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Rough fuzzy ideals in near-rings

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ABSTRACT
The purpose of this paper is to introduce the idea of rough fuzzy sets in nearrings. We introduce the notion of rough fuzzy sets with respect to an ideal of a nearring which is an extended notion of an ideal of a ring and we derive some properties of the lower and upper approximations in a nearring.

1. Introduction

The rough set theory was introduced by Pawlak in 1982. Rough set which deals with uncertainty is an extension of set theory, in which a subset of an universe is described by a pair of ordinary sets called the lower and upper approximations. Some authors [10, 11, 12, 16, 17] have studied the algebraic properties of rough sets. Aslam et al. [2], Biswas and Nanda [3], Chinram [4], Davvaz [5], Jun [10], Kuroki and Mordeson [11], Kuroki [12] applied roughness in di erent algebraic structures. In [8], as a generalization of ideals in BCK-algebras, the notion of rough ideals is discussed. In [5, 6, 7, 8], Davvaz applied the concept of approximations in the theory of algebraic structures and derived a relationship between rough sets and ring theory and considered a ring as a universal set and introduced the notion of rough ideals and rough subrings with respect to an ideal of a ring. Since Rosenfeld [18] applied the notion of fuzzy sets to algebra and introduced the notion of fuzzy sub groups. Liu [13] introduced and examined the notion of a fuzzy ideal of a ring. Subsequently, among others, Liu, Mukherjee and Sen [14] fuzzi ed certain standard concepts and results on rings and ideals. Based on an equivalence relation, in 1990, Dubois and Prade [9], investigated the lower and upper approximations of fuzzy sets in a Pawlak approximation space to obtain an extended notion called rough fuzzy sets. Since then the relations between rough sets, fuzzy sets and algebraic systems have been considered by many mathematicians. In [8]], the lower and upper approximations are formulated in the context of ring theory. In this paper we substitute a nearring instead of the universe in Pawlak approximation space and derive some of the the properties of rough fuzzy sets in nearrings.

2. Preliminaries

Throughout this paper R is a nearring, is a congruence relation on R and A is a non-empty subset of R. Then, the sets

$\rho^{-}(A) = \{x \in R \mid [x]\rho \subseteq A\}$,

$\rho_{-}(A) = \{x \in R \mid [x]\rho \cap A \neq \varphi\}$

are called respectively, the ρ –lower and ρ-upper approximation of the set A. ρ(A) = (ρ_(A), ρ−(A)) is called a rough set with respect to ρ if ρ_(A) ≠ ρ−(A). A congruence ρ on R is called complete if [a]ρ[b]ρ=[ab]ρ for any a,b∈ R.

Definition 2.1. [16] Let ρ be an equivalence relation on R. Then, ρ is called a full congruence relation if (a, b) ∈ρ implies (a+x, b+x), (ax, bx) and (xa, xb) ∈ρ, for all x ∈ R.

Definition 2.2. [16] Let ρ be an equivalence relation on R. Then,(a, b)∈ρ and (c, d) ∈ρ imply (a + c, b + d) ∈ ρ, (ac, bd) ∈ ρ and (−a, −b) ∈ ρ, for all a, b, c, d ∈ R.

Lemma 2.3. [16] Let ρ be a full congruence relation on a ring R. If a, b ∈ R, then

(1) $[a]\rho + [b]\rho = [a + b]\rho$;

(2) $[-a]\rho = -[a]\rho$;

(3) $\{xy \mid x \in [a]\rho, y \in [b]\rho\} \subseteq [ab]\rho$.

Definition 2.4. [15] A non-empty set R with two binary operations + and · is called a nearring if

(1) (R, +) is a group;

(2) (R, ·) is a semigroup;

(3) $x \cdot (y + z) = x \cdot y + x \cdot z$, for all x, y, z ∈ R.

To be more precise, these are left near-rings because they satisfy the left dis- tributive law. We will use the word 'nearring' to mean 'left nearring'.

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Definition 2.5. [2] Let (R, +, ·) be a nearring. An ideal of R is a subset I of R such that
(1) (I , +) is a normal subgroup of (R, +);
(2) RI ⊂ I ;
(3) (r + i)s − rs ∈ I , for all i ∈ I and r, s ∈ R
Note that if I satisfies (1) and (2), then it is called a left ideal of R. If I satisfies (1) and (3), then it is called a right ideal of R.

