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Stability and Consistency Analysis for Alternating Direction Explicit Scheme for the Two-Dimensional Sine-Gordon Equation

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ABSTRACT

In this paper, we consider the stability of the *An Alternating Direction Explicit Scheme* for solving two-dimensional *Sine-Gordon Equation* .The derivation of the *An Alternating Direction implicit Scheme* schemes was presented. The stability and consistency of the scheme are described . The scheme is found to be unconditionally stable and convergent.

1.0 Introduction

Apart from very special cases, partial differential equations (PDEs) can only be solved numerically; the construction of their numerical solutions is a fundamental task in science and engineering. Among three classical numerical methods that are widely used for numerical solving of PDEs the finite difference method is the oldest one and is based upon the application of a local Taylor expansion to approximate the differential equations by difference ones defined on the chosen computational grid. The difference equations that approximate differential equations in the system of PDEs form its finite difference approximation (FDA) which together with discrete approximation of initial or/and boundary conditions is called finite-difference scheme (FDS).

1.1 The model equation

A Josephson junction model consists of two superconducting layers separated by isolated barriers. The (2+1) dimensional sine-Gordon equation that governs current flow through Josephson junction is given by;

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}-\frac{1}{\bar{C}^2}\frac{\partial^2 u}{\partial t^2}-\frac{\beta}{\bar{C}^2}\frac{\partial u}{\partial t}=\frac{1}{\lambda_J^2}\sin u \qquad (1)$$

The function $u = u(x,y)$ represents the Josephson current flow density at position (x,y) and at time t along Josephson junction. The model in Equation (1) has various applications in physics; electronics etc. With the following initial conditions

1.2 Literature review

Tinega and Oduor [23] solved the (1+1) Sine-Gordon equation that governs the vibrations of the rigid pendula attached to a stretched wire using Finite Difference Method where Forward Difference and Crank Nicholson numerical schemes were developed. Stability of these schemes were analysed using matrix method..Nyachwaya et al [63] solved third order seepage parabolic partial differential equation (which models the fluid flows) and analysed stability of the schemes developed by two types of finite differences methods, which are Alternating Direction Explicit (ADE) method and Alternating Direction Implicit (ADI) method subject to some boundary and initial conditions. The numerical results for the two methods were compared. They derived the finite differential form of ADE and ADI methods for the given model and then presented an algorithm for each method. They studied the numerical stability of both methods by matrix Method. It was observed that both schemes are conditionally stable.

1.2.1 Consistency of An Alternating Direction Explicit Scheme

To solve the equation, spatial and temporal domains are discretized by the grids of points and partial derivatives occurring in the equation are replaced by approximations based on the Taylor series expansions of the function near the point or points of interest [9,14,17]. Since convergence is difficult to prove directly, we use an equivalent result known as the Lax Equivalence Theorem which stated that, for a given properly posed linear consistent finite difference approximation to Partial differential

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equation (PDE), stability is necessary and sufficient for convergence [22]. Thus, showing the consistency and stability of the finite difference scheme is sufficient for convergence. Doyo and Gofe [24] considered the convergence rates and stability of the Forward Time, Centered Space (FTCS) and Backward Time Centered Space (BTCS) schemes for solving one-dimensional, time-dependent diffusion equation with Neumann boundary condition. The derivation of the schemes and development of a computer program to implement them were presented. The consistency and the stability of the schemes were described and used numerical problems to determine convergence rates of the schemes. It was found that both methods are first order accurate in the spatial dimension. We use Gerschgorin's Theorem to determine the stability of the methods [17], and show that An Alternating Direction Explicit Scheme is stable if the modulus of the Eigenvalues of the Amplification Matrix should be less than or equal to one. The method is unconditionally stable. Since finite difference discretization converges at the rate of the Truncation Error (TE) (determined by the order of the spatial and temporal discretization) if the exact solution is smooth enough, we expand the exact solution at the mesh points of the scheme with a Taylor series and insert the Taylor expansions in the scheme to calculate the TE (difference between the resulting equation and the original PDE) and determine its order in the approximation used. Then, we see that as the discrete step sizes approach to zero, their TE also approaches to zero which indicates that the difference approximations are consistent. For the remainder of this paper, we give the details of the numerical algorithms to solve application problems involving diffusion equation. In section 2, a difference schemes for one-dimensional, time-dependent diffusion equation is derived. In section 3, convergence of the schemes is described. Finally, numerical problems are given to verify the validity of the theoretical results.

