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On the Effect of Topological Fixed Point Theory

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ABSTRACT

The article presents the concept of fixed point theorem with the Banach contraction principle which is the main source of fixed point theory. We give definitions, theories and examples of Banach contraction principles, complete lattice, homeomorphism between spaces. The metric fixed point theory is given more importance due to its simplicity of application. To ensure existence of the fixed point, continuity property is not a necessary condition, however some order – monotonic conditions must be satisfied by a given mapping.

Introduction

In topology and analysis, the fixed point is of the essence in many applications. The Banach Contraction Principle, is a fixed point theorem in analysis. The concept was introduced by Stephan Banach in 1922 which serves as the main source of metric fixed point theory. The fixed point theory exists on the continuity of the function g. Consider a given interval I= [0, 1] and g weakly increasing, the fixed point continues to exist even if there is discontinuity of g. If M is a set, we define an end of function over M to be a function g: M $\rightarrow$ M, thus g (m) = m, $\forall$ m $\in$ M is a fixed point of g [18]. We can define a fixed point to be both prefixed and post fixed points, if there is a set M which is partially ordered by a relation $\leq$, such that for m $\in$ M, thus g is a prefixed point, if g(m) $\leq$ m and dually m $\leq$ g(m) if g is a post fixed point, see [8], [9], [12], [13], [18], [20], [21], [22].

Definition 1.1 Given a set V. A function g: $V \rightarrow V$ and v $\in$ V is a fixed point of h if and only if g (v) = v

Definition 1.2 Let v^* be a fixed point of g: R $\rightarrow$ R. The point v^* is an attracting fixed point if there exist an open interval w= [0, 1] which contains v^* such that there is u $\in$ w, then g^n (u)$\rightarrow$ v^* as n $\rightarrow\infty$.

Definition 1.3 Let v^* be a fixed point of g: R $\rightarrow$ R. The point v^* is said to be a repelling fixed point if there is an open interval w= [0, 1], such that u $\in$ w and u$\neq$ v^* then g^n (u) $\notin$ w for n an integer.

Definition 1.4 Given that v^* is a fixed point of g, such that g: R $\rightarrow$ R is continuous. Then the point v^* is said to be neutral fixed point, if it is neither attracting nor repelling.

Example.1.5 Consider a map $\boldsymbol{U_{n+1}}=\boldsymbol{u_n^3}$. Let u_0 be an initial condition, then the sequence that will be generated by $\boldsymbol{U_{n+1}}$ is an infinite sequence of values.

Taking u_0= 0.2, with $\boldsymbol{U_{n+1}}= \boldsymbol{u_n^3}$, such that

{0.2}, $\{0.2\}^3$, $\{0.2\}^9$, $\{0.2\}^{27}$,

Given the following initial values u_0 with the map $\boldsymbol{U_{n+1}}=\boldsymbol{u_n^3}$.

(a) Taking $u_0 = 0$,

{{0}, $\{0\}^3$, $\{0\}^9$,…} given u_n = 0 which is fixed

(b)With u_n= 1,

{{1}, $\{1\}^3$,$\{1\}^9$,..} u_n is fixed

(c) With u_0 = 1.1,

{{1.1}, $\{1.1\}^3$, $\{1.1\}^9$,} given $u_n \rightarrow \infty$, unstable

Contraction

We take a look at the **Banach Fixed Point Theorem** which generally proves the uniqueness and existence result, see [3],[4], [5], [7], [11], [14], [16], [17].

Theorem 2.1[4] Let (V, d) be a complete metric space and g: V $\rightarrow$ V be a contraction with Lipschitzian constact c, such that 0 $\leq$ c $\leq$ 1 then for all v_0, v $\in$ V

$$d(g(v_0), g(v)) \leq cd(v_0, v)$$

then the function g has a unique fixed point v = g(v) of V.

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Proof

As in Lipschitz function, contraction guarantees continuity. And to show the existence, let's take $v_0 \in V$ and given that $\{v_p\}$ is a Cauchy sequence, such that v_0, $g(v_0)$, $g^2(v_0)$, …, $g^p(v_0) = v$ forms the sequence, $\forall$ g: V $\rightarrow$ V, for all p $\in \{0,1,2,\ldots\}$ then $v_0 \in V$, we have

$d(g^p(v_0), g^{p+1}(v_0)) \leq cd(g^{p-1}(v_0), g^p(v_0),) \leq \ldots \leq c^p d(v_0, g^p(v_0))$

thus for $q > p$,

$d(g^p(v_0), g^q(v_0)) \leq d(g^p(v_0), g^{p+1}(v_0)) + d(g^{p+1}(v_0), g^{p+2}(v_0)) + \ldots + d(g^{q-1}(v_0), g^q(v_0))$

