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Non-wandering Sets in Topological Dynamical Systems

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ABSTRACT

In this research we use the logistic function to demonstrate the existence of a stationary point as a type of a non-wandering set in topological dynamical systems or dynamical systems without any loss of generality.

1.0 Introduction

A discrete-time dynamical system *(X,T)* is a continuous map T on a non-empty topological space X [10][8]. This dynamics is obtained by iterating the map T. The discrete logistic function operates within a range by a control parameter. This function changes in state as the parameter is being altered. If we are to take the set of points in the given space *X* and upon operation by iterating with an initial point, it comes close or exactly back to the points in the set where there is a change which is either, stationary (fixed points), cycles (periodic) or chaos. This state (set) is what we termed as the non-wandering set.

2.0 Preliminaries

Definition 2.1: DISCRETE-TIME DYNAMICAL SYSTEM (See [6],[8]): Let $\boldsymbol{X}$ be a non-empty topological space and *T* be a continuous map. A discrete-time dynamical system (X,T) is defined as; T:X→X, where the dynamics are obtained by iterating the map T, hence, a dynamical system (X,T) induces an action on $\boldsymbol{X}$ by $\boldsymbol{n} \to \boldsymbol{T^n}$, where $\boldsymbol{T^n(x)}$= x and $\boldsymbol{T^{n+1}(x) = T(T^n(x))}$ for all $\boldsymbol{x \in X \text{ and } n \in N}$.

Illustration 2.2: (see[1]) Let f be a continuous function on X such that $\boldsymbol{x \in X}$ i.e. f:X→X, $\boldsymbol{x \in X}$, then ; {…,f^(-2) (x),f^(-1) (x),x,f^1 (x),f^2 (x),…} are the orbit sequence of x which are bi infinite sequence and form the discrete-time of the solution [1]. But since we are interested in the set type, we let f be a function from the Z of discrete-time to the state space X, with parameter and initial point x, then the orbit relation is defined as,

$$\boldsymbol{\vartheta f = \cup_{n=1}^{\infty} f^n} \tag{1.0}$$

Hence, the following is the set of the relation but not a sequence; ϑf(x)= ⟦{f⟧^1 (x),f^2 (x),…} made up of the states and the initial point x in time [9].

2.3 Non-wandering Set: It is the set of points in the phase space for which all points beginning from a point of this set come arbitrarily close and arbitrarily often to any point of the set. In [3], the following shows the types and the existence of non-wandering set:

1. fixed points (stationary)
2. periodic solutions (limit cycles)
3. quasi-periodic orbits
4. chaotic orbits

2.4 Logistic Function: The logistic function is a difference equation which is non-linear system. It is a function that transform into different state or phenomenon depending on changes of the parameter α.

Definition 2.5: if $\boldsymbol{X_n}$is a state with discrete-time n, then the function is defined as;

$$\boldsymbol{X_{n+1} = \alpha x_n(1 - x_n)} \tag{1.1}$$

where $\boldsymbol{n \in N^+}$and for $\boldsymbol{X_n \in [0,1]}$and $\boldsymbol{\alpha \in [1,4]}$

2.6 Using the logistic function to illustrate the fixed points and the periodic solutions as the types of the non-wandering set

2.6.1 fixed points (stationary) (see [2], [9])

If a point $\boldsymbol{x \in X}$, its orbit or trajectory is; $\boldsymbol{\vartheta(x) = \cup_{n+1}^{\infty} X^n}$ (1.2)

Hence, a point x is said to be fixed or stationary if $\boldsymbol{\vartheta(x) = x}$ (1.3)

That is $\boldsymbol{f(X_n) = x_n}$ (1.4)

Also for the logistic, let $\boldsymbol{f: X_n \to X_n}$be defined as;

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$f(X_n) = \alpha\, x_n(1 - x_n)$, *where* $n \in N, X \in [0, 1]$ and $\alpha \in [1, 4]$
(1.5)

Theorem 2.7: A non-wandering set is a fixed point (stationary) if a point x in a space Xcomes arbitrarily back to the starting point after iterating it for a number of times. That is if $n \in N$ and $x \in X$ then $\vartheta f^n(x) = \{x\}$**,** is a stationary non-wandering set.

