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Applications of Matroid Theory in Combinatorial Optimization and Projective Geometry

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ABSTRACT

A Matroid is a set with an independent structure defined on it. A Matroid abstracts and generalizes the notion of linear independence in vector spaces and independence in graphs. Matroids unite the concepts of graph theory, linear algebra, projective geometry, transversal theory, and combinatorial optimization. Applications of Matroids involve different areas such as combinatorial optimization, network theory, coding theory and many other areas. Matroids can be found in projective geometry; the fano plane of order 2 gives rise to a Matroid. An important application of Matroids in optimization involves the greedy algorithm. Kruskal's algorithm for finding a minimal spanning tree which is an example of the greedy algorithm can be used to understand how Matroids can be involved in the greedy algorithm. Consider a network of vertices with weighted links between the vertices. Our goal is to find a collection of links that connect all vertices using the smallest weight. That is a spanning tree with minimal weights. Kruskal's algorithm can be generalized to a Matroid by taking a Matroid $\boldsymbol{M}$ and a function $\boldsymbol{w}: \boldsymbol{M} \to \mathbb{R}$ which assigns weights to each element. The goal is to find the basis $\boldsymbol{B}$ of $\boldsymbol{M}$ such that $\sum \boldsymbol{w}(\boldsymbol{x})$ where $\boldsymbol{x} \in \boldsymbol{B}$ is minimized. The greedy algorithm is a characterization of the Matroid. Matroids are the structures in which the greedy algorithm works successfully.

Introduction

Matroids is a generalization of independence in graphs and linear independence in vector spaces. It was first introduced by the American Mathematician Hassler Whitney in 1935. There are several equivalent ways to define a Matroid. The two most important definitions of Matroids are in terms of independent sets and bases. Proof of each of the properties of these definitions can be found in Oxley [2], Whitney [4], and Wilson [5].

Applications of Matroids involve different areas such as combinatorial optimization, projective geometry, network theory, coding theory and many other areas. Matroids are connected to many branches of Mathematics such as linear algebra, graph theory, combinatorial optimization, finite geometry and abstract algebra.

Preliminaries

Definitions (as in Wilson [5])

The definition of a Matroid in terms of independent sets is as follows. A Matroid $\boldsymbol{M}$ consists of a non-empty finite set $\boldsymbol{E}$ and a non-empty collection $\boldsymbol{I}$ of subsets of $\boldsymbol{E}$ (called independent sets) satisfying the following properties:

(i) Any subset of an independent set is independent;

(ii) If $\boldsymbol{I}$ and $\boldsymbol{J}$ are independent sets with $|\boldsymbol{J}| > |\boldsymbol{I}|$, then there is an element $\boldsymbol{e}$ contained in $\boldsymbol{J}$ but not in $\boldsymbol{I}$, such that $\boldsymbol{I} \cup \{\boldsymbol{e}\}$ is independent.

With this definition a base is defined to be a maximal independent set and a cycle is defined to be a minimal dependent set.

A Matroid can be defined in terms of bases as well. A Matroid $\boldsymbol{M}$ consists of a non-empty finite set $\boldsymbol{E}$ and a non-empty collection $\boldsymbol{B}$ of subsets of **E**, called bases, satisfying the following properties:

(i).no base properly contains another base;

(ii). if $\boldsymbol{B_1}$ and $\boldsymbol{B_2}$ are bases and if $\boldsymbol{e}$ is any element of $\boldsymbol{B_1}$, then there is an element $\boldsymbol{f}$ of $\boldsymbol{B_2}$ such that $(\boldsymbol{B_1} - \{\boldsymbol{e}\}) \cup \{\boldsymbol{f}\}$ is also a base.

Examples of Matroids

A $\boldsymbol{k}$-uniform on $\boldsymbol{E}$ is the Matroid whose bases are those subsets of $\boldsymbol{E}$ with exactly $\boldsymbol{k}$ elements. Independent sets are those subsets of $\boldsymbol{E}$ with not more than $\boldsymbol{k}$ elements. A cyclic Matroid of a graph $\boldsymbol{G}$ is the Matroid defined by taking $\boldsymbol{E}$ as the set of edges of $\boldsymbol{G}$ and bases as the edges of the spanning forest of $\boldsymbol{G}$. It is denoted by $\boldsymbol{M}(\boldsymbol{G})$. The independent sets of $\boldsymbol{M}(\boldsymbol{G})$ are those sets of edges of $\boldsymbol{G}$ that contain no cycle. If $\boldsymbol{E}$ is a finite set of vectors in a vector space $\boldsymbol{V}$, then the vector Matroid is defined by taking as bases all linearly independent subsets of $\boldsymbol{E}$ that span the same subspace as E.

