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Buckling Analysis of Thin Laminated Composite Plates with Exact Displacement Functions

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ABSTRACT

This work presents the buckling analysis of thin laminated composite plates using Euler-Bernoulli equilibrium equation. The project aims at obtaining the exact equation which will not depend on assumed shape function which charcterised the anlaysis of composite plates. The present study is based on classical plate theory which is widely used for analysis of thin plates. The governing equation for the thin laminated plate were obtain considering the total potential energy function which was in turn minimized to obtain equation for analysis of buckling. Numerical work examples were performed considering different aspect ratios and elastic moduli for SSSS laminated plates at orientation 0/90/90/0. The results were compared with the works of Reddy and that of Osman and Sulieman. The maximum percentage difference between the present work to the work of Reddy for SSSS laminated plate is 0.06 while the difference with Osman and Sulieman work has maximum of 1.81. Reddy presented also an exact method base on the SSSS shape function, recorded minimal difference may be as result of round off errors introduced along the computation line while the difference with that of Osman and Sulieman is because he used finite element method with assumed shape function which gives approximate result.

Introduction

According to Ventsel & Krauthammer [1], thin plates are initially flat structural members bounded by two parallel planes, called faces, and a cylindrical surface, called an edge or boundary. The generators of the cylindrical surface are perpendicular to the plane faces. The distance between the plane faces is called the thickness (h) of the plate. Most often, the plate thickness is always assumed to be very small compared with other characteristic dimensions of the faces (length, width, diameter, etc.). Thin Plates are widely used in Engineering work; hence production of new plates is inevitable.

Presently, new engineering materials are being produced with the following characteristics: - lighter weight, higher strength, high modular ratio and high temperature resistance. Combinations of materials with the above special properties to produce different plates and shells have gone a long way to create the field of laminated composite materials in which laminated composite plate is a member. Composite materials consist of two or more materials which together produce desirable properties that cannot be achieved with any of the constituents alone [2].

Thin laminated composite plate analysis is based on Classical plate theory and it gives an acceptable result. Classical plate theories (CPT) are based on Kirchhoff's hypothesis which assumes that normal to the mid–surface of the plate before deformation remains straight and normal to the mid–surface after deformation. These theories are widely used for the analysis of thin plates [3].

Buckling is one of the failure modes experienced when thin laminated composite plates are in use which needs to be evaluated; Other forms of failure modes include pure bending and free vibration [4]. There are different methods of analyzing buckling of thin laminated composite plates but most of the methods are based on assumed displacement function ([2]; [4]; [5]; [6]; [7]; [8]). Some also, assume displacement function as well as finite element method [4]. The assumed function is either based on Navier or levy method. It is widely agreed that assumed displacement function will always give results that differ from the exact ones except where the displacement function assumed is the exact displacement function.

Due to the fact that most buckling analyses on thin laminated composite plates are done using assumed displacement function, the present research aims at using the Euler-Bernoulli equilibrium equation which has been accepted as the deflected shape of beam strip to analyzed thin laminated composite plate.

2.0. Displacement Field and Kinematics of a Lamina of Thin Laminated Plate

Some governing assumptions in this study are the plane stress assumption (normal stress along z-axis, x-z plane and y-z plane shear stresses are zeros), another assumption is normal strain along z axis is so small that neglecting it shall not affect the gross response of the plate. Itemizing the assumptions gives.

i. $\boldsymbol{\sigma_{zz} = 0}$

ii. $\boldsymbol{\tau_{xz} = 0}$

iii. $\boldsymbol{\tau_{yz} = 0}$

iv. $\boldsymbol{\varepsilon_{zz} = 0}$

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Two in-plane displacements and one out-of-plane displacement (u, v and w respectively) constitute the displacement field.

From the fourth assumption it is taken that the out-of-plane displacement (deflection) is constant along z-axis, which means it is not a function of z. However, the two in-plane displacements (u and v) are functions of all coordinates (x, y and z). From assumption ii and iii, it is taken that corresponding x-z and y-z planes' shear strains are zeros. Thus, the in-plane displacements are given as:

$$\mathbf{u} = -\mathbf{z}\frac{\mathbf{dw}}{\mathbf{dx}} + \mathbf{u_0} \qquad 1$$

$$\mathbf{v} = -\mathbf{z}\frac{\mathbf{dw}}{\mathbf{dy}} + \mathbf{v_0} \qquad 2$$

The in-plane displacements of the middle surface (u_0 and v_0) are not constants [9]. Using equations 1 and 2 and the no-constant values of u_0 and v_0, in-plane strains are defined as:

$$\varepsilon_{xx} = \frac{du}{dx} = \varepsilon_{xx}^0 + \varepsilon_{xx}^i = \frac{du_0}{dx} - z\frac{d^2w}{dx^2} \qquad 3$$

$$\varepsilon_{yy} = \frac{dv}{dy} = \varepsilon_{yy}^0 + \varepsilon_{yy}^i = \frac{dv_0}{dy} - z\frac{d^2w}{dy^2} \qquad 4$$

$$\gamma_{xy} = \varepsilon_{xy} + \varepsilon_{yx} = \left[-z\frac{d^2w}{dxdy} + \frac{du_0}{dy}\right] + \left[-z\frac{d^2w}{dxdy} + \frac{dv_0}{dx}\right]$$

$$\textit{That is}:$$

$$\gamma_{xy} = \gamma_{xy}^0 + \gamma_{xy}^i = \left(\frac{du_0}{dy} + \frac{dv_0}{dx}\right) - 2z\frac{d^2w}{dxdy} \qquad 5$$