Definition 2.6. [17] Let ρ be an equivalence relation on R and μ a fuzzy subset of R.Then, we define the fuzzy sets $\rho_{-}(\mu)$ and $\rho^{-}(\mu)$ as follows:

$$\rho_{-}(\mu)(x) = \bigwedge_{a\in[x]_\rho} \mu(a) \text{ and } \rho^{-}(\mu)(x) = \bigvee_{a\in[x]_\rho} \mu(a)$$

The fuzzy sets $\rho_{-}(\mu)$ and $\rho^{-}(\mu)$ are called, respectively the ρ-lower and ρ-upper approximations of the fuzzy set μ. ρ(μ) = ($\rho_{-}(\mu)$, $\rho^{-}(\mu)$) is called a rough fuzzy set with respect to ρ if $\rho_{-}(\mu) = \rho^{-}(\mu)$
$\rho^{-}(\mu)$ is called a rough fuzzy set with respect to ρ if $\rho_{-}(\mu) \neq \rho^{-}(\mu)$

Definition 2.7. [19] Let R be a nearring and μ be a fuzzy subset of R. We say μ a fuzzy sub nearring of R if
(1) μ(x − y) ≥ min{μ(x), μ(y)};
(2) μ(xy) ≥ min{μ(x), μ(y)}, for all x, y ∈ R. μ is called a fuzzy ideal of R if μ is a sub nearring of R and
(3) μ(x) = μ(y + x − y);
(4) μ(xy) ≥ μ(y);
(5) μ((x + i)y − xy) ≥ μ(i), for any x, y, i ∈ R.
Note: μ is a fuzzy left ideal of R if it satisfies (1), (3), (4) and fuzzy right ideal of R if it satisfies (1), (2), (3), (4).

Definition 2.8. [19] If a fuzzy subset μ of R satisfies the property (1) of 2.7, then
(1) $\mu(0_R) \geq \mu(x)$;
(2) μ(−x) = μ(x), for all x, y ∈ R.

3. Rough fuzzy ideals in nearrings:

Definition 3.1. Let μ be fuzzy ideal of a nearring R. Then, for each α ∈ [0, 1], the set
U (μ, α) = {(a, b) ∈ R × R | μ(a − b) ≥ α} is called a α-cut of μ.

Proposition 3.2. If μ and λ be two fuzzy sets of a universe set R. Then the following holds
(1) Uα (μ) ⊆ Uβ (μ) if α ≥ β;
(2) μ ⊆ λ implies Uα (μ) ⊆ Uα (λ);
(3) Uα (μ ∩ λ) = Uα (μ) ∩ Uα (λ);
(4) Uα (μ ∪ λ) ⊇ Uα (μ) ∪ Uα (λ)

Proof. (1) If (a, b) ∈ Uα (μ), then μ(a − b) ≥ α. Since α ≥ β, μ(a − b) ≥ α ≥ β. implies μ(a − b) ≥ β and so (a, b) ∈ Uβ (μ). Hence, Uα (μ) ⊆ Uβ (μ) if α ≥ β.
(2) Let (a, b) ∈ Uα (μ) implies μ(a − b) ≥ α. As λ ⊇ μ so λ(a − b) ≥μ(a − b) ≥ α. This implies that λ(a − b) ≥ α and so (a, b) ∈ Uα (λ). Hence Uα (μ) ⊆ Uα (λ).
(3) Since μ ∩ λ ⊆ μ and μ ∩ λ ⊆ μ by (1) we have, Uα (μ ∩ λ) ⊆ Uα (μ) and Uα (μ ∩ λ) ⊆ Uα (λ) which implies that Uα (μ ∩ λ) ⊆ Uα (μ) ∩ Uα (λ). Now, let (a, b) ∈ Uα (μ) ∩ Uα (λ) which implies (a, b) ∈ Uα (μ) and (a, b) ∈ Uα (λ) implies μ(a − b) ≥ α and λ(a − b) ≥ α implies μ(a − b) ∩ λ(a − b) ≥ α
implies (μ ∩ λ)(a − b) ≥ α which implies that (a − b) ∈ Uα (μ ∩ λ).Hence Uα (μ ∩ λ) = Uα (μ) ∩ Uα (λ)
(4) Since μ ⊆ (μ ∪ λ) and λ ⊆ (μ ∪ λ) by (1) we have Uα (μ) ⊆ Uα (μ ∪ λ) and Uα (λ) ⊆ Uα (μ ∪ λ) implies that Uα (μ) ∪ Uα (λ) ⊆ Uα (μ ∪ λ). Hence, the proof.