On the basis of the literature review, it appears that no work was reported on analysing the stability of (2+1) model Sine-Gordon equation with first time derivative that governs the current flow density through the Josephson junction using Finite Difference Method for An Alternating Direction implicit Scheme. The objective of this paper is to analyse the stability of An Alternating Direction implicit Scheme for the two-dimensional Sine-Gordon equation that describe the Josephson current density through the long Josephson junction subject to some prescribed boundary conditions. The rest of the paper is organized as follows. Section II is Method of solution of (2+1) SGE. After a brief discussion of the numerical Methods; Section III describes the numerical schemes employed. Section IV addresses numerical results and discussion while the last Section V is about conclusion and recommendation.

2.0 Numerical Schemes

The numerical methods can be categorized as Finite Difference, Finite element, Finite volume and Boundary element. The method of Finite Difference is one of the most valuable methods of approximating numerical solutions of PDEs. In this study, Finite Differences Method is used to solve a (2+1) dimensional Sine-Gordon Equation (1) with first time derivative. Before numerical computations are made, there is an important property of finite difference equations that must be considered is Stability of the scheme developed. The difference between a partial differential equation and the equivalent finite difference expression is referred to as truncation error. A numerical process is said to be stable if it limits amplification of all components of the initial conditions. The use of finite difference techniques for the solution of partial differential equation is a three step process .These steps are;

i) The partial differential equations are approximated by a set of linear equations relating to the values of the functions at each mesh point.

ii) The set of the algebraic equations, generated for equation must be solved and

iii) An iteration procedure has to be developed which takes into account the non-linear character of the equation. In our endeavor to solve the (2+1) dimensional sine-Gordon equation, the stability of the scheme developed for the equation is analyzed.

2.1 Alternating Direction Explicit Scheme (ADES)

In this scheme we advance the solution of the (2+1) sine-Gordon partial differential Equation (1) from n^{th} plane to $(n+1)^{th}$ plane by replacing u_{xx} by implicit finite difference approximation at the $(n+1)^{th}$ plane. Similarly, u_{tt} , u_{yy} and u_t are replaced by an explicit central finite difference approximation at the n^{th} plane as in equations . With these approximations substituted into Equation (1), the following scheme is obtained. The Alternating Direction Explicit (ADE) method for generating numerical solutions to the hyperbolic partial differential equations is stable for some time because it is an explicit method; it holds a speed advantage over implicit methods for computations over a single time level [43], the explicit methods in which the solution at the new time step is formed by a combination of the previous time step solutions. In Equation (1) , u_{xx} , u_{yy} and u_{tt} are replaced by the central difference scheme for the derivative with respect to x, y and t respectively. For the derivative with respect to t i.e u_t , it is replaced by the forward scheme and the nonlinear term Sinu is replaced with a central finite approximations (Refer [3,20,21]). With these approximations substituted into Equation (1), the following scheme is obtained

$$\frac{U_{i+1,j}^{n+1}-2U_{i,j}^{n+1}+U_{i-1,j}^{n+1}}{(\Delta x)^2}+\frac{U_{i,j+1}^{n}-2U_{i,j}^{n}+U_{i,j-1}^{n}}{(\Delta y)^2}-\left(\frac{U_{i,j}^{n+1}-2U_{i,j}^{n}+U_{i,j}^{n-1}}{\bar{c}^2(\Delta t)^2}\right)-\frac{\beta}{\bar{c}^2}\left(\frac{U_{i,j}^{n+1}-U_{i,j}^{n}}{(\Delta t)}\right)=\left[\frac{U_{i+1,j+1}^{n}-U_{i+1,j-1}^{n}+U_{i-1,j-1}^{n}-U_{i-1,j+1}^{n}}{4(\Delta x\Delta y)\lambda_J^2}\right] \quad (2)$$