2.1. Constitutive relations for a lamina of thin laminated plate.

The Hook's law equation for one lamina in laminated plate is given as:

$$\begin{bmatrix}\sigma_{11}\\ \sigma_{22}\\ \tau_{12}\end{bmatrix} = E_0\begin{bmatrix}e_{11} & e_{12} & 0\\ e_{12} & e_{22} & 0\\ 0 & 0 & e_{33}\end{bmatrix}\begin{bmatrix}\varepsilon_{11}\\ \varepsilon_{22}\\ \gamma_{12}\end{bmatrix} \qquad 6$$

$$\textit{Where}: e_{11} = \frac{E_{11}/E_0}{1-\mu_{xy}\mu_{yx}};$$

$$e_{12} = \frac{\mu_{21}.E_{11}/E_0}{1-\mu_{xy}\mu_{yx}} = \frac{\mu_{12}.E_{22}/E_0}{1-\mu_{xy}\mu_{yx}}$$

$$e_{22} = \frac{E_{22}/E_0}{1-\mu_{xy}\mu_{yx}};\ e_{33} = \frac{G_{12}}{E_0};$$

E_0 is the reference Elastic modulus. it can be E_{11} or E_{22}. E_{ij} are the modulii of elasticity and μ_{ij} are the and Poisson's ratios of an anisotropic lamina.

Using the transformation matrix [T], equation 6 is transformed from (1-2 local) coordinate system to (x-y global) coordinate system as [10].

$$\begin{bmatrix}\sigma_{xx}\\ \sigma_{yy}\\ \tau_{xy2}\end{bmatrix} = E_0\left\{[T]^{-1}\begin{bmatrix}e_{11} & e_{12} & 0\\ e_{12} & e_{22} & 0\\ 0 & 0 & e_{33}\end{bmatrix}[T]^{-T}\right\}\begin{bmatrix}\varepsilon_{xx}\\ \varepsilon_{yy}\\ \gamma_{xy}\end{bmatrix} \qquad 7$$

Here the transformation matrix, [T] is defined as:

$$[T] = \begin{bmatrix}m^2 & n^2 & 2mn\\ n^2 & m^2 & -2mn\\ -mn & mn & (m^2-n^2)\end{bmatrix} \qquad 8$$

$$\textit{Where}: \quad m = Cos\theta \text{ and } n = Sin\theta$$

Substituting equation 8 into equation 7 gives:

$$\begin{bmatrix}\sigma_{xx}\\ \sigma_{yy}\\ \tau_{xy}\end{bmatrix} = E_0\begin{bmatrix}a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{bmatrix}\begin{bmatrix}\varepsilon_{xx}\\ \varepsilon_{yy}\\ \gamma_{xy}\end{bmatrix} \qquad 9$$

Where:

$$a_{11} = m^4e_{11} + 2m^2n^2(e_{12}+2e_{33}) + n^4e_{22}$$
$$a_{12} = e_{12}(n^4+m^4) + m^2n^2(e_{11}+e_{22}-4e_{33})$$
$$a_{13} = m^3n(e_{11}-e_{12}-2e_{33}) + mn^3(e_{12}-e_{22}+2e_{33})$$
$$a_{22} = n^4e_{11} + 2m^2n^2(e_{12}+2e_{33}) + m^4e_{22}$$
$$a_{23} = mn^3(e_{11}-e_{12}-2e_{33}) + m^3n(-e_{22}+e_{12}+2e_{33})$$
$$a_{33} = m^2n^2(e_{11}-2e_{12}+e_{22}-2e_{33}) + e_{33}(m^4+n^4)$$
$$a_{21} = a_{12}, \quad a_{31} = a_{13} \text{ and } a_{32} = a_{23}$$

Substituting equations 3, 4 and 5 into equation 9 gives:

$$[\sigma] = \begin{bmatrix}\sigma_{xx}\\ \sigma_{yy}\\ \tau_{xy}\end{bmatrix} = E_0[a_{ij}][\varepsilon] \qquad 10a$$

Where:

$$[a_{ij}] = \begin{bmatrix}a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{bmatrix} \qquad 11a$$

$$[\varepsilon] = \begin{bmatrix}\varepsilon_{xx}\\ \varepsilon_{yy}\\ \gamma_{xy}\end{bmatrix} = \begin{bmatrix}\frac{du_0}{dx} - z\frac{d^2w}{dx^2}\\ \frac{dv_0}{dy} - z\frac{d^2w}{dy^2}\\ \left(\frac{du_0}{dy} + \frac{dv_0}{dx}\right) - 2z\frac{d^2w}{dxdy}\end{bmatrix} \qquad 11b$$

2.2. Total potential energy functional for a laminated thin rectangular plate.

The total potential energy functional for a laminated thin rectangular plate is given as [1]:

$$\Pi = \frac{1}{2}\iiint[\sigma][\varepsilon]\ dx.dy.dz - \frac{N_x}{2}\iint\left(\frac{dw}{dx}\right)^2\ dx\,dy \qquad 12$$

Substituting equations 11a and 11b into equation 12 gives:

$$\Pi = \frac{E_0}{2}\iiint[\varepsilon]^T[a_{ij}][\varepsilon]\ dxdydz - \frac{N_x}{2}\iint\left(\frac{dw}{dx}\right)^2 dxdy \qquad 13$$