Proof: To show that a stationary (fixed point) is a non-wandering set, we take equations (1.4) and (1.5) and Let $n = 0$.Then equations (1.4) and (1.5) become;

$$f(x_0) = x_0 \qquad (1.6)$$

$$f(x_0) = \alpha\, x_0(1 - x_0) \qquad (1.7)$$

Equating, Eqn (1.6) to Eqn (1.7)

$$\alpha\, x_0(1 - x_0) = x_0$$

Algebraically, $x_0 = 0$ *and* $x_0 = \frac{\alpha-1}{\alpha}$ tends to be the solutions for this logistic function.

Illustration 2.8 : Taking $x_0 = 0$ *and* $x_0 = \frac{\alpha-1}{\alpha}$ to show the existence of the stationary non-wandering set.

At $x_0 = 0$ Trivial

At $x_0 = \frac{\alpha-1}{\alpha}$,

$$\text{Let } \lim_{x_0 \to \frac{\alpha-1}{\alpha}} f(X_0) = \lim_{x_0 \to \frac{\alpha-1}{\alpha}} [\alpha\, x_0(1 - x_0)]$$

$$= \alpha \lim_{x_0 \to \frac{\alpha-1}{\alpha}} [\, x_0(1 - x_0)]$$

$$= \alpha \left[\frac{\alpha-1}{\alpha}\left(1 - \frac{\alpha-1}{\alpha}\right)\right]$$

$$= \alpha \left[\frac{\alpha-1}{\alpha}\left(\frac{\alpha-\alpha+1}{\alpha}\right)\right]$$

$$= \frac{\alpha-1}{\alpha}$$

Clearly, if $x = \{0\} \in X$ and $x = \left\{\frac{\alpha-1}{\alpha}\right\} \in X$, where $X \in [0, 1]$ and $\alpha \in [1, 4]$, a discrete value for α gives a point x in the space X,by iterating comes back to that same x in the space X, hence a non-wandering set.

Example 2.9

Given $f(x_0) = \alpha\, x_0(1 - x_0)$, then at α= [1,4], and $X \ni \left\{x_0 = \frac{\alpha-1}{\alpha}\right\}$. Then a non- wandering set is stationary or fixed point if; $f\left(x_0 = \frac{\alpha-1}{\alpha}\right) = \{x_0 = \frac{\alpha-1}{\alpha}\} \in X$

Solution: Let $\alpha = 2$**,** implies $\left\{x_{0 = \frac{2-1}{2}}\ \frac{1}{2}\right\}, n = N$

Then $f(x_0) = 2\, x_0(1 - x_0)$,implies that at $x_0 = \frac{1}{2}$

$$f\left(\frac{1}{2}\right) = 2\left(\frac{1}{2}\right)\left(1 - \frac{1}{2}\right) = \frac{1}{2} = f\left(f\left(f\left(\frac{1}{2}\right)\right)\right) = \frac{1}{2}$$

$$\therefore f^n\left(\frac{1}{2}\right) = \left\{\frac{1}{2}\right\} \in X$$

Let $\alpha = 3$, implies $\left\{x_{0 = \frac{3-1}{3}}\ \frac{2}{3}\right\}$

Then $f(x_0) = 3\, x_0(1 - x_0)$, implies that at $x_0 = \frac{2}{3}$

$$f\left(\frac{2}{3}\right) = 3\left(\frac{2}{3}\right)\left(1 - \frac{2}{3}\right) = \frac{2}{3} = f\left(f\left(f\left(\frac{2}{3}\right)\right)\right) = \frac{2}{3}$$

$$\therefore f^n\left(\frac{2}{3}\right) = \left\{\frac{2}{3}\right\} \in X$$

Let α=4, implies $\left\{x_0 = \frac{4-1}{4} = \frac{3}{4}\right\}$

Then $f(x_0) = 4\, x_0(1 - x_0)$,implies that at $x_0 = \frac{3}{4}$

$$f\left(\frac{3}{4}\right) = 4\left(\frac{3}{4}\right)\left(1 - \frac{3}{4}\right) = \frac{3}{4} = f\left(f\left(f\left(\frac{3}{4}\right)\right)\right) = \frac{3}{4}$$

$$\therefore f^n\left(\frac{3}{4}\right) = \left\{\frac{3}{4}\right\} \in X$$

Let α=1, implies $\left\{x_0 = \frac{1-1}{1} = 0\right\}$

Then $f(x_0) = x_0(1 - x_0)$, implies that at $x_0 = 0$

$$\therefore f(0) = \{0\} \in X$$

Thus, for discrete α value within [1, 4] all $X \ni \left\{x_0 = \frac{\alpha-1}{\alpha}\right\}$ tends to be a fixed point irrespective of the number of iteration, therefore forming their own constant orbit $\vartheta f^n(x) = \{x\}$ where change in this parameter affect the $X \ni \left\{x_0 = \frac{\alpha-1}{\alpha}\right\}$ and the behavior making it stable or unstable. □

Theorem 2.10: A non-wandering set can be either stable or unstable. Let $X_n \in [0, 1]$ and α be a parameter of the system. Then a change in α of the system can change the stability of a non-wandering set.