We let $\Delta x=\Delta y=\Delta t=s$, $\bar{c}^2=1$ and multiplying Equation (2) by $4\bar{c}^2(\Delta x)^2$, we obtain the scheme;

$$4U_{i+1,j}^{n+1}-(-12-4s)U_{i,j}^{n+1}-4U_{i-1,j}^{n+1}=-4sU_{i,j}^{n}-4U_{i,j+1}^{n}-4U_{i,j-1}^{n}+U_{i+1,j-1}^{n}-U_{i+1,j-1}^{n}-U_{i-1,j+1}^{n}+4U_{i,j}^{n-1}+U_{i-1,j-1}^{n} \quad (3)$$

2.2 Stability Analysis of Alternating Direction Explicit Scheme (ADES)

We use also the matrix method to analyze stability of the scheme (3). This is done by expanding the scheme in equation (3) by taking $i=1,2,3,...............(N-2),(N-1)$.We get the system of linear algebraic equations as

$$\left.\begin{array}{l} 4U_{2,j}^{n+1}-(-12-4s)U_{1,j}^{n+1}-4U_{0,j}^{n+1}=-4sU_{1,j}^{n}-4U_{1,j+1}^{n}-4U_{1,j-1}^{n}+U_{2,j-1}^{n}-U_{2,j-1}^{n}-U_{0,j+1}^{n}+4U_{1,j}^{n-1}+U_{0,j-1}^{n} \\ 4U_{3,j}^{n+1}-(-12-4s)U_{2,j}^{n+1}-4U_{1,j}^{n+1}=-4sU_{2,j}^{n}-4U_{2,j+1}^{n}-4U_{2,j-1}^{n}+U_{3,j-1}^{n}-U_{3,j-1}^{n}-U_{1,j+1}^{n}+4U_{2,j}^{n-1}+U_{1,j-1}^{n} \\ : \\ : \\ 4U_{N-1,j}^{n+1}-(-12-4s)U_{N-2,j}^{n+1}-4U_{N-2,j}^{n+1}=-4sU_{N-2,j}^{n}-4U_{N-2,j+1}^{n}-4U_{N-2,j-1}^{n}+U_{N-1,j-1}^{n}-U_{N-1,j-1}^{n} \\ -U_{N-3,j+1}^{n}+4U_{N-1,j}^{n-1}+U_{N-3,j-1}^{n} \\ 4U_{N,j}^{n+1}-(-12-4s)U_{N-1,j}^{n+1}-4U_{N-1,j}^{n+1}=-4sU_{N-1,j}^{n}-4U_{N-1,j+1}^{n}-4U_{N-1,j-1}^{n}+U_{N,j-1}^{n}-U_{N,j-1}^{n} \\ -U_{N-2,j+1}^{n}+4U_{N,j}^{n-1}+U_{N-2,j-1}^{n} \end{array}\right\} \quad (4)$$

Writing the system of algebraic Equations (4) in matrix-vector form;