Carrying out the multiplication and closed domain integration of equation 13 with respect to z gives:

$$= \frac{E_0t^3}{2}\iint\Bigg\{\left(\frac{A_{11}}{t^2}\left[\frac{du_0}{dx}\right]^2 + 2\frac{A_{12}}{t^2}\frac{du_0}{dx}\frac{dv_0}{dy} + \frac{A_{33}}{t^2}\left[\frac{du_0}{dy}\right]^2 + 2\frac{A_{33}}{t^2}\frac{du_0}{dy}\frac{dv_0}{dx} + \frac{A_{33}}{t^2}\left[\frac{dv_0}{dx}\right]^2 + \frac{A_{22}}{t^2}\left[\frac{dv_0}{dy}\right]^2\right)$$

$$-2\left(\frac{B_{11}}{t}\frac{du_0}{dx}.\frac{d^2w}{dx^2} + \frac{(B_{12}+2B_{33})}{t}\frac{du_0}{dy}\frac{d^2w}{dxdy} + \frac{(B_{12}+2B_{33})}{t}\frac{dv_0}{dx}\frac{d^2w}{dxdy} + \frac{B_{22}}{t}\frac{dv_0}{dy}\frac{d^2w}{dy^2}\right)$$

$$+2\left(\frac{A_{13}}{t^2}\frac{du_0}{dx}\frac{du_0}{dy} + \frac{A_{13}}{t^2}\frac{du_0}{dx}\frac{dv_0}{dx} - 3\frac{B_{13}}{t}\frac{du_0}{dx}\frac{d^2w}{dxdy} - \frac{B_{13}}{t}\frac{dv_0}{dx}\frac{d^2w}{dx^2} + 2D_{13}\frac{d^2w}{dx^2}\frac{d^2w}{dxdy}\right)$$

$$+\left(D_{11}\left[\frac{d^2w}{dx^2}\right]^2 + 2(D_{12}+2D_{33})\left[\frac{d^2w}{dxdy}\right]^2 + D_{22}\left[\frac{d^2w}{dy^2}\right]^2\right)$$

$$+2\left(\frac{A_{23}}{t^2}\frac{du_0}{dy}\frac{dv_0}{dy}+\frac{A_{23}}{t^2}\frac{dv_0}{dy}\frac{dv_0}{dx}-3\frac{B_{23}}{t}\frac{dv_0}{dy}\frac{d^2w}{dxdy}-\frac{B_{23}}{t}\frac{du_0}{dy}\frac{d^2w}{dy^2}+2D_{23}\frac{d^2w}{dy^2}\frac{d^2w}{dxdy}\right)\}\,dx\,.dy$$

$$-\frac{N_x}{2}\iint\left(\frac{dw}{dx}\right)^2 dx\,dy \qquad 14$$

Where:

$$A_{ij}=\frac{\overline{A_{ij}}}{t}\ \ and\ \ \overline{A_{ij}}=t\sum_{m=1}^{m=n}a_{ij}(s_m-s_{m-1}) \quad 15$$

$$B_{ij}=\frac{\overline{B_{ij}}}{t^2}\ and\ \overline{B_{ij}}=\frac{t^2}{2}\sum_{m=1}^{m=n}a_{ij}(s_m^2-s_{m-1}^2) \quad 16$$

$$D_{ij}=\frac{\overline{D_{ij}}}{t^3}\ and\ \overline{D_{ij}}.=\frac{t^3}{3}\sum_{m=1}^{m=n}a_{ij}(s_m^3-s_{m-1}^3) \quad 17$$

"m" stands for the number of a lamina in the laminated plate, n is the total number of laminas "s" is the non dimensional coordinate along z-axis defined as s = z/t.

Let the summation of the following three constants be one. That is:

$$n_1+n_2+n_3=1 \qquad 18$$

Substituting equation 18 into equation 14 to multiply the in-plane load, N_x (that is: $Nx = n_1N_x + n_2N_x + n_3N_x$) rearranging the resulting quation gives:

$$\Pi=\Pi_1+\Pi_2+\Pi_3 \qquad 19$$

Where:

$$\Pi_1=\frac{E_0t^3}{2}\iint\{\left(\frac{A_{11}}{t^2}\left[\frac{du_0}{dx}\right]^2+2\frac{A_{12}}{t^2}\frac{du_0}{dx}\frac{dv_0}{dy}+\frac{A_{33}}{t^2}\left[\frac{du_0}{dy}\right]^2+2\frac{A_{33}}{t^2}\frac{du_0}{dy}\frac{dv_0}{dx}+\frac{A_{33}}{t^2}\left[\frac{dv_0}{dx}\right]^2+\frac{A_{22}}{t^2}\left[\frac{dv_0}{dy}\right]^2\right)$$

$$-2\left(\frac{B_{11}}{t}\frac{du_0}{dx}.\frac{d^2w}{dx^2}+\frac{(B_{12}+2B_{33})}{t}\frac{du_0}{dy}\frac{d^2w}{dxdy}+\frac{(B_{12}+2B_{33})}{t}\frac{dv_0}{dx}\frac{d^2w}{dxdy}+\frac{B_{22}}{t}\frac{dv_0}{dy}\frac{d^2w}{dy^2}\right)$$

$$+\left(D_{11}\left[\frac{d^2w}{dx^2}\right]^2+2(D_{12}+2D_{33})\left[\frac{d^2w}{dxdy}\right]^2+D_{22}\left[\frac{d^2w}{dy^2}\right]^2\right)-n_1\frac{N_x}{2}\iint\left(\frac{dw}{dx}\right)^2 dx\,dy \qquad 20a$$

$$\Pi_2=\frac{2E_0t^3}{2}\iint\left[\frac{A_{13}}{t^2}\frac{du_0}{dx}\frac{du_0}{dy}+\frac{A_{13}}{t^2}\frac{du_0}{dx}\frac{dv_0}{dx}-3\frac{B_{13}}{t}\frac{du_0}{dx}\frac{d^2w}{dxdy}-\frac{B_{13}}{t}\frac{dv_0}{dx}\frac{d^2w}{dx^2}+2D_{13}\frac{d^2w}{dx^2}\frac{d^2w}{dxdy}\right]dx\,.dy-n_2\frac{N_x}{2}\iint\left(\frac{dw}{dx}\right)^2 dx\,dy \qquad 20b$$

$$\Pi_3=\frac{2E_0t^3}{2}\iint\left[\frac{A_{23}}{t^2}\frac{du_0}{dy}\frac{dv_0}{dy}+\frac{A_{23}}{t^2}\frac{dv_0}{dy}\frac{dv_0}{dx}-3\frac{B_{23}}{t}\frac{dv_0}{dy}\frac{d^2w}{dxdy}-\frac{B_{23}}{t}\frac{du_0}{dy}\frac{d^2w}{dy^2}+2D_{23}\frac{d^2w}{dy^2}\frac{d^2w}{dxdy}\right]dx\,.dy-n_3\frac{N_x}{2}\iint\left(\frac{dw}{dx}\right)^2 dx\,dy \qquad 20c$$

