationship between constructed complete bipartite graphs and normalized Hadamard matrices. A Computer program has been developed using this algorithm to obtained complete bipartite graphs with large vertices. Java Programming language has been used to write above program. The following figures are obtained using above computer program.

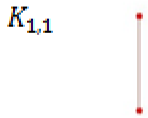

Fig. 4

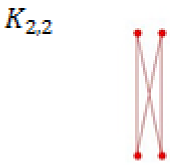

Fig. 5

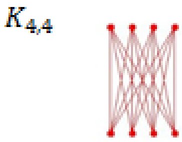

Fig. 6

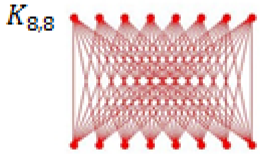

Fig. 7

Table 01. Complete bipartite graphs and normalized Hadamard matrices in the construction.

Order($n=2^k$;k≥ **1**)	Normalized Hadamard Matrices		Constructed Complete bipartite graph	Vertices of the graph	Adjacency matrix = A	Adjacency matrix = A
		Notation				notation
n	$\boldsymbol{n \times n}$	$\boldsymbol{H_n}$	$\boldsymbol{H_{n,n}}$	n+n	(2n) ×(2n)	$\boldsymbol{A_{2n}}$
n=2	$\boldsymbol{2 \times 2}$	$\boldsymbol{H_2}$	$\boldsymbol{H_{2,2}}$	4	4×4	$\boldsymbol{A_4}$
N=4	$\boldsymbol{4 \times 4}$	$\boldsymbol{H_4}$	$\boldsymbol{H_{4,4}}$	8	8×8	$\boldsymbol{A_8}$

$K_{16,16}$

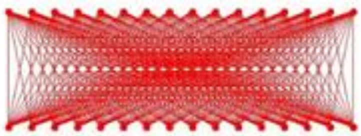

Fig. 8

$K_{64,64}$

Fig. 9

IV. Conclusion

Using this method, we have obtained the complete bipartite graph order $\boldsymbol{K_{n,n}}$ from normalized Hadamard matrix of order $\boldsymbol{n = 2^k, k} \in \mathbb{N}$ Furthermore, obtained the adjacency matrix of the relevant graph which has the order $\boldsymbol{2n}$**.** The adjacency matrix of each complete bipartite graph is symmetric and the diagonal elements are all zeros. Hence, using this method, complete bipartite graphs can be constructed for all $\boldsymbol{n = 2^k, k \in N}$.

References

[1]Anthony, B. (2001). Hadamard matrices and strongly regular graphs with the 3-e-c adjacency property. The electronic Journal of Combinatorics(8).

[2]Biggs, Norman. (1994). Algebraic Graph Theory (2nd ed.). Cambridge Mathematical Library: Cambridge University Press.

[3] Bondy, J. (1976). Graph Theory with Applications. North Holland.

[4]Charles, J. (2007). Handbook of Combinatorial designs. CRC Press.

[5]Craigen, R. (1996). Hadamard matrices and designs. In J.H.C.J. Colbourn, CRC Handbook of Cominatorial designs (pp. 370-376). CRC Press.

[6]Geramita, A. (1979). orthogonal designs. In Quadratic Forms and Hadamard matrices. New York: Marcal Dekker.

[7]Hedayat, A. (1978). Hadamard matrices and their applications. Annals of Statistics. 28

[8]Horadam, K. (2007). Hadamard matrices and generalizations. In Hadamard matrices and their applications (pp. 10-12). New Jercy: Princeton Univrsity Press.

[9]Jayathilake, A. (2013, February). Generation of strongly Regular Graphs from Normalized Hadamard Matrices. International Journal of Scientific and Technology research, 2(2).

[10]Jennifer Seberry, M. (1992). Hadamard matrices, sequences and block designs. In D.J.H.Dinitz, Contemporary design theory-A collection of surveys (pp. 431-560). John Wiley and sons.