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A Recurrence Relation to Construct 1- Factors of Complete Graphs

W.V. Nishadi[1], K.D.E. Dhananjaya[1], A.A.I. Perera[1] and P.Gunathilake[2]

[1]Department of Mathematics, Faculty of Science, University of Peradeniya, Sri Lanka.

[2]Department of Statistics and Computer Science, Faculty of Science, University of Peradeniya, Sri Lanka.

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ABSTRACT

Prior researches found several methods to construct *1*- factorization using Steiner triple systems [1], the staircase method of Bileski [2], and etc. But not given any method of constructing *1*- factors in complete graphs. In our previous work, we briefly explained this construction and published in an Abstract form in the *i*PURSE 2017. Generalization of that work is given in this paper. For complete graphs whose number of vertices is a multiple of 2, we implement our finding using Java program.

1. Introduction

A factor of a graph *G* is a spanning subgraph of *G* which is not totally disconnected. The union of edge disjoint factors which form *G* is called factorization of graph *G* [3]. An *n*-factor is regular of degree *n*. If *G* is the sum of *n*-factors, their union is called an n-factorization [4]. The graph which admits *n*-factorization is called an *n*-factorable graph.

A *1*-factor is a set of pair wise disjoint edges of *G* that between them contain every vertex. The necessary conditions to be a *1*-factorable graph are that the graph must have an even number of vertices and it should be regular [5]. So, it is conjectured that a regular graph with *2n* vertices and degree greater than *n* will always have a *1*-factorization [6].

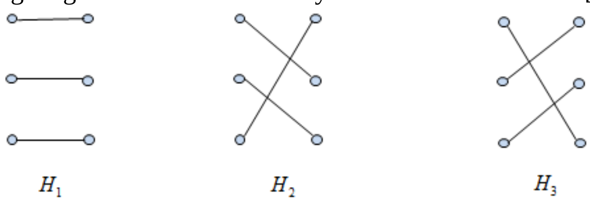


Fig 1. A *1*- factorization of $K_{3,3}$.

Complete Graphs K_n is a simple undirected graph such that every pair of distinct vertices is connected by a unique edge and total number of edges is $n(n-1)/2$.

Theorem 1: The complete graph K_{2n} is *1*-factorable.

We need to prove a partition of the set *Y* of lines of K_{2n} into $(2n-1)$ *1*-factors. Label the points of *G* by $v_1, v_2, \ldots, v_{2n}$, and define, for $i = 1, 2, \ldots, (2n-1)$, the sets of lines $Y_i = \{v_i v_{2n}\} \cup \{v_{i-j} v_{i+j}\ ;\ j = 1, 2, \ldots, (n-1)\}$, where each $i+j$ and $i-j$ is expressed as one of the numbers $1, 2, \ldots, (2n-1)$ modulo $(2n-1)$. The collection $\{Y_i\}$ is displayed to give a suitable partition of *Y*, and the union of the subgraphs G_i induced by Y_i is a *1*-factorization of K_{2n}.

The study of *1*-factorization is used in various combinatorial applications. An instantaneous application of *1*-factorization is that of edge coloring [7]. Also, in scheduling tournament, especially round-robin tournaments [8], study of *1*-factorization is used. Other applications of *1*-factorization include block designs, 3-designs, and Room square and Steiner system [9], [10].

2. Methodology

In this paper, we produce a recursive method of constructing at *1*-factors of K_{2n} by presenting an algorithm.

Steps of the proposed algorithm

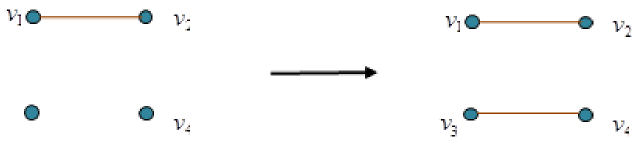
2.1. When $n = 1$; Complete graph of 2 vertices. Clearly, it has one *1*-factor.