For easy understanding of the meanings for m, n and z is illustrated with a four-lamina laminated plate shown on Figure 1

$z_0 = t\,s_0 = t/2$

Lamina 1: $m=1$; $z_{m-1}=z_0$; $z_m=z_1$

$z_1 = t\,s_1 = t/4$

Lamina 2: $m=2$; $z_{m-1}=z_1$; $z_m=z_2$

$z_2 = t\,s_2 = 0$

Lamina 3: $m=3$; $z_{m-1}=z_2$; $z_m=z_3$

$z_3 = t\,s_3 = -t/4$

Lamina 4: $m=4$; $z_{m-1}=z_3$; $z_m=z_4$

$z_4 = t\,s_4 = -t/2$

Figure 1: A laminated plate that is made of four laminas

2.3.General and direct Variation of Total potential energy functional for a laminated thin rectangular plate.

Minimizing equations 20a, 20b and 20c with respect to w, u_0 and v_0 and making some rearrangements shall give the respective equations:

$$\frac{\partial\Pi_1}{\partial w}=0=\iint\left[-\frac{1}{t}\left(B_{11}\frac{\partial^3u_0}{\partial x^3}+(B_{12}+2B_{33})\frac{\partial^3u_0}{\partial x\partial y^2}+(B_{12}+2B_{33})\frac{\partial^3v_0}{\partial x^2\partial y}+B_{22}\frac{\partial^3v_0}{\partial y^3}\right)+\left(D_{11}\frac{\partial^4w}{\partial x^4}+2(D_{12}+2D_{33})\frac{\partial^4w}{\partial x^2\partial y^2}+D_{22}\frac{\partial^4w}{\partial y^4}\right)\right]dxdy+\frac{n_1N_x}{E_0t^3}\iint\frac{d^2w}{dx^2}dx\,dy \qquad 21a$$

$$\frac{\partial\Pi_2}{\partial w}=0=\frac{1}{t}\iint\frac{\partial^2}{\partial x^2}\left(-3B_{13}\frac{\partial u_0}{\partial y}-B_{13}\frac{\partial v_0}{\partial x}+4D_{13}\frac{\partial^2w}{\partial x\partial y}\right)dxdy+\frac{n_2N_x}{E_0t^3}\iint\frac{d^2w}{dx^2}dx\,dy \qquad 21b$$

$$\frac{\partial \Pi_3}{\partial w} = 0 = \frac{1}{t}\iint \frac{\partial^2}{\partial y^2}\left(-3B_{23}\frac{\partial v_0}{\partial x} - B_{23}\frac{\partial u_0}{\partial y} + 4D_{23}\frac{\partial^2 w}{\partial x \partial y}\right)\, dxdy + \frac{n_3 N_x}{E_0 t^3}\iint \frac{d^2 w}{dx^2}\,dx\,dy \qquad 21c$$

$$\frac{\partial \Pi_1}{\partial u_0} = 0 = \frac{1}{t^2}\iint \left(\frac{d^2}{dx^2}\left[A_{11}u_0 - B_{11}\frac{dw}{dx}\right] + \frac{d^2}{dxdy}\left[A_{12}v_0 - B_{12}\frac{dw}{dy}\right] + \frac{d^2}{dxdy}\left[A_{33}v_0 - B_{33}\frac{dw}{dy}\right] + \frac{d^2 u_0}{dy^2}\left[A_{33}u_0 - B_{33}\frac{dw}{dx}\right]\right) dxdy \qquad 22a$$

$$\frac{\partial \Pi_2}{\partial u_0} = \iint \left(2\frac{d^2}{dxdy}\left[\frac{A_{13}}{t^2}u_0 - \frac{B_{13}}{t}\frac{dw}{dx}\right] + \frac{d^2}{dx^2}\left[\frac{A_{13}}{t^2}v_0 - \frac{B_{13}}{t}\frac{dw}{dy}\right]\right)\, dxdy = 0 \qquad 22b$$

$$\frac{\partial \Pi_3}{\partial u_0} = \iint \frac{d^2}{dy^2}\left[\frac{A_{23}}{t^2}v_0 - \frac{B_{23}}{t}\frac{du_w}{dy}\right]\, dxdy = 0 \quad 22c$$

$$\frac{\partial \Pi_1}{\partial v_0} = \iint \left[\frac{d^2}{dxdy}\left(\left[\frac{A_{12}}{t^2}u_0 - \frac{B_{12}}{t}\frac{dw}{dx}\right] + \frac{d^2}{dy^2}\left[\frac{A_{22}}{t^2}v_0 - \frac{B_{22}}{t}\frac{dw}{dy}\right] + \frac{d^2}{dxdy}\left[\frac{A_{33}}{t^2}u_0 - \frac{B_{33}}{t}\frac{dw}{dx}\right] + \frac{d^2}{dx^2}\left[\frac{A_{33}}{t^2}v_0 - \frac{B_{33}}{t}\frac{dw}{dy}\right]\right)\right] dxdy = 0 \quad 23a$$

$$\frac{\partial \Pi_2}{\partial v_0} = \iint \frac{d^2}{dx^2}\left[\frac{A_{13}}{t^2}u_0 - \frac{B_{13}}{t}\frac{dw}{dx}\right]\, dxdy = 0 \qquad 23b$$

$$\frac{\partial \Pi_3}{\partial v_0} = \iint \left(\frac{d^2}{dy^2}\left[\frac{A_{23}}{t^2}u_0 - \frac{B_{23}}{t}\frac{dw}{dx}\right] + 2\frac{d^2}{dxdy}\left[\frac{A_{23}}{t^2}v_0 - \frac{B_{23}}{t}\frac{dw}{dy}\right]\right)\, dxdy = 0 \qquad 23c$$