$= \sum_{k=0}^{q-p-1} d(g^{p+k}(v_0), g^{p+k+1}(v_0))$

$\leq \sum_{k=0}^{q-p-1} c^{p+k} d(v_0, g(v_0))$

$\leq c^p(v_0, g(v_0)) + \ldots + c^{m-1} d(v_0, g(v_0))$

$= c^p \frac{1}{1-c} d(v_0, g(v_0))$

Given the function g, the sequence $\{g^n(v_0)\}$ converges to an element v_0 in V, since the sequence is cauchy and V is complete, there exist the

$\lim_{n\to\infty} g^n(v_0) = v_0$

Consider

$g(v_0) = g(\lim_{n\to\infty} g^n(v_0)) = \lim_{n\to\infty} g(g^n(v_0)) = \lim_{n\to\infty} g^{n+1}(v_0) = v_0$

This shows g is continuous, hence point v_0 is a fixed point of this continuous function g [16][15], [11],[10],

Theorem 2.2:[12, 14,13, 9] Banach Fixed Point Theorem: A contraction v in a complete metric space (V, d) and g: V→ V is said to have exactly one unique fixed point.

Proof: Let $v \in V$ and for all $n \geq 0$ there is $v_n = g(v_{n-1})$. Since the sequence $\{v_n\}_n$ is a Cauchy sequence, g is a contraction which is complete, then there is a limit in g where $\{v_n\}_n$ converges to, such that g(g(v))= g(v) = v, implies g has one fixed point, then v is a proper contraction.

By induction, let's consider v, v_0, as two fixed points of the function g, with $0 < c < 1$.Then we have $d(v, v_0) = d(g(v), g(v_0)) \leq c \bullet d(v, v_0)$, hence this follows that $d(v, v_0) = 0$

thus, $v = v_0$,as it is not possible to have two fixed points. However, this implies g has only one unique fixed point.

Given set V, an end of unction g over V is a function that takes V $\rightarrow$ V. Thus, $v^* \in V$ is a fixed point of the function g, if there exist $g(v^*) = v^*$.

Given a relation ≤ which partially orders a set V, thus, a prefixed point of g is said to be a $\in$ V for $g(a) \leq a$. However, $a \leq g(a)$ is a post fixed point of g $\forall$ a $\in$ V. At this point, we defined the fixed point as both prefixed and post fixed points.

Remarks: The end of unction here is defined as a function whose domain and the range are equivalence.

With the existence of duality of prefixed point, given the properties of the prefixed point implies the post fixed point, see [17]

Preposition 2.3: [18] If g is a monotonic function over a lattice (S, ≤), then, x ≤ y implies g(x) ≤ f(y), for any x and y in L, such that, the fixed point g(x) = x, is both a prefixed g(x) ≤ x and a post-fixed points x ≤ g(x).

Proposition 2.4. [18] Let C be a partially ordered set and g: C→C be monotone, such that g contains a least prefixed point, then such least prefixed point is the least fixed point of g. And if the function contains a greatest post-fixed point, then it contains the greatest fixed point of g.

Proof: Let c be the least pre-fixed point of g. If there exist a unique fixed point, say $c \subseteq d$

Then g(c) ≤ c and g (g(c)) ≤ g(c) for monotonicity of g, thus, g(g(c)) ≤ g(c) ≤ c. Therefore g (c) is a pre-fixed point.

Conversely, given that c is the greatest post-fixed point of g. Since g is monotone, with d ≤ g (d) implies g(d) ≤ g(g(d)), hence g(d) is the post-fixed point. Thus $c \subseteq \mathbf{d}$ is both pre-fixed and post-fixed point of g, if c = d and hence c is a fixed point.

Theorem 2.5: If X is complete lattice, then for any a, b $\in$ X with a ≤ b, the interval $\{x \in X: a \leq x \leq b\}$ is a complete lattice

Proof: Consider a, b $\in$ X, given that a ≤ b and let $R = \{x \in X: a \leq x \leq b\}$. Let any $S \subseteq R$, S contains a least upper bound $s^* \in X$, as X is complete, we show that $s^* \in R$.

Since b is an upper bound of $S \subseteq X$ and there fore $s^* \leq b$. Given w_n, a sequence which is Cauchy in S and as the sequence increases, converges to s^*, thus $a \leq w_n \leq s^*$. And since a is a lower bound for $S \subseteq R$, this $a \leq s^* \leq b$, hence $s^* \in R$. And by this, consider $S \subseteq R$ contains a greatest lower bound and a least upper bound in R, but since X is complete, hence R is a complete lattice.

Remark: Let $S \subseteq R$, if S is an incomplete metric space this will imply R is incomplete.

Example: Given V to be an incomplete metric space, a contraction g from the space into itself has no fixed point.