Proposition 3.3. Let ρ be a full congruence relation on R. If μ is a fuzzy ideal of R, then
(1) $\rho^{-}(\mu)$ is a fuzzy ideal of R.
(2) $\rho_{-}(\mu)$ is a fuzzy ideal of R.
The proof follows from Theorem 3.3, 3.4 of [16].

Corollary 3.4. If μ is fuzzy ideal of R, then ($\rho_{-}(\mu)$, $\rho^{-}(\mu)$) is a rough fuzzy ideal of R.

Proposition 3.5. Let ρ be a complete congruence relation on a nearring R. If μ is a fuzzy ideal of R, then it is a upper rough fuzzy ideal of R.

Proof. Since ρ is a complete congruence relation on R, $[a]_\rho[b]_\rho = [ab]_\rho$, forall a, b ∈ R. Let μ be a fuzzy ideal of a near ring R and x, y ∈ R. Then, we have

(1) $$\rho^{*}\mu(x-y) = \bigvee_{z\in[x-y]_\rho} \mu(z) = \bigvee_{z\in[x]_\rho-[y]_\rho} \mu(z) = \bigvee_{a-b\in[x]_\rho-[y]_\rho} \mu(a-b)$$

$$\geq \bigvee_{a\in[x]_\rho, b\in[y]_\rho} [\mu(a)\wedge\mu(b)],$$ since μ is a fuzzy ideal of R.

$$= [\bigvee_{a\in[x]_\rho} \mu(a)] \wedge [\bigvee_{b\in[y]_\rho} \mu(b)] = \rho^{*}\mu(x) \wedge \rho^{*}\mu(y)$$

(2) $$\rho^{*}\mu(y+x-y) = \bigvee_{z\in[y+x-y]_\rho} \mu(z) = \bigvee_{z\in[y+x]_\rho-[y]_\rho} \mu(z) = \bigvee_{a-d\in[y+x]_\rho-[y]_\rho} \mu(a-d)$$

$$= \bigvee_{d\in[y+x]_\rho, c\in[y+x], d\in[y]_\rho} \mu(d+c-d) = \bigvee_{d+c\in[y+x]_\rho, d\in[y]_\rho} \mu(d+c-d)$$

$= \bigvee_{d\in[y]_\rho, c\in[x]_\rho} \mu(c)$, since μ is a fuzzy ideal of R.

$= \bigvee_{c\in[x]_\rho} \mu(c) = \rho^*\mu(x)$

(3) $\rho^*\mu(xy) = \bigvee_{z\in[xy]_\rho} \mu(z) = \bigvee_{ab\in[x]_\rho[y]_\rho} \mu(ab) = \bigvee_{a\in[x]_\rho, b\in[y]_\rho} \mu(ab)$

$\geq \bigvee_{b\in[y]_\rho} \mu(b)$, since μ is a fuzzy ideal of R.

$= \rho^*\mu(y)$

(4) $\rho^*\mu((x+z)y - xy) = \bigvee_{z\in[(x+z)y-xy]_\rho} \mu(z) = \bigvee_{ab-c\in[(x+z)y-xy]_\rho} \mu(ab-c) = \bigvee_{ab\in[(x+z)y]_\rho, c\in[xy]_\rho} \mu(ab-c)$

$= \bigvee_{a\in[x+z]_\rho, b\in[y]_\rho, c\in[xy]_\rho} \mu(ab-c) = \bigvee_{d\in[x]_\rho, e\in[z]_\rho, b\in[y]_\rho} \mu((d+e)b-db)$

$\geq \bigvee_{e\in[z]_\rho} \mu(e)$, since μ is a fuzzy ideal of R.

$= \rho^*\mu(z)$

This shows that μ is an upper rough fuzzy ideal of R.

Proposition 3.6. Let ρ be a complete congruence relation on a nearing R.
If μ is a fuzzy ideal of R, then it is a lower rough fuzzy ideal of R.