For equations 22a, 22b, 22c, 23a, 23b and 23c to be true, the following shall hold (where c and d are yet to be determined constants):

$$u_0 = t\frac{B_{ij}}{A_{ij}}\frac{\partial w}{\partial x} = c.t\frac{\partial w}{\partial x} \qquad 24a$$

$$v_0 = t\frac{B_{ij}}{A_{ij}}\frac{\partial w}{\partial y} = d.t\frac{\partial w}{\partial y} \qquad 24b$$

Substituting equations 24a and 24b into equation 21a and making some rearrangements and observing that an integral can only be zero if its integrand is gives:

$$\iint \left([D_{11} - cB_{11}]\frac{\partial^4 w}{\partial x^4} + 2[D_{12} - cB_{12} - dB_{12} + 2D_{33} - 2cB_{33} - 2dB_{33}]\frac{\partial^4 w}{\partial x^2 \partial y^2} + [D_{22} - dB_{22}]\frac{\partial^4 w}{\partial y^4} + \frac{n_1 N_x}{E_0 t^3}.\frac{d^2 w}{dx^2}\right) dxdy = 0 \quad 25$$

Dividing equation 25 by $[D_{22} - dB_{22}]$ gives:

$$\iint \left[f_1\frac{\partial^4 w}{\partial x^4} + f_2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} + \frac{n_1 N_x}{E_0 t^3}.\frac{d^2 w}{dx^2}\right] dxdy = 0 \qquad 26$$

$$Where:\ f_1 = \frac{[D_{11} - cB_{11}]}{[D_{22} - dB_{22}]};$$

$$f_2 = \frac{2[D_{12} - cB_{12} - dB_{12} + 2D_{33} - 2cB_{33} - 2dB_{33}]}{[D_{22} - dB_{22}]}$$

The exact solutions to equation 26 (in terms of non-dimensional coordinates) for pure bending analysis, buckling analysis and free vibration analysis were obtained to be (see [Ibearugbulem 2016] for details):

$$w = (a_0 + a_1 R + a_2 \cos kR + a_3 \sin kR)\,(b_0 + b_1 Q + b_2 \cos gQ + b_3 \sin gQ) \qquad 27a]$$

$$Where: k = \sqrt{\frac{n_1 N_x a^2}{D}}$$

From equations 27a, it was gathered that:

$$w = \beta_1 h \qquad 27b$$

Substituting equation 27d into equations 24a and 24b gives:

$$u_0 = c.t.\beta_1\frac{\partial h}{\partial x} = \beta_2\frac{\partial h}{\partial x} \qquad 28a$$

$$v_0 = d.t.\beta_1\frac{\partial h}{\partial y} = \beta_3\frac{\partial h}{\partial y} \qquad 28b$$

$$\beta_2 = c.t.\beta_1 \ \ and\ \ \beta_3 = d.t.\beta_1 \qquad 28c$$

Substituting equations 27b, 28a and 28b into equations 20a, 20b and 20c and writing the outcomes in terms of non dimensional coordinates gives:

$$\Pi_1 = \frac{E_0 t^3 ab}{2a^4}\iint \left[\left(A_{11}{\beta_2}^2\left(\frac{\partial^2 h}{\partial R^2}\right)^2 + \frac{1}{\alpha^2}\left[2A_{12}\beta_2\beta_3 + 2A_{33}\beta_2\beta_3 + A_{33}{\beta_2}^2 + A_{33}{\beta_3}^2\right]\left(\frac{\partial^2 h}{\partial R \partial Q}\right)^2 + A_{22}\frac{{\beta_3}^2}{\alpha^4}\left(\frac{\partial^2 h}{\partial Q^2}\right)^2\right) - 2\left(B_{11}\beta_1\beta_2\left(\frac{\partial^2 h}{\partial R^2}\right)^2 + (B_{12} + 2B_{33})\frac{\beta_1\beta_2}{\alpha^2}\left(\frac{\partial^2 h}{\partial R \partial Q}\right)^2 + (B_{12} + 2B_{33})\frac{\beta_1\beta_3}{\alpha^2}\left(\frac{\partial^2 h}{\partial R \partial Q}\right)^2 + B_{22}\frac{\beta_1\beta_3}{\alpha^4}\left(\frac{\partial^2 h}{\partial Q^2}\right)^2\right) + \left(D_{11}{\beta_1}^2\left(\frac{\partial^2 h}{\partial R^2}\right)^2 + 2(D_{12} + 2D_{33})\frac{{\beta_1}^2}{\alpha^2}\left(\frac{\partial^2 h}{\partial R \partial Q}\right)^2 + D_{22}\frac{{\beta_1}^2}{\alpha^4}\left(\frac{\partial^2 h}{\partial Q^2}\right)^2\right)\right] dRdQ$$

$$-\frac{n_1 N_x ab}{2a^2}{\beta_1}^2\iint \left(\frac{dh}{dR}\right)^2 dR\, dQ$$