Proof: Let $g(v) = 0.5 + \frac{v^3}{2}$ be a contraction. Let g : V → V and $v \in V \subseteq R$, if there exist $\{v_n\}$ a Cauchy sequence, given $\lim_{n\to\infty} v_n$ does not converge to an element in V. Consider $\{g(v_0), g(v_1), g(v_2), \ldots v_n = g(v_{n-1}), \ldots\}$. By implication, g will contract if $-1 \notin V$ but will fail to have fixed point since the space does not converge in V. Hence V is incomplete.

Theorem 2.6 If a, b $\in$ X and $S \subseteq R$, given that g: X → X is continuous then g has a fixed point.

Theorem 2.7: [2] Consider set X a complete lattice. If g: X → X is weakly increasing, then the set of fixed point of g, is a complete lattice.

Proof: Let X denote the set of fixed points. We show that X contains the largest element y^* and a smallest element x^*. Dually, if for x^* is the greatest fixed point, then it is the post fixed points of g. For the fixed point to form a complete lattice see proof of theorem 2.5 above.

Theorem 2.8 [2]: Given a complete lattice (C, ≤), if there exist the least fixed and the prefixed points of g and the greatest fixed and the post-fixed points of g. Then, there exist a fixed point that forms a complete lattice.

Proof: consider b to be the least prefixed point and w for any prefixed point, b exist since C is a complete lattice. For the fixed point to form a complete lattice see proof of theorem 2.5 above

Let's first show b is the least prefixed point. For b ≤ w, since g is monotone, implies g(b) ≤ g(w) and as w is the prefixed point , then g(w) ≤w then g(b) ≤ w, thus, g(b) ≤ g(w) ≤w, then g(b) ≤ w

Thus g(b) is a lower bound of prefixed points and b is the greatest lower bound of prefixed point. Hence g(b) ≤ b

Given g(b) ≤ b, to show b is the least fixed point. Let for monotonicity of g, thus g (g(b)) ≤ g(b) implies g(b) is a prefixed point and b is a lower bound over prefixed point, then b ≤ g(b) therefore if b is a prefixed point , then g(b) ≤ b, hence b= g(b) is a fixed point.

With the existence of duality of prefixed point, given the properties of the prefixed point [17] implies the post fixed point and therefore b is a fixed point, see[6], [19].

The Concept of Retracts, Retractions, Homeomorphisms and the Fixed Point Property

A retraction g is a continuous function that takes a metric space X, to a subspace V ⊂ X , a continuous function g: X → V, g is said to be a retraction, if g(v) = v for all v∈V and v is a retract of X see [13].

Given that there are two spaces or sets, say (X, d) and (Y, d), a property of the spaces X and Y is topological given that one of the spaces has the property of topology, implies the spaces are home omorphic. Here, the spaces will fail to be home omorphic, if they do not contain the same number of element, thus , if there exist different cardinalty between the spaces. For instance, the space [0, 1] which is closed is not home omorphic to (0, 1) since their cardinalities are not the same.

However, properties such as the fixed point property is topological. Consider the following definitions about the fixed point property.

Preposition 3.1: If X has the fixed point property, then V has a fixed point property.

Proof: The continuous function f: V→V , thus, v_{n-1} is a fixed point for the composite function fg, where v_{n-1} ∈V. Thus, $g^n(v_{n-1}) = v_{n-1}$, which also implies $f^n(g^n(v_{n-1})) = v_{n-1}$ is a fixed point of f.

Preposition 3.2: [13] If X is contractible, then V is a contractible.

Proof: Let c ∈ [0, 1] and V be a set, then g: V× c → V is a contraction V and so is x∈ V, thus (x, c) → h(g(x, c)).

Definition 3.3: Let X be a metric space. A function f: X→X is continuous, given that X has a fixed point property such that it is a point of f.

Definition 3.4: A metric spaces X and Y are said to be homeomorphic if and only if there exist a bijection g: X →Y and both the function and its inverse are continuous.

Theorem 3.5: Given (X, d) and (Y, d) a metric spaces. If the two spaces are homeomorphic such that X has a fixed point property then implies Y.

Proof: Given v: X→Y to be homeomorphism. For composition of continuous functions, we let a function g: X→X and h: Y→Y. Take a∈ X and b∈ Y, such that v (a) = b. By defining a composition of the two functions as v^{-1}(h (b)) = a, which then implies h(v(a))=v(a)=b. Hence v (a) is a fixed point of g since X contains the fixed point property.

Conclusion

We introduced in this work the concept of fixed point theory by exploring the Banach contraction principle, completeness in terms of the metric spaces. We explored spaces or sets and in that sets or spaces can be home omorphic to one another. In this article the uniqueness and existence of the fixed point.

Conflict of interest

The authors declare that there is no conflict of interest regarding the publication of this paper.

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