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Prime Labeling of Complete Tripartite Graphs of the Form $K_{1,m,n}$

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ABSTRACT

A graph with n vetices is said to have a prime labeling if the vertices can be labeled with first n positive integers such that each pair of adjacent vertices are relatively prime. The present work in prime labeling focuses on finite simple undirected graphs, in particular on complete bipartite graphs. Our results are analogous to those stated by A. H. Berlineer et al., where they considered prime and coprime labeling of complete bipartite graphs while ours are for prime labeling of complete tripartite graphs. In our work, we have proved prime labeling for the general case $K_{1,m,n}$, where m, n are positive integers. Further, some non-existing cases have been proved.

Introduction

Let $G \equiv (V, E)$ be an ordered pair of vertex set $V = V(G)$and an edge set $E = E(G)$. Graph labeling or *vertex (*graph) labeling is the process of assigning values to the vertices of a graph subject to certain conditions. It is one of the most flourishing areas in Graph theory let alone in mathematics. Novel results are persistently being discovered and published at a rapid rate due to the enormous number of open problems and conjectures. Any graph labeling problem follows three noteworthy features; a set of numbers from which vertex labels are chosen, a rule that assigns a value to each edge, and a condition that these values must satisfy. The notion of prime labeling was introduced by Roger Entringer around 1980's in the setting of *trees* where he stated the conjecture; *every tree is prime*. The theory was developed to the current setting by A. Tout et al. in 1982 [4].A recent survey of a variety of graph labeling problems can be found in [2].

Definition 1

A simple graph or multigraph is tripartite if its vertices can be partitioned into three subsets in such a way that no edge joins two vertices in the same set. Simple tripartite graph is complete tripartite if each vertex in one partite set is adjacent to all the vertices in the other two sets. If the three partite sets have cardinalities l, $\mathbf{m}$ and n, then the complete tripartite graph is denoted by $K_{l,m,n}$.Complete tripartite graph is a generalization of complete bipartite graph.

Definition 2

A bijective map $f: V \to \{1, 2, 3, \dots, |V|\}$ is called a *prime labeling* if for each $e = uv$ in E, $\gcd(f(u), f(v)) = 1$. A graph that admits prime labeling is called a prime graph. (Here $\gcd(a, b)$ denotes the *greatest* common divisor of a and b).

Methodology

In our work, two Theorems have been proved which necessary and sufficient conditions for the existence for prime labeling of $K_{1,m,n}$.

As stated in [1], let $P(t, v)$ be the set of all primes x such that $t < x \le v$.

Theorem 1

Let m, n be positive integers. Then, $K_{1,m,n}$ has prime labeling if and only if

$$1+m \le \left| P\left(\frac{1+m+n}{2}, 1+m+n\right) \right| + 1,$$

where $P\left(\frac{1+m+n}{2}, 1+m+n\right)$ denotes the set of all primes p such that $\frac{1+m+n}{2} < p \le 1+m+n$.

Proof. First, consider the set of primes given by $X = P\left(\frac{1+m+n}{2}, 1+m+n\right)$. Let $Y = \{1, \dots, 1+m+n\}$, $Z = \{1\}$ and $Y' = Y \setminus \{X \cup Z\}$. Since$1$ is relatively prime to elements in Y' and X, elements in Z and X are relatively prime to Y'. Now, consider set of $1+m$ points labeling $\{1, \dots, 1+m\}$. Join vertex 1 to all vertices in X. Next, 1 to all vertices in Y'. Then join vertices in$X$ to all vertices in Y'. The resulting graph is tripartite and it is $K_{1,m,n}$. Furthermore, it is a prime graph. ∎

As mentioned in [1, p.10], let $\pi(x)$ denote the number of primes less than or equal to x, then the n^{th} *Ramanujan prime* is the least integer R_n for which $\pi(x) - \pi\left(\frac{x}{2}\right) \ge n$ holds for all $x \ge R_n$. The first few Ramanujan primes are $R_1 = 2, R_2 = 11, R_3 = 17$ etc.

Our next result establishes a lower bound for n in order $K_{1,m,n}$ to be a prime graph.

Theorem 2

The complete tripartite graph $K_{1,m,n}$ is prime if $n \ge R_m - (m+1)$.

Proof. There are at least m primes in the interval $\left(\frac{1+m+n}{2}, 1+m+n\right]$ for $n \ge R_m - (m+1)$.

