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Constructing Strongly Regular Graphs from Skew-Hadamard Matrices

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ABSTRACT

In this work, a new algorithm is proposed and that can be used to construct strongly regular graphs from skew-Hadamard matrices. The proposed method is based on matrix manipulations and is similar to the construction given by Jayathilaka A.A.C.A. et al., where they considered Hadamard matrices. Two strongly regular graphs have been constructed by using skew-Hadamard matrices of order $\mathbf{2}$ and $\mathbf{4}$ and are drawn using a Java programme. Considering skew-Hadamard matrices of higher order, larger strongly regular graphs can be obtained and those can be used as networks with several nodes.

Introduction

A Hadamard matrix which is named after the French mathematician Jacques Hadamard is an nxn matrix H with entries ± 1 such that $HH^T = H^T H = nI_n$. This matrix is said to be skew Hadamard if $H + H^T = 2I_n$. Study of constructions of skew-Hadamard matrices is an active research area due to its usage in constructing orthogonal designs. A short survey on constructions of skew-Hadamard matrices can be found in [2].

In our work, two strongly regular graphs are constructed by using skew-Hadamard matrices of order $\mathbf{2}$ and $\mathbf{4}$ respectively. In [2], Skew-Hadamard matrix of order n has been constructed for some n values which are multiples of 4 such that $n-1$ is a prime. Yamada-Williamson construction has been used to construct those matrices. According to [2], skew-Hadamard matrices of order $\mathbf{2}$ and $\mathbf{4}$ are,

$$\begin{bmatrix} 1 & - \\ 1 & 1 \end{bmatrix} \text{ and } \begin{bmatrix} 1 & 1 & 1 & 1 \\ - & 1 & 1 & - \\ - & - & 1 & 1 \\ - & 1 & - & 1 \end{bmatrix}$$

respectively, where $-$ is written for -1. By considering the columns of these skew-Hadamard matrices, adjacency matrix of the corresponding strongly regular graph has been constructed similar to the construction given in [1]. Further, using a Java programming, corresponding strongly regular graphs for $n = 2$ and 4 cases are obtained.

Definition 1. A graph is said to be regular if each vertex has the same number of neighbors. A regular graph with vertices of degree k is called a k-regular graph.

Definition 2. A k-regular graph on v nodes is strongly k-regular if there exist positive integers k, λ and μ such that every vertex has k neighbors, every adjacent pair of vertices has λ common neighbors and every non-adjacent pair μ has common neighbors. It is denoted by $SRG(v, k, \lambda, \mu)$.

Definition 3. The adjacency matrix of a graph having vertices $v_1, v_2, \dots, v_n$ is the $n \times n$ matrix whose (i,j) entry is the number of edges connecting v_i and v_j.

Methodology

The proposed method constructs Strongly Regular Graph from skew-Hadamard matrices.

Consider the skew-Hadamard matrix of order $\mathbf{2}$ and label each column vector as c_i for $i = 1, 2$.

$$\begin{bmatrix} 1 & - \\ 1 & 1 \end{bmatrix} = [c_1 \quad c_2]$$

Next, find C_i matrices such that $C_i = c_i c_i^T$ where $i = 1, 2$.

$$C_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} . [1 \quad 1] = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

$$C_2 = \begin{bmatrix} 1 \\ - \end{bmatrix} . [1 \quad -] = \begin{bmatrix} 1 & - \\ - & 1 \end{bmatrix}$$

Now consider a latin square of order 2, $\begin{bmatrix} C_1 & C_2 \\ C_2 & C_1 \end{bmatrix}$ which is an order 4 marix in ± 1. Then,

$$\begin{bmatrix} C_1 & C_2 \\ C_2 & C_1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & - \\ 1 & 1 & - & 1 \\ 1 & - & 1 & 1 \\ - & 1 & 1 & 1 \end{bmatrix}$$

replacing -1 by 0 will leads to the following adjacency matrix.

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}$$

Figure 1. The adjacency matrix obtained from the skew-Hadamard matrix of order 2

By considering a skew-Hadamard matrix of order $\mathbf{4}$ and labeling each column vector as c_i for $i = 1, 2, 3, 4$ the following order 4 skew-Hadamard marix can be obtained.

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ - & 1 & 1 & - \\ - & - & 1 & 1 \\ - & 1 & - & 1 \end{bmatrix} = \begin{bmatrix} c_1 & c_2 & c_3 & c_4 \end{bmatrix}$$

Following the same procedure, the Latin square of order 4 whose entries are matrices $\boldsymbol{C_i}$ such that $\boldsymbol{C_i = c_i c_i^T}$ for $\boldsymbol{i = 1, 2, 3, 4}$ can be obtained.

$$\begin{bmatrix} C_1 & C_2 & C_3 & C_4 \\ C_2 & C_3 & C_4 & C_1 \\ C_3 & C_4 & C_1 & C_2 \\ C_4 & C_1 & C_2 & C_3 \end{bmatrix}$$

Replace $-\mathbf{1}$ in the resultant matrix by $\mathbf{0}$ will leads to the adjacency matrix illustrated below.

$$\begin{bmatrix}
1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 1 \\
0 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 1 \\
0 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 \\
1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\
1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 \\
1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 \\
1 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\
1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 1 \\
1 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 1 \\
1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 \\
1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 \\
1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 1
\end{bmatrix}$$

Figure 2. The adjacency matrix of the strongly regular graph obtained from the skew-Hadamard matrix of order 4.

Results and Discussion

Using Java programme strongly regular graphs from skew-Hadamard matrices are obtained and the algorithm was described under the methodology.

By using Java programme, the corresponding strongly regular graph $\boldsymbol{SRG(4, 4, 0, 2)}$, for the adjacency matrix in figure **1** have been obtained which is a graph consisting of 4 points in which each vertex has the degree 4 and each pair of adjacency vertices have no common neighbours and each non-adjacency pair of vertices have 2 common neighbours.

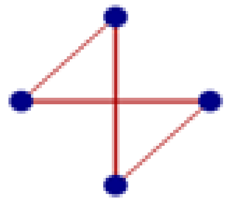

Figure 3. Strongly regular obtained from the skew-Hadamard matrix of order 2.

Moreover, $\boldsymbol{SRG(16, 11, 4, 6)}$ can be obtained from the skew-Hadamard matrix of order 4 is shown below.

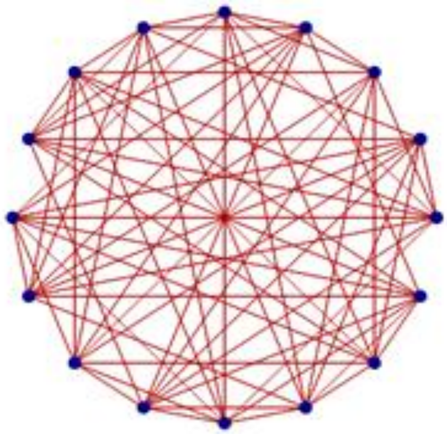

Figure 4. Strongly regular graph obtained from the skew-Hadamard matrix of order 4.

In the above two figures coloured circle indicates a loop at each point.

Conclusion

Strongly regular graphs have been constructed from skew-Hadamard matrices of order 2 and 4. Java programming was used to draw the graphs. This construction method can be used to construct strongly regular graphs of higher degree. For that, one could construct higher order skew-Hadamard matrices using the methods given by Yamada-Williamson or any other method. These graphs can be used in computer networks, telecommunication, data structures and to solve many real world problems.

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