$$\Pi_2 = \frac{2E_0t^3ab}{2a^4}\iint\Bigg[A_{13}\frac{{\beta_2}^2}{\alpha}\frac{\partial^2 h}{\partial R^2}.\frac{\partial^2 h}{\partial R\partial Q} + A_{13}\frac{\beta_2\beta_3}{\alpha}\frac{\partial^2 h}{\partial R^2}.\frac{\partial^2 h}{\partial R\partial Q} - 3B_{13}\frac{\beta_1\beta_2}{\alpha}\frac{\partial^2 h}{\partial R^2}\frac{\partial^2 h}{\partial R\partial Q} - B_{13}\frac{\beta_1\beta_3}{\alpha}.\frac{\partial^2 h}{\partial R^2}\frac{\partial^2 h}{\partial R\partial Q} + 2D_{13}\frac{{\beta_1}^2}{\alpha}\frac{\partial^2 h}{\partial R^2}\frac{\partial^2 h}{\partial R\partial Q}\Bigg]\,dR\,.dQ - \frac{n_2N_xab}{2a^2}{\beta_1}^2\iint\left(\frac{dh}{dR}\right)^2 dR\,dQ \quad 29b$$

$$\Pi_3 = \frac{2E_0t^3ab}{2a^4}\iint\Bigg[A_{23}\frac{\beta_2\beta_3}{\alpha^3}\frac{\partial^2 h}{\partial R\partial Q}.\frac{\partial^2 h}{\partial Q^2} + A_{23}\frac{{\beta_3}^2}{\alpha^3}\frac{\partial^2 h}{\partial R\partial Q}.\frac{\partial^2 h}{\partial Q^2} - 3B_{23}\frac{\beta_1\beta_3}{\alpha^3}\frac{\partial^2 h}{\partial R\partial Q}.\frac{\partial^2 h}{\partial Q^2} - B_{23}\frac{\beta_1\beta_2}{\alpha^3}\frac{\partial^2 h}{\partial R\partial Q}.\frac{\partial^2 h}{\partial Q^2} + 2D_{23}\frac{{\beta_1}^2}{\alpha^3}\frac{\partial^2 h}{\partial R\partial Q}.\frac{\partial^2 h}{\partial Q^2}\Bigg]\,dR\,.dQ - \frac{n_3N_xab}{2a^2}{\beta_1}^2\iint\left(\frac{dh}{dR}\right)^2 dR\,dQ \quad 29c$$

Minimizing equations 29a, 29b and 29c with respect to β_1 and rearrange gives respectively:

$$\frac{d\Pi_1}{d\beta_1} = 0 = -\left(B_{11}\beta_2k_x + (B_{12}+2B_{33})\frac{\beta_2}{\alpha^2}k_{xy} + (B_{12}+2B_{33})\frac{\beta_3}{\alpha^2}k_{xy} + B_{22}\frac{\beta_3}{\alpha^4}k_y\right) + \beta_1\left(D_{11}k_x + \frac{2}{\alpha^2}(D_{12}+2D_{33})k_{xy} + \frac{D_{22}}{\alpha^4}k_y\right) - \frac{n_1N_xa^2}{E_0t^3}\beta_1k_N \quad 30a$$

$$\frac{d\Pi_2}{d\beta_1} = 0 = \left(4D_{13}\frac{\beta_1}{\alpha} - 3B_{13}\frac{\beta_2}{\alpha} - B_{13}\frac{\beta_3}{\alpha}\right)k_{xxy} - \frac{n_2N_xa^2}{E_0t^3}\beta_1k_N \quad 30b$$

$$\frac{d\Pi_3}{d\beta_1} = 0 = \left(4D_{23}\frac{\beta_1}{\alpha^3} - B_{23}\frac{\beta_2}{\alpha^3} - 3B_{23}\frac{\beta_3}{\alpha^3}\right)k_{xyy} - \frac{n_3N_xa^2}{E_0t^3}\beta_1k_N \quad 30c$$

Minimizing equations 29a, 29b and 29c with respect to β_2 gives:

$$\frac{d\Pi_1}{d\beta_2} = \frac{ab}{a^4}\left[\left(A_{11}\beta_2k_x + \frac{1}{\alpha^2}[A_{12}\beta_3 + A_{33}\beta_2 + A_{33}\beta_3]k_{xy}\right) - B_{11}\beta_1k_x - (B_{12}+2B_{33})\frac{\beta_1}{\alpha^2}k_{xy}\right] = 0 \quad 31a$$

$$\frac{d\Pi_2}{d\beta_2} = \frac{ab}{a^4}\left[2A_{13}\frac{\beta_2}{\alpha}k_{xxy} + A_{13}\frac{\beta_3}{\alpha}k_{xxy} - 3B_{13}\frac{\beta_1}{\alpha}k_{xxy}\right] = 0 \quad 31b$$

$$\frac{d\Pi_3}{d\beta_2} = 0 = \frac{ab}{a^4}\left[A_{23}\frac{\beta_3}{\alpha^3}k_{xyy} - B_{23}\frac{\beta_1}{\alpha^3}k_{xyy}\right] \quad 31c$$

Minimizing equations 29a, 29b and 29c with respect to β_3 gives:

$$\frac{d\Pi_1}{d\beta_3} = \frac{ab}{a^4}\left[A_{22}\frac{\beta_3}{\alpha^4}k_y + \frac{1}{\alpha^2}[A_{12}\beta_2 + A_{33}\beta_2 + A_{33}\beta_3]k_{xy} - B_{22}\frac{\beta_1}{\alpha^4}k_y - (B_{12}+2B_{33})\frac{\beta_1}{\alpha^2}k_{xy}\right] = 0 \quad 32a$$

$$\frac{d\Pi_2}{d\beta_3} = 0 = \frac{ab}{a^4}\left[A_{13}\frac{\beta_2}{\alpha}k_{xxy} - B_{13}\frac{\beta_1}{\alpha}.k_{xxy}\right] \quad 32b$$

$$\frac{d\Pi_3}{d\beta_3} = 0 = \frac{ab}{a^4}\left[A_{23}\frac{\beta_2}{\alpha^3}k_{xyy} + 2A_{23}\frac{\beta_3}{\alpha^3}k_{xyy} - 3B_{23}\frac{\beta_1}{\alpha^3}k_{xyy}\right] \quad 32c$$