$$\begin{bmatrix} (-12-4s) & -4 & ... & 0 & 0 \\ -4 & (-12-4s) & -4 & 0 & 0 \\ \vdots & -4 & \ddots & -4 & \vdots \\ 0 & 0 & -4 & (-12-4s) & -4 \\ 0 & 0 & \cdots & -4 & (-12-4s) \end{bmatrix}\begin{bmatrix} U_{1,j+1}^{n+1} \\ U_{2,j}^{n+1} \\ U_{2,j}^{n+1} \\ \vdots \\ U_{N-2,j}^{n+1} \\ U_{N-1,j}^{n+1} \end{bmatrix}=\begin{bmatrix} -4s & -4 & ... & 0 & 0 \\ -4 & -4s & -4 & 0 & 0 \\ \vdots & -4 & \ddots & -4 & \vdots \\ 0 & 0 & -4 & -4s & -4 \\ 0 & 0 & \cdots & -4 & -4s \end{bmatrix}\begin{bmatrix} U_{1,j+1}^{n} \\ U_{2,j+1}^{n} \\ \vdots \\ U_{N-2,j+1}^{n+1} \\ U_{N-1,j+1}^{n+1} \end{bmatrix}+\begin{bmatrix} -4sU_{1,j}^{n} & -4U_{1,j-1}^{n} & -U_{2,j-1}^{n} & +U_{0,j-1}^{n} & +4U_{1,j}^{n-1} \\ -4sU_{2,j}^{n} & -4U_{2,j-1}^{n} & -U_{2,j-1}^{n} & +U_{0,j-1}^{n} & +4U_{2,j}^{n-1} \\ : & : & : & : & : \\ -4sU_{N-2,j}^{n} & -4U_{N-2,j-1}^{n} & -U_{N-1,j-1}^{n} & +U_{N-3,j-1}^{n} & +4U_{N-2,j}^{n-1} \\ -4sU_{N-1,j}^{n} & -4U_{N-1,j-1}^{n} & -U_{N,j-1}^{n} & +U_{N-2,j-1}^{n} & +4U_{N-1,j}^{n-1} \end{bmatrix} \quad (5)$$

The system in Equation (5) can be written as

$$\left((-12-4s)I_{N-1}-4A_{N-1}\right)U_{N-1,j}^{n+1}=\left[-4I_{N-1}+A_{N-1}\right]U_{N-1,j}^{n}+\vec{f} \quad (6)$$

where I_{N-1} and are A_{N-1} identity and square matrices respectively of order $(N-1)\times(N-1)$, $\vec{f}$ is a constant vector. The matrices and a vector are given as;

$$B_{N-1}=\begin{bmatrix} 0 & 1 & ... & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ \vdots & 1 & \ddots & 1 & \vdots \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}, \quad I_{N-1}=\begin{bmatrix} 1 & 0 & ... & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ \vdots & 0 & \ddots & 0 & \vdots \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & \cdots & 0 & 1 \end{bmatrix} \text{and} \quad \vec{f}=\begin{bmatrix} -4sU_{1,j}^{n} & -4U_{1,j-1}^{n} & -U_{2,j-1}^{n} & +U_{0,j-1}^{n} & +4U_{1,j}^{n-1} \\ -4sU_{2,j}^{n} & -4U_{2,j-1}^{n} & -U_{2,j-1}^{n} & +U_{0,j-1}^{n} & +4U_{2,j}^{n-1} \\ : & : & : & : & : \\ -4sU_{N-2,j}^{n} & -4U_{N-2,j-1}^{n} & -U_{N-1,j-1}^{n} & +U_{N-3,j-1}^{n} & +4U_{N-2,j}^{n-1} \\ -4sU_{N-1,j}^{n} & -4U_{N-1,j-1}^{n} & -U_{N,j-1}^{n} & +U_{N-2,j-1}^{n} & +4U_{N-1,j}^{n-1} \end{bmatrix}$$

Equation (11) can again be written as

$$U_{N-1,j}^{n+1}=\left[-4I_{N-1}+A_{N-1}\right]\left((-12-4s)I_{N-1}-4A_{N-1}\right)^{-1}U_{N-1,j+1}^{n}+\left((-12-4s)I_{N-1}-4A_{N-1}\right)^{-1}\vec{f} \quad (7)$$

Therefore, Equation (7) is compactly written as

$$U_{N-1,j}^{n+1}=EU_{N-1,j+1}^{n}+\vec{g} \quad (8)$$

Where $E=\left[-4I_{N-1}+A_{N-1}\right]\left((-12-4s)I_{N-1}-4A_{N-1}\right)^{-1}$ and $\vec{g}=\left((-12-4s)I_{N-1}-4A_{N-1}\right)^{-1}\vec{f}$.