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If we denote the first $\boldsymbol{m}$ of these primes by $\boldsymbol{p_1, p_2, \dots, p_m}$, then the sets given by $\boldsymbol{A_{1,m,n}} = \{\boldsymbol{1}\}$, $\boldsymbol{B_{1,m,n}} = \{\boldsymbol{P_1, P_2, \dots, P_m}\}$, and $\boldsymbol{C_{1,m,n}} = \{\boldsymbol{1, \dots, 1+m+n}\} \setminus (\boldsymbol{A_{1,m,n}} \cup \boldsymbol{B_{1,m,n}})$ give a prime labeling of $\boldsymbol{K_{1,m,n}}$ for $\boldsymbol{n \geq R_m - (m+1)}$. ■

Results and Discussion

In this section, we illustrate our results (which are analogous to Propositions 20 and 21 in ([1], p.11) with proofs for $\boldsymbol{K_{1,m,n}}$ when $\boldsymbol{n = 2}$ and $\boldsymbol{3}$ by applying Theorem 2.

Corollary 1

(i) $\boldsymbol{K_{1,2,n}}$ is a prime graph if $\boldsymbol{n = 2, 4, 5, 6}$ ***or*** $\boldsymbol{n \geq 8}$.

(ii) $\boldsymbol{K_{1,2,n}}$ is not a prime graph if $\boldsymbol{n = 3}$ or $\boldsymbol{7}$.

Proofs. **(i)** By inspection, if $\boldsymbol{n = 2}$, then $\boldsymbol{A_{1,2,2}} = \{\boldsymbol{1}\}$, $\boldsymbol{B_{1,2,2}} = \{\boldsymbol{3, 5}\}$ and $\boldsymbol{C_{1,2,2}} = \{\boldsymbol{2, 4}\}$ give the prime labeling of $\boldsymbol{K_{1,2,2}}$:

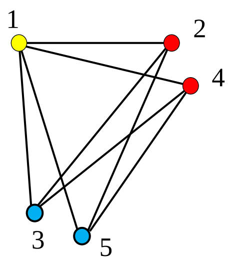

Figure 1: Prime labeling of $\boldsymbol{K_{1,2,2}}$

If $\boldsymbol{n = 4, 5, 6}$, then $\boldsymbol{A_{1,2,n}} = \{\boldsymbol{1}\}$, $\boldsymbol{B_{1,2,n}} = \{\boldsymbol{5, 7}\}$ and $\boldsymbol{C_{1,2,n}} = \{\boldsymbol{2,3,4,6}\}, \{\boldsymbol{2,3,4,6,8}\}, \{\boldsymbol{2,3,4,6,8,9}\}$ respectively, we give the prime labelings of $\boldsymbol{K_{1,2,4}}$, $\boldsymbol{K_{1,2,5}}$, and $\boldsymbol{K_{1,2,6}}$ as illustrated.

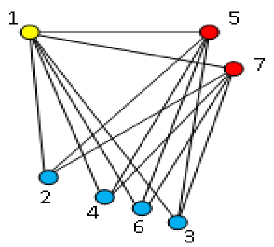

Figure 2: Prime labeling of $\boldsymbol{K_{1,2,4}}$

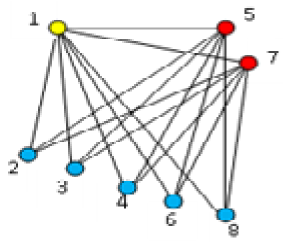

Figure 3: Prime labeling of $\boldsymbol{K_{1,2,5}}$

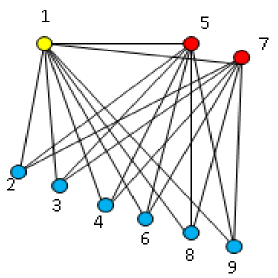

Figure 4: Prime labeling of $\boldsymbol{K_{1,2,6}}$

Since $\boldsymbol{R_2 = 11}$, there are at least two primes $\boldsymbol{p_1, p_2}$ in the interval $\left(\left\lfloor \frac{n+3}{2} \right\rfloor, n+3\right]$ for $\boldsymbol{n \geq 8}$. Hence, the sets $\boldsymbol{A_{1,2,n}} = \{\boldsymbol{1}\}$, $\boldsymbol{B_{1,2,n}} = \{\boldsymbol{p_1, p_2}\}$ and $\boldsymbol{C_{1,2,n}} = \{\boldsymbol{1, \dots, n+3}\} \setminus \left(\boldsymbol{A_{1,2,n}} \cup \boldsymbol{B_{1,2,n}}\right)$ give a prime labeling of $\boldsymbol{K_{1,2,n}}$ for $\boldsymbol{n \geq 8}$. ■

(ii) When $\boldsymbol{n = 3}$, we *cannot* obtain a subset with two vertices of prime labeling such that the vertices of this subset is relatively prime with labeling of vertices of other subsets.