Two Matroid $\boldsymbol{M_1}$ and $\boldsymbol{M_2}$ are said to be isomorphic if there is a one-one correspondence between their underlying sets $\boldsymbol{E_1}$ and $\boldsymbol{E_2}$. When a Matroid is isomorphic to the cycle Matroid of some graph, it is said to be a graphic Matroid. Given a Matroid $\boldsymbol{M}$ on a set $\boldsymbol{E}$, $\boldsymbol{M}$ is said to be representable over a field $\boldsymbol{F}$, if $\boldsymbol{M}$ is isomorphic to $\boldsymbol{M}(\boldsymbol{G})$ for some graph $\boldsymbol{G}$.

A transversal Matroid is obtained by taking a non-empty finite set $\boldsymbol{E}$ and the partial transversals of $\boldsymbol{F}$, where $\boldsymbol{F} = (\boldsymbol{S_1}, \boldsymbol{S_2}, \dots, \boldsymbol{S_m})$ is a family of non-empty subsets of $\boldsymbol{E}$, as the independent sets.

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Results

One of the most important application of the Matroids is in the field of combinatorial optimization. It is the association between Matroids and greedy algorithms. One important property of Matroids is that they are the structures in which the greedy algorithm works successfully.

Consider a network of vertices with links between the vertices, and each link has a weight. Then goal is to find a collection of links that connect all vertices, using the smallest weight. That is a spanning tree with minimal weights. For this problem, Kruskal's algorithm for finding a minimal spanning tree which is an example of the greedy algorithm, guarantees to find a minimal spanning tree. This algorithm can be generalized to any Matroid. Consider a Matroid $\boldsymbol{M}$ and a function $\boldsymbol{w}: \boldsymbol{M} \to \mathbb{R}$ which assigns weights to each element. The goal is to find the basis $\boldsymbol{B}$ of $\boldsymbol{M}$ such that $\sum \boldsymbol{w}(\boldsymbol{x})$ where $\boldsymbol{x} \in \boldsymbol{B}$ is minimized. The optimal solution will necessarily be a basis. The algorithm is as follows:

1. Initialize a set of elements $\boldsymbol{I}$ to be the empty set $\emptyset$.
2. Sort the elements of $\boldsymbol{M}$ according to weight.
3. Run through the elements, starting with the smallest weighted elements. For each element $\boldsymbol{x}$, add $\boldsymbol{x}$ to $\boldsymbol{I}$ unless $\boldsymbol{I} \cup \{\boldsymbol{x}\}$ is dependent.

Any time the greedy algorithm guarantees an optimal solution for all weight functions, the underlying structure must be a Matroid. Moreover, only when the structure is a Matroid, the greedy algorithm guarantees to return an optimal solution.

Other examples for which the greedy algorithm and Matroids can be involved are the task scheduling problem and assignment problem. The latter can be solved by applying the greedy algorithm to the transversal Matroid.

Let $\{\boldsymbol{T_i}\}$ be a set of works ordered by their importance (priority) and let $\{\boldsymbol{E_i}\}$ be a set of employees capable to do one or various of these works. We suppose that the works will be done at the same time (and thus each employee can do just one work each time). The problem is to assign the works to the employees in an optimal way, that is to maximize the priorities.

The problem can be solved by applying the greedy algorithm to the transversal Matroid where $\boldsymbol{E} = \{\boldsymbol{T_i}\}$ is the non-empty finite set and $\boldsymbol{F} = \{\boldsymbol{S_1}, \ldots, \boldsymbol{S_k}\}$ with $\boldsymbol{S_i}$ the set of works for which employee is qualified. It can be noticed that the maximal number of works that can be done at the same time is equal to the biggest partial transversal of $\boldsymbol{F}$ with the function $\boldsymbol{w}: \boldsymbol{E} \to \mathbb{R}$ corresponding to the importance of the work.

Using a similar attempt, determining an optimal schedule can be done by identifying the transversal Matroid and by applying the greedy algorithm (the solution to the scheduling problem is an example of a system of distinct representatives or SDR [1]).

Matroids are also applied in projective geometry. To understand the connection between Matroid theory and projective geometry the *Fano plane* of order 2 can be considered. Here, it gives rise to a Matroid named the *Fano Matroid*.

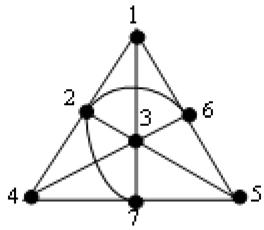

Figure 1. Fano Matroid.

Conclusion(s)

Matroids unite the concepts of graph theory, linear algebra, projective geometry, transversal theory, and combinatorial optimization. This research work considered only certain applications of Matroid theory which shows the close connection between Matroids and greedy algorithm in combinatorial optimization, and projective geometry. Kruskal's algorithm which is an example of the greedy algorithm can be used to understand how the greedy algorithm can be generalized using Matroids. Fano plane of order 2 can be considered to understand the connection between Matroid theory and projective geometry. There are many applications of Matroids in different areas. To understand the concepts of these applications further study in Matroids is needed. Future work includes the study of the definition of Matroid for finite geometries of dimension beyond 2.

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