Proof: let $\in [\mathbf{0}, \mathbf{1}]$ $\boldsymbol{\alpha} = [\mathbf{1}, \mathbf{4}]$, and $\boldsymbol{X} \ni \left\{\boldsymbol{x_0} = \frac{\alpha-1}{\alpha}\right\}$where α is the parameter and define the logistic, let $\boldsymbol{f}: \boldsymbol{X_n} \rightarrow \boldsymbol{X_n}$ as; $\boldsymbol{f}(\boldsymbol{X_n}) = \boldsymbol{\alpha}\, \boldsymbol{x}(\mathbf{1} - \boldsymbol{x})$.

Then for stable non-wandering set, the fixed point must be stable or attracting that is $||\boldsymbol{f}'(\boldsymbol{x_0})| < \mathbf{1}$ that is absolute derivative of the function is less than one [6], where $\boldsymbol{f}'(\boldsymbol{x_0}) = \boldsymbol{\alpha} - \mathbf{2}\boldsymbol{\alpha}\boldsymbol{x_0}$

$|\boldsymbol{f}'(\boldsymbol{x_0})| = |\boldsymbol{\alpha} - \mathbf{2}\boldsymbol{\alpha}\boldsymbol{x_0}| < \mathbf{1}$,

$-\mathbf{1} < \boldsymbol{\alpha} - \mathbf{2}\boldsymbol{\alpha}\boldsymbol{x_0} < \mathbf{1}$ At $\boldsymbol{x_0} = \frac{\alpha-1}{\alpha}$

$-\mathbf{1} < \boldsymbol{\alpha} - \mathbf{2}\boldsymbol{\alpha}(\frac{\alpha-1}{\alpha}) < \mathbf{1}$

$-\mathbf{1} < \boldsymbol{\alpha} - \mathbf{2}\boldsymbol{\alpha} + \mathbf{2} < \mathbf{1}$

$\therefore \{\boldsymbol{\alpha}: \mathbf{1} < \boldsymbol{\alpha} < \mathbf{3}\}$

Thus α ∈ (1,3) is where the function is asymptotically stable. That is attracting fixed point where there is convergence and stability of the state. Hence the non-wandering set is stable at $\boldsymbol{\alpha} \in (\mathbf{1}, \mathbf{3})$and attracting since it is true for the fixed point/stationary point. □

Also, for unstable non-wandering set, the fixed point/stationary point must be unstable as the control parameter is altered, at a repelling fixed point. And in [6] the way to this is $|\boldsymbol{f}'(\boldsymbol{x_0})| > \mathbf{1} \cdot \boldsymbol{f}'(\boldsymbol{x_0}) = \boldsymbol{\alpha} - \mathbf{2}\boldsymbol{\alpha}\boldsymbol{x_0}$

$|\boldsymbol{f}'(\boldsymbol{x_0})| = |\boldsymbol{\alpha} - \mathbf{2}\boldsymbol{\alpha}\boldsymbol{x_0}| > \mathbf{1}$,

$\boldsymbol{\alpha} - \mathbf{2}\boldsymbol{\alpha}\boldsymbol{x_0} > \mathbf{1}$ or $\boldsymbol{\alpha} - \mathbf{2}\boldsymbol{\alpha}\boldsymbol{x_0} < -\mathbf{1}$ At $\boldsymbol{x_0} = \frac{\alpha-1}{\alpha}$

$\boldsymbol{\alpha} - \mathbf{2}\boldsymbol{\alpha}\left(\frac{\alpha-1}{\alpha}\right) > \mathbf{1}$ or $\boldsymbol{\alpha} - \mathbf{2}\boldsymbol{\alpha}(\frac{\alpha-1}{\alpha}) < -\mathbf{1}$

$\therefore \{\, \boldsymbol{\alpha} < \mathbf{1}\, \}$ or $\{\boldsymbol{\alpha} > \mathbf{3}\}$

Thus, for unstable non-wandering set $\boldsymbol{\alpha} < \mathbf{1}$ or $\boldsymbol{\alpha} > \mathbf{3}$.

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