Similar proof can be given for $\boldsymbol{n = 7}$. ■

Corollary 2

(i)$\boldsymbol{K_{1,3,n}}$ is a prime graph if $\boldsymbol{n = 9}$ **or** $\boldsymbol{n \geq 13}$.

(ii)$\boldsymbol{K_{1,3,n}}$ is nota prime graph if $\boldsymbol{2 \leq n \leq 12}$ excluding $\boldsymbol{9}$.

Proofs.(i) By inspection, If $\boldsymbol{n = 9}$, then $\boldsymbol{A_{1,3,9}} = \{\boldsymbol{1}\}$, $\boldsymbol{B_{1,3,9}} = \{\boldsymbol{7, 11, 13}\}$ and $\boldsymbol{C_{1,3,9}} = \{\boldsymbol{2,3,4,5,6,8,9,10,12}\}$ give a prime labeling of $\boldsymbol{K_{1,3,9}}$.

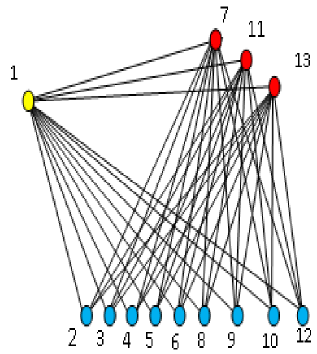

Figure 5: Prime labeling of $\boldsymbol{K_{1,3,9}}$

Since $\boldsymbol{R_3 = 17}$, there are at least three primes $\boldsymbol{p_1, p_2, p_3}$ in the interval $\left(\left\lfloor \frac{n+4}{2} \right\rfloor, n+4\right]$ for $\boldsymbol{n \geq 13}$. Hence, the sets $\boldsymbol{A_{1,3,n}} = \{\boldsymbol{1}\}$, $\boldsymbol{B_{1,3,n}} = \{\boldsymbol{p_1, p_2, p_3}\}$ and $\boldsymbol{C_{1,3,n}} = \{\boldsymbol{1, \dots, n+4}\} \setminus \left(\boldsymbol{A_{1,3,n}} \cup \boldsymbol{B_{1,3,n}}\right)$ give a prime labeling of $\boldsymbol{K_{1,3,n}}$ for $\boldsymbol{n \geq 13}$. ■

(ii) A similar proof as for Corollary 1. (ii) can be given. ■

Table1. The prime labeling of tripartite graphs for $\boldsymbol{2 \leq m \leq 12}$.

$\boldsymbol{K_{1,m,n}}$	$\boldsymbol{n}$ values for which prime graphs can be obtained
$\boldsymbol{K_{1,2,n}}$	$\boldsymbol{n = 4, 5, 6}$, and $\boldsymbol{n \geq 8}$
$\boldsymbol{K_{1,3,n}}$	$\boldsymbol{n = 9}$ and $\boldsymbol{n \geq 13}$
$\boldsymbol{K_{1,4,n}}$	$\boldsymbol{n = 14, 15, 16, 18, 19, 20}$, and $\boldsymbol{n \geq 24}$
$\boldsymbol{K_{1,5,n}}$	$\boldsymbol{n = 25, 26, 27, 31}$, and $\boldsymbol{n \geq 35}$
$\boldsymbol{K_{1,6,n}}$	$\boldsymbol{n = 36, 37, 38}$, and $\boldsymbol{n \geq 40}$
$\boldsymbol{K_{1,7,n}}$	$\boldsymbol{n = 45, 46, 47, 48, 49}$, and $\boldsymbol{n \geq 51}$
$\boldsymbol{K_{1,8,n}}$	$\boldsymbol{n = 52}$ and $\boldsymbol{n \geq 58}$
$\boldsymbol{K_{1,9,n}}$	$\boldsymbol{n \geq 61}$
$\boldsymbol{K_{1,10,n}}$	$\boldsymbol{n = 62, 68, 69, 70, 72, 73, 74, \ 78, 79, 80, 81, 82}$ and $\boldsymbol{n \geq 86}$
$\boldsymbol{K_{1,11,n}}$	$\boldsymbol{n \geq 89}$
$\boldsymbol{K_{1,12,n}}$	$\boldsymbol{n = 90, 91, 92}$ and $\boldsymbol{n \geq 94}$

Conclusion

Prime labeling of bipartite graphs are well known. In our work, we have proved that prime labeling of tripartite graphs $\boldsymbol{K_{1,m,n}}$ exist for positive integers $\boldsymbol{m}$ and $\boldsymbol{n}$. Furthermore, some non-existing cases have been discussed and some examples for prime graphs are given under the results. As for future work we are planning to generalize our results for $\boldsymbol{K_{l,m,n}}$, where $\boldsymbol{l}, \boldsymbol{m}$, and $\boldsymbol{n}$ are positive integers.

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