$$Where: k_x = \iint\left(\frac{\partial^2 h}{\partial R^2}\right)^2 dR\,.dQ\ :$$

$$k_{xy} = \iint\left(\frac{\partial^2 h}{\partial R\partial Q}\right)^2 dR\,.dQ\ :\ k_y = \iint\left(\frac{\partial^2 h}{\partial Q^2}\right)^2 dR\,.dQ$$

$$k_{xxy} = \iint\frac{\partial^2 h}{\partial R^2}\frac{\partial^2 h}{\partial R\partial Q}\ dR\,.dQ$$

$$k_{xyy} = \iint\frac{\partial^2 h}{\partial R\partial Q}.\frac{\partial^2 h}{\partial Q^2}\ dR\,.dQ\ :\ k_q = \iint h\ dR\,.dQ$$

$$k_N = \iint\left(\frac{dh}{dR}\right)^2 dR\,.dQ\ :\ k_\lambda = \iint h^2\ dR\,.dQ$$

Adding the equations 30a, 30b and 30c together and rearranging the outcome gives:

$$\frac{d\Pi}{d\beta_1} = \frac{d\Pi_1}{d\beta_1} + \frac{d\Pi_2}{d\beta_1} + \frac{d\Pi_3}{d\beta_1} = 0.\quad \textit{That is}:$$

$$\frac{d\Pi}{d\beta_1} = \beta_1\left(D_{11}k_x + \frac{2}{\alpha^2}(D_{12}+2D_{33})k_{xy} + \frac{D_{22}}{\alpha^4}k_y + 4\frac{D_{13}}{\alpha}k_{xxy} + 4\frac{D_{23}}{\alpha^3}k_{xyy}\right) - \beta_2\left(B_{11}k_x + (B_{12}+2B_{33})\frac{k_{xy}}{\alpha^2} + 3B_{13}\frac{k_{xxy}}{\alpha} + B_{23}\frac{k_{xyy}}{\alpha^3}\right) - \beta_3\left((B_{12}+2B_{33})\frac{k_{xy}}{\alpha^2} + B_{22}\frac{k_y}{\alpha^4} + B_{13}\frac{k_{xxy}}{\alpha} + 3B_{23}\frac{k_{xyy}}{\alpha^3}\right) - \frac{a^4}{E_0t^3}(n_1+n_2+n_3)\frac{N_x}{\alpha^2}\beta_1k_N \quad 33a$$

Substituting equation 18 into equation 33a and rearranging the outcome gives:

$$\frac{N_xa^2}{E_0t^3}\beta_1k_N = \beta_1\left(D_{11}k_x + \frac{2}{\alpha^2}(D_{12}+2D_{33})k_{xy} + \frac{D_{22}}{\alpha^4}k_y + 4\frac{D_{13}}{\alpha}k_{xxy} + 4\frac{D_{23}}{\alpha^3}k_{xyy}\right) - \beta_2\left(B_{11}k_x + (B_{12}+2B_{33})\frac{k_{xy}}{\alpha^2} + 3B_{13}\frac{k_{xxy}}{\alpha} + B_{23}\frac{k_{xyy}}{\alpha^3}\right) - \beta_3\left((B_{12}+2B_{33})\frac{k_{xy}}{\alpha^2} + B_{22}\frac{k_y}{\alpha^4} + B_{13}\frac{k_{xxy}}{\alpha} + 3B_{23}\frac{k_{xyy}}{\alpha^3}\right) \quad 33b$$

Adding the equations 31a, 31b and 31c together and rearranging the outcome gives:

$$\beta_2\left(A_{11}k_x + A_{33}\frac{k_{xy}}{\alpha^2} + 2A_{13}\frac{k_{xxy}}{\alpha}\right) + \beta_3\left(A_{12}\frac{k_{xy}}{\alpha^2} + A_{33}\frac{k_{xy}}{\alpha^2} + A_{13}\frac{k_{xxy}}{\alpha} + A_{23}\frac{k_{xyy}}{\alpha^3}\right) = \beta_1\left(B_{11}k_x + (B_{12}+2B_{33})\frac{k_{xy}}{\alpha^2} + 3B_{13}\frac{k_{xxy}}{\alpha} + B_{23}\frac{k_{xyy}}{\alpha^3}\right) \qquad 34a$$

Adding the equations 32a, 32b and 32c together and rearranging the outcome gives:

$$\beta_2\left(A_{12}\frac{k_{xy}}{\alpha^2} + A_{33}\frac{k_{xy}}{\alpha^2} + A_{13}\frac{k_{xxy}}{\alpha} + A_{23}\frac{k_{xyy}}{\alpha^3}\right) + \beta_3\left(A_{22}\frac{k_y}{\alpha^4} + A_{33}\frac{k_{xy}}{\alpha^2} + 2A_{23}\frac{k_{xyy}}{\alpha^3}\right) = \beta_1\left(B_{22}\frac{k_y}{\alpha^4} + (B_{12}+2B_{33})\frac{k_{xy}}{\alpha^2} + B_{13}\frac{k_{xxy}}{\alpha} + 3B_{23}\frac{k_{xyy}}{\alpha^3}\right) \qquad 34b$$

Solving equations 34a and 34b simultaneously gives:

$$\beta_2 = T_2\beta_1 = \beta_1\frac{(d_{12}.d_{23} - d_{13}.d_{22})}{({d_{12}}^2 - d_{11}d_{22})} \qquad 35a$$

$$\beta_3 = T_3\beta_1 = \beta_1\frac{(d_{12}.d_{13} - d_{11}d_{23})}{({d_{12}}^2 - d_{11}d_{22})} \qquad 35b$$