Tele: +94776003272

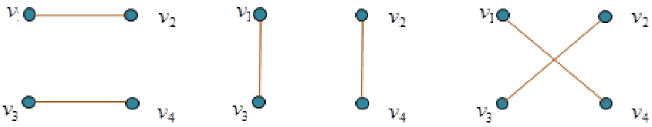
E-mail address: vindinisha@hotmail.com

2.2. When $n=2$; Complete graph of 4 vertices.

Label 4 vertices as v_1, v_2, v_3 and v_4 . Take any vertex (say) v_1 and join it to any other vertex (say) v_2. Then join the remaining two vertices.

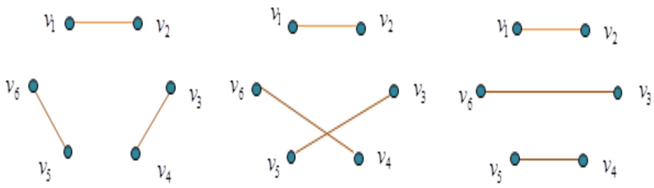


There are 3 ways to join the vertex v_1 to other vertices. So, we can construct 3 types of *1*-factors.

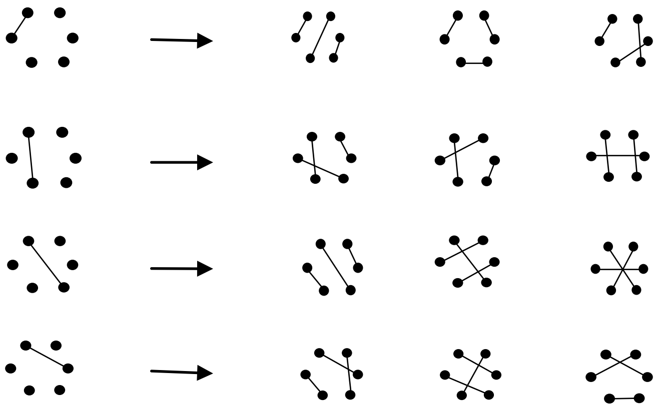


2.3. When $n=3$; Complete graph of 6 vertices.

Label 6 vertices as v_1, v_2, v_3, v_4, v_5 and v_6 .Taking any vertex (say) v_1 and join it to any other vertex (say) v_2 . The remaining 4 vertices could be constructed connected as in K_4.



There are 5 ways to join the vertex v_1 to other vertices. So, we can construct 15 types of *1*-factors. $(=3\times 5)$



By repeating this algorithm, *1*-factors corresponding to the complete graph K_{2n} can be constructed.

3. Results and Discussion

Table 01 illustrate the relationship between number of 1-factors K_2, K_4 and K_6 .

Table 1. Tabulation of the results.

Value of n	Complete Graph	Construction	Number of 1-factors
1	K_2	$x_1 = 1\times 1$	1
2	K_4	$x_2 = (1\times 3) = x_1 \times (2.2-1)$	3
3	K_6	$x_3 = (3\times 5) = x_2 \times (2.3-1)$	15

Consider the complete graph of $2n$ vertices which has x_n number of *1*-factors. Fix one vertex and connect with another vertex. Then there are $(2n-2)$ remaining vertices. There are x_{n-1} number of *1*-factors corresponding to $(2n-2)$ vertices. Also, there are $(2n-1)$ ways of connecting fixed vertex with other vertices.

Using this algorithm a recurrence relation $x_n = (2n-1)x_{n-1}$ with $x_1 = 1$, where x_n is the number of *1*-factors corresponding to the complete graph K_{2n} can be obtained.

Solving the recurrence relation recursively we can obtain x_n.

$$\left.\begin{aligned}
x_2 &= 3\times x_1\\
x_3 &= 5\times x_2\\
x_4 &= 7\times x_3\\
&\vdots\\
x_{n-2} &= (2n-5)\times x_{n-3}\\
x_{n-1} &= (2n-3)\times x_{n-2}\\
x_n &= (2n-1)\times x_{n-1}
\end{aligned}\right\downarrow \quad \textbf{(x)}$$

$$\Rightarrow x_n = (3.5.7\ldots(2n-3).(2n-1))x_1$$

$$\Rightarrow x_n = \frac{(2n)!}{2^n.n!}$$

Thus K_8 has 105 *1*-factors and K_{10} has 945.
Alternative proof is given by the Principle of Mathematical Induction.
When $n=1$, number of 1-factors in $K_2 = 1$