E is the amplification matrix . According to Wen-Chyuan [30], the Eigenvalue of a tridiagonal (N-1) by (N-1) matrix

$$M=\begin{bmatrix} b & c & .. & 0 \\ a & b & \ddots & : \\ : & \ddots & \ddots & c \\ 0 & .. & a & b \end{bmatrix} \quad (9)$$

Is given by

$$\lambda_M=b+2\sqrt{ac}\cos\left(\frac{m\pi}{N+1}\right) \quad (10)$$

Using the formula in (10), the eigenvalues of A_{N-1} and I_{N-1} matrix are

Eigenvalue o Eigenvalue of

$$A_{N-1}=2\cos\left(\frac{m\pi}{N+1}\right)=2-4\sin^2\left(\frac{m\pi}{2(N+1)}\right)$$

Eigenvalue of A_{N-1} is 1

Hence Eigenvalue of the amplification matrix E is given as

$$|\lambda_E|=\left|\frac{4I+-4A_{N-1}}{\left((-12-4s)I_{N-1}-4A_{N-1}\right)}\right|=\left|\frac{4+2-4\sin^2\left(\frac{m\pi}{2(N+1)}\right)}{-12-4s-8+16\sin^2\left(\frac{m\pi}{2(N+1)}\right)}\right|$$

$$\left|\lambda_E\right| = \left|\frac{6-4\sin^2\left(\frac{m\pi}{2(N+1)}\right)}{-20-4s+16\sin^2\left(\frac{m\pi}{2(N+1)}\right)}\right| \tag{11}$$

For a tridiagonal matrix, the modulus of the eigenvalue of the amplification matrix E should be less than or equal to unity

$$\left|\lambda_E\right| = \left|\frac{6-4\sin^2\left(\frac{m\pi}{2(N+1)}\right)}{-20-4s+16\sin^2\left(\frac{m\pi}{2(N+1)}\right)}\right| \le 1 \tag{12}$$

a) For $\sin^2\left(\frac{m\pi}{2(N+1)}\right)=0$, Equation (12) becomes;

$$\left|\lambda_E\right| = \left|\frac{6}{-20-4s}\right| \le 1 \tag{13}$$

b) For $\sin^2\left(\frac{m\pi}{2(N+1)}\right)=1$, Equation (12) becomes;

$$\left|\lambda_E\right| = \left|\frac{6-4}{-20-4s+16}\right| \le 1 \tag{14}$$

$$\left|\lambda_E\right| = \left|\frac{1}{-1-s}\right| \le 1$$

Both expressions in equations (13) and (14) are unconditionally stable

3.0 Consistency Analysis of Alternating Direction Explicit Scheme (ADES)

Consistency requires that the original equation can be recovered from the algebraic equations. Obviously this is a minimum requirement for any discretization. In the following we will illustrate how this can be done in terms of a Taylor's series expansion for the alternating direction explicit scheme (3).

If we re-arrange equation (8), we get

$$4\left(U^n_{i+1,j}+U^n_{i-1,j}\right)+(-4-4s)U^{n+1}_{i,j}-(-8-4s)4U^n_{i,j}+4(U^n_{i,j-1}+U^n_{i,j+1})+(U^n_{i+1,j-1}+U^n_{i-1,j+1})+(U^n_{i-1,j+1}-U^n_{i-1,j-1})-4U^{n-1}_{i,j}=0 \tag{20}$$

If we let $\Delta x=\Delta y=\Delta z=\Delta t=s=h$, and expand every term of the equation (20) using Taylor's series expansion

$$4U^n_{i,j+1}=4U^n_{i,j}+4hu_y+\frac{4h^2}{2!}u_{yy}+\frac{4h^3}{3!}u_{yyy}+\frac{4h^4}{4!}u_{yyyy}+... \tag{21}$$

$$4U^n_{i,j-1}=4U^n_{i,j}-4hu_y+\frac{4h^2}{2!}u_{yy}-\frac{4h^3}{3!}u_{yyy}+\frac{4h^4}{4!}u_{yyyy}+... \tag{22}$$