Proof. Since ρ is a complete congruence relation on R, [a]ρ[b]ρ = [ab]ρ, for all a,b ∈ R.Let μ be a fuzzy ideal of a near ring R and x, y ∈ R. Then, we have

(1) $\rho_*\mu(x-y) = \bigwedge_{z\in[x-y]_\rho} \mu(z) = \bigwedge_{z\in[x]_\rho-[y]_\rho} \mu(z) = \bigwedge_{a-b\in[x]_\rho-[y]_\rho} \mu(a-b)$

$\geq \bigwedge_{a\in[x]_\rho, b\in[y]_\rho} [\mu(a)\wedge\mu(b)]$, since μ is a fuzzy ideal of R.

$= [\bigwedge_{a\in[x]_\rho} \mu(a)] \wedge [\bigwedge_{b\in[y]_\rho} \mu(b)] = \rho_*\mu(x) \wedge \rho_*\mu(y)$

(2) $\rho_*\mu(y+x-y) = \bigwedge_{z\in[y+x-y]_\rho} \mu(z) = \bigwedge_{z\in[y+x]_\rho-[y]_\rho} \mu(z) = \bigwedge_{a-d\in[y+x]_\rho-[y]_\rho} \mu(a-d)$

$= \bigwedge_{d\in[y+x]_\rho, c\in[y+x], d\in[y]_\rho} \mu(d+c-d) = \bigwedge_{d+c\in[y+x]_\rho, d\in[y]_\rho} \mu(d+c-d)$

$= \bigwedge_{d\in[y]_\rho, c\in[x]_\rho} \mu(c)$, since μ is a fuzzy ideal of R.

$= \bigwedge_{c\in[x]_\rho} \mu(c) = \rho_*\mu(x)$

(3) $\rho_*\mu(xy) = \bigwedge_{z\in[xy]_\rho} \mu(z) = \bigwedge_{ab\in[x]_\rho[y]_\rho} \mu(ab) = \bigwedge_{a\in[x]_\rho, b\in[y]_\rho} \mu(ab)$

$\geq \bigwedge_{b\in[y]_\rho} \mu(b)$, since μ is a fuzzy ideal of R.

$= \rho_*\mu(y)$

(4) $\rho_*\mu((x+z)y - xy) = \bigwedge_{z\in[(x+z)y-xy]_\rho} \mu(z) = \bigwedge_{ab-c\in[(x+z)y-xy]_\rho} \mu(ab-c) = \bigwedge_{ab\in[(x+z)y]_\rho, c\in[xy]_\rho} \mu(ab-c)$

$= \bigwedge_{a\in[x+z]_\rho, b\in[y]_\rho, c\in[xy]_\rho} \mu(ab-c) = \bigwedge_{d\in[x]_\rho, e\in[z]_\rho, b\in[y]_\rho} \mu((d+e)b-db)$

$\geq \bigwedge_{e\in[z]_\rho} \mu(e)$, since μ is a fuzzy ideal of R.

$= \rho_*\mu(z)$

This shows that μ is an lower rough fuzzy ideal of R.

Proposition 3.7. Let μ be a fuzzy ideal of an abelian nearring R and α ∈ [0, 1]. If A is an ideal of an abelian nearring R, then A is an upper rough ideal of an abelian nearring R.

Proof. Let a,b ∈ $\overline{U}$ Then, [a](μ,α) ∩ A ≠ φ and [b](μ,α)∩ A ≠ φ. So,there exists elements x and y in R such that

x ∈ [a](μ,α) ∩ A and y ∈ [b](μ,α) ∩ A and x, y ∈ A.

Since U (μ, α) is a congruence relation on R, it follows that

x ± y ∈ [a](μ,α) ± [b](μ,α) = [a ± b](μ,α)

Since A is an ideal of an abelian nearring R, A is a subgroup of R. So, x+y ∈ A. Thus,

x + y ∈ [a + b](μ,α) ∩ A,

which implies that x+ y ∈ $\overline{U}$ (μ, α).

Let a be any element of $\overline{U}$ (μ, α). Then, we have [a](μ,α) ∩ A ≠ φ

Hence, there exists an element x such that x ∈ [a](μ,α) ∩ A.

Thus, x ∈ [a](μ,α) and x ∈ A. Since A is a subgroup, it follows that −x ∈ A. Since U (μ, α) is a congruence relation, it follows that (x, a) ∈ U (μ, α). This implies that

(−a, −x) = (−a + x − x, −a + a − x) ∈ U (μ, α).