$$= x_1 = \frac{2!}{2.1!}$$

Thus the result is true for $n=1$.
Assume that the result is true for $n = p$,

Number of 1-factors in $K_{2p} = x_p = \dfrac{(2p)!}{2^p.p!}$

We must prove that the result is true for $n = p+1$

Number of 1-factors in $K_{2(p+1)} = [2(p+1)-1]\times$ (number of 1-factors of K_{2p})

$$\begin{aligned}
&= (2p+1)\,\frac{(2p)!}{2^p.p!}\\
&= \frac{2(p+1)(2p+1)(2p)!}{2(p+1)2^p\,p!}\\
&= \frac{(2p+2)!}{2^{p+1}(p+1)!}
\end{aligned}$$

The result is true for $n = p+1$

By the Principle of Mathematical Induction the result is true for all $n \in Z^+$.
In addition, Java program is used to implement our results.

```
for (int c = 0; c < initialEdges.size(); c++) {

    for(int k = 0; k < allEdges.size(); k++) {

        for(int i = k + 1; i < allEdges.size(); i++) {
            selectedEdges = new LinkedHashSet<>();
            selectedEdges.add(initialEdges.get(c));

            for (int j = i + 1; j < allEdges.size(); j++) {
                if (!exists(selectedEdges, allEdges.get(j))) {
                    selectedEdges.add(allEdges.get(j));
                }
            }

            if(n / (float)selectedEdges.size() == 2){
                combinations.add(selectedEdges);
            }
        }
    }
}
```

Fig 2. The user input interface.

Hence, the obtained *1*-factors corresponding to the complete graph of order 4 and order 6 are as follows:

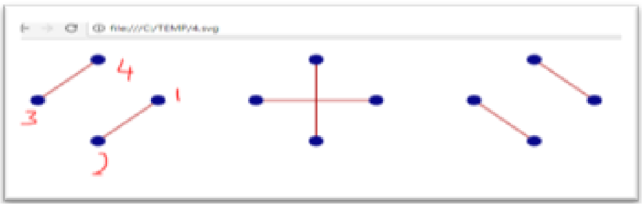


Fig 3. Resulting *1*-factors corresponding to K_4.

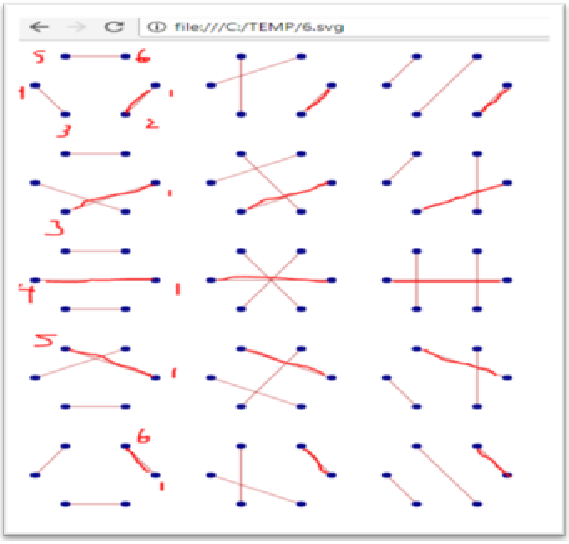
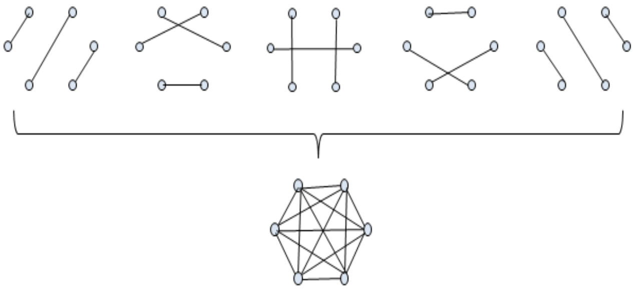


Fig 4. Resulting *1*-factors corresponding to K_6.

4. Conclusion

The *1*-factors of complete graphs have been constructed using the above generalized algorithm. Recurrence relation of *1*-factors of complete graphs has been proved using the Principle of Mathematical Induction. Further, the complete graphs can be constructed using line disjoint 1-factors. This construction is illustrated using K_6.



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