$$-4U^{n-1}_{i,j}=-4U^n_{i,j}+4hu_t-\frac{4h^2}{2!}u_{tt}+\frac{4h^3}{3!}u_{ttt}-\frac{4h^4}{4!}+... \tag{23}$$

$$(-4-4s)U^{n+1}_{i,j}=(-4-4s)[U^n_{i,j}+hu_t+\frac{h^2}{2!}u_{tt}+\frac{h^3}{3!}u_{ttt}+\frac{h^4}{4!}u_{tttt}]+... \tag{24}$$

$$-U^n_{i+1,j+1}=-U^n_{i,j}-(hu_x+hu_y)-\frac{1}{2!}(h^2u_{xx}+2h^2u_{xy}+h^2u_{yy})-\frac{1}{3!}(h^3u_{xxx}+3h^3u_{xxy}+3h^3u_{xyy}+h^3u_{yyy}) \tag{25}$$

$$U^n_{i+1,j-1}=U^n_{i,j}+(-hu_x+hu_y)+\frac{1}{2!}(h^2u_{xx}-2h^2u_{xy}+h^2u_{yy})\frac{1}{3!}(-h^3u_{xxx}+3h^3u_{xxy}-3h^3u_{xyy}+h^3u_{yyy}) \tag{26}$$

$$-U^n_{i-1,j-1}=-U^n_{i,j}+(hu_x+hu_y)=\frac{1}{2!}(-h^2u_{xx}-2h^2u_{xy}-h^2u_{yy})+\frac{1}{3!}(-h^3u_{xxx}-3h^3u_{xxy}-3h^3u_{xyy}-h^3u_{yyy}) \tag{27}$$

$$U^n_{i-1,j+1}=U^n_{i,j}+(-hu_x+hu_y)+\frac{1}{2!}(h^2u_{xx}-2h^2u_{xy}+h^2u_{yy})+\frac{1}{3!}(-h^3u_{xxx}+3h^3u_{xxy}-3h^3u_{xyy}+h^3u_{yyy}) \tag{28}$$

$$(-8-4s)U^n_{i,j}=-8U^n_{i,j}-4h^2U^n_{i,j} \tag{29}$$

substituting (21), (22), (23), (24), (25), (26), (27), (28), (29) into (20) we get

$$4h^2u_{xx}+4h^2u_{yy}-4h^2u_t-h^2u_{tt}+\frac{h^2}{3}u_{xxxx}-2h^5u_{xx}-4h^3u_y+h^2u_{yy}+...=0 \tag{30}$$

Dividing (30) throughout by $4h^2$, we obtain

$$u_{xx} + u_{yy} - u_{tt} - u_t - u_{xy} + \frac{h^2}{12} u_{xxx} + \frac{1}{12} u_{yyy} + ... = 0 \quad (31)$$

Where the error is

$$E^n{}_{i,j} = \frac{h^2}{12} u_{xxx} + \frac{h^2}{12} u_{yyy} + ... \quad (32)$$

It is noted that the first five terms in equation (31) are for the recovered PDE that is (two- dimension Sine Gordon equation) and the other terms is the truncation error, since the beam equation has been recovered from the algebraic equation of the alternating direction explicit scheme developed, we therefore conclude that the scheme is consistent with the SGE.

4. Discussion

The equation (18) and (19) satisfies the stability conditions. The condition on the right is always satisfied as the left inequality requires. All the eigenvalues in equations (18) and (19) are bounded by 1 since the denominator larger than the numerator. Thus the ADES scheme is unconditionally stable. The Equation (31) satisfies the consistency conditions. The sine-Gordon partial differential equation (1) has been recovered from the Alternating Direction implicit Scheme (2).

5.0 Conclusion

It can be concluded that the stability of the *An Alternating Direction implicit Scheme* developed for the two-dimensional Sine-Gordon equation that govern Josephson junction current flowing through the long Josephson junctions is uncondionally stable. It is also noted that the scheme is consistent.

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