Thus, −x ∈ [−a](μ,α). Therefore, −x ∈ [−a](μ,α)∩A . Also, −a ∈ $\overline{U}$ (μ, α). Hence, $\overline{U}$ (μ, α) is a subgroup of R. Assume that A is an ideal of an abelian nearring R. Then, A is a normal subgroup of R. Let a and x be any elements of $\overline{U}$ (μ, α) and R. Then, [a](μ,α) ∩ A ≠ φ and [x](μ,α) ∩ A ≠ φ.

Then, there exists y ∈ [a](μ,α) ∩ A for some y ∈ R. Thus, we have y ∈ [a](μ,α) and y ∈ A.

Since A is normal, it follows that (x + y − x) ∈ x + A − x ⊆ A. Since U (μ, α)
is a congruence relation on R, (y, a) ∈ U (μ, α) implies (x + y − x, x + a − x) ∈ U (μ, α).

Thus, we obtain (x + y − x) ∈ [x + a − x](μ,α). Therefore, we have x + y − x ∈ [x + a − x](μ,α) ∩ A and x + y − x ∈ $\overline{U}$ (μ, α).

Thus, $\overline{U}$ (μ, α) is a normal subgroup of R. Let r ∈ R and a ∈ $\overline{U}$. Then, [a](μ,α) ∩ A ≠ φ. Thus x ∈ [a](μ,α) and x ∈ A.

Since A is an ideal, it follows that rx ∈ RA ⊆ A. So, (rx, ra) ∈ U (μ, α). This implies that rx ∈ [ra] ∩ A. Therefore, rx ∈ $\overline{U}$. This completes the proof.

Proposition 3.8. Let μ be an fuzzy ideal of an abelian nearring R and α ∈ [0, 1]. If A is a fuzzy ideal of R, then $\underline{U}$ is an ideal of R.

Proof. Let a,b ∈ $\underline{U}$. Then, [a](μ,α) ⊆ A and [b](μ,α) ⊆ A.

Since U (μ, α) is a congruence relation, [a + b](μ,α) = [a](μ,α) +[b](μ,α) ⊆ A + A ⊆ A
implies a + b ∈ $\underline{U}$ (A). Let a be an element of $\underline{U}$. Then, we get [a](μ,α) ⊆ A.

Let x be any element of [−a](μ,α). Then, (x, −a) ∈ U (μ, α), and so (−x, a) ∈ U (μ, α).

Then, we obtain −x ∈ [a](μ,α) ⊆ A.

Since A is an ideal of R, it is a subgroup of R. Thus, x ∈ A and so [−a](μ,α) ⊆ A.

Hence, −a ∈ $\underline{U}$ .Let a and x be any element of $\underline{U}$. Then, [a](μ,α) ⊆ A.

Let z be any element of [x + a − x](μ,α). Then, (z, (x + a − x)) ∈ U (μ, α).

Since U (μ, α) is a congruence relation on R,

(−x + z + x, a) ∈ U (μ, α) and so − x + z + x ∈ [a](μ,α) ⊆ A.

Thus −x + z + x = b for some b ∈ A.

Since A is normal, z = x+b −x ∈ x+A−x ⊆ A and so we have, [a+b −x](μ,α) ⊆ A.

Therefore x +b − x ∈ $\underline{U}$.

Which means that $\underline{U}$ is normal. If r ∈ R and a ∈ $\underline{U}$, then [a](μ,α) ⊆ A

If z = rx is any element of [ra](μ,α), then

(z, ra) ∈ U (μ, α) implies (rx, ra) ∈ U (μ, α),

which implies that rx ∈ [ra]μ,α ⊆ A.

That is [ra](μ,α)⊆B implies ra ∈ $\underline{U}$

Corollary 3.9. If I is a fuzzy left ideal of an abelian nearring R, then ($\underline{U}$, $\overline{U}$) is a rough left ideal of R.

Lemma 3.10. Let μ be a fuzzy ideal of an abelian nearring R and α ∈ [0, 1]. If $\underline{U}$ (μ, α) is a non-empty set, then [0](μ,α) ⊆ A.

Proposition 3.11. Let μ be a fuzzy ideal of an abelian nearring R and α ∈ [0, 1]. Let A be an ideal of R.
If $\underline{U}$ (μ, α) is a non-empty set, then it is equal to A.

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