Where:

$$d_{11} = A_{11}k_x + A_{33}\frac{k_{xy}}{\alpha^2} + 2A_{13}\frac{k_{xxy}}{\alpha} \qquad 36a$$

$$d_{12} = A_{12}\frac{k_{xy}}{\alpha^2} + A_{33}\frac{k_{xy}}{\alpha^2} + A_{13}\frac{k_{xxy}}{\alpha} + A_{23}\frac{k_{xyy}}{\alpha^3} \qquad 36b$$

$$d_{22} = A_{22}\frac{k_y}{\alpha^4} + A_{33}\frac{k_{xy}}{\alpha^2} + 2A_{23}\frac{k_{xyy}}{\alpha^3} \qquad 36c$$

$$d_{13} = B_{11}k_x + (B_{12}+2B_{33})\frac{k_{xy}}{\alpha^2} + 3B_{13}\frac{k_{xxy}}{\alpha} + B_{23}\frac{k_{xyy}}{\alpha^3} \qquad 36d$$

$$d_{23} = B_{22}\frac{k_y}{\alpha^4} + (B_{12}+2B_{33})\frac{k_{xy}}{\alpha^2} + B_{13}\frac{k_{xxy}}{\alpha} + 3B_{23}\frac{k_{xyy}}{\alpha^3} \qquad 36e$$

Substituting equations 35a and 35b into equation 33b and rearranging gives:

Substituting equations 35a and 35b into equation 33b and rearranging gives:

$$k_N = E_0\left\{\left(\frac{D_{11}}{D_0}k_x + \frac{2}{\alpha^2}\left(\frac{D_{12}}{D_0} + 2\frac{D_{33}}{D_0}\right)k_{xy} + \frac{D_{22}}{D_0}.\frac{k_y}{\alpha^4} + 4\frac{D_{13}}{D_{22}}.\frac{k_{xxy}}{\alpha} + 4\frac{D_{23}}{D_{22}\alpha^3}k_{xyy}\right) - \frac{T_2}{D_0}\left(B_{11}k_x + \frac{(B_{12}+2B_{33})}{\alpha^2}k_{xy} + \frac{3B_{13}}{\alpha}k_{xxy} + \frac{B_{23}}{\alpha^3}k_{xyy}\right) - \frac{T_3}{D_0}\left(\frac{(B_{12}+2B_{33})}{\alpha^2}k_{xy} + \frac{B_{22}}{\alpha^4}k_y + \frac{B_{13}}{\alpha}k_{xxy} + \frac{3B_{23}}{\alpha^3}k_{xyy}\right)\right\} \qquad 37$$

Rearranging equations 37 gives:

$$\frac{N_x a^2}{D_0 t^3} = \frac{N_x a^2}{\overline{D_0}} = E_0\left(\frac{k_{T1} + k_{T2} + k_{T3}}{k_N}\right) \quad 38$$

Where:

$$k_{T1} = \left(\frac{D_{11}}{D_0}k_x + \frac{2}{\alpha^2}\left(\frac{D_{12}}{D_0} + 2\frac{D_{33}}{D_0}\right)k_{xy} + \frac{D_{22}}{D_0}.\frac{k_y}{\alpha^4} + 4\frac{D_{13}}{D_{22}}.\frac{k_{xxy}}{\alpha} + 4\frac{D_{23}}{D_{22}\alpha^3}k_{xyy}\right) \qquad 40c$$

$$k_{T2} = -\frac{T_2}{D_0}\left(B_{11}k_x + \frac{(B_{12}+2B_{33})}{\alpha^2}k_{xy} + \frac{3B_{13}}{\alpha}k_{xxy} + \frac{B_{23}}{\alpha^3}k_{xyy}\right) \qquad 40d$$

$$k_{T3} = -\frac{T_3}{D_0}\left(\frac{(B_{12}+2B_{33})}{\alpha^2}k_{xy} + \frac{B_{22}}{\alpha^4}k_y + \frac{B_{13}}{\alpha}k_{xxy} + \frac{3B_{23}}{\alpha^3}k_{xyy}\right) \qquad 40e$$

3.0 Numerical Example

A thin rectangular plate simply supported along all the four edges is made of four laminas that are laminated together. The laminas were arranged as 0/90/90/0. Some of the material properties include: G_{12}/E_2 = 0.5; V_{12} = 0.25; E_1/E_2 varies from 5 to 40. It is required to determine the deflection of the plate under uniformly distributed lateral load, critical buckling load under uniaxial in-plane load and fundamental natural frequency when the plate is undergoing free vibration. The reference elastic modulus, E_0 is taken to be E_2. Hence,

$$\frac{E_0 t^3}{qa^4}.\beta_1 = \frac{E_2 t^3}{qa^4}.\beta_1 = \frac{k_q}{k_{T1} + k_{T2} + k_{T3}} \qquad 39a$$

$$\frac{N_x a^2}{D_0 t^3} = \frac{N_x a^2}{\overline{D_0}} = \frac{N_x a^2}{\overline{D_2}} = E_2\left(\frac{k_{T1} + k_{T2} + k_{T3}}{k_N}\right) \qquad 40a$$

$$\frac{a^4 m\lambda^2}{D_0 t^3} = \frac{a^4 m\lambda^2}{\overline{D_0}} = \frac{a^4 m\lambda^2}{\overline{D_2}} = E_2\left(\frac{k_{T1} + k_{T2} + k_{T3}}{k_\lambda}\right) \qquad 40b$$

If the aspect ratio is a/b and the parameters are in terms of long length "b" then:

$$N_x b^2 = [N_x a^2] \times [b/a]^2$$

The exact deflection function for buckling analyses of SSSS plate after satisfying the boundary condition using equations 27a is:

$$w = A\, Sin\, m\pi R\, .\, Sin\, n\pi Q$$

The stiffness coefficients from the obtained using the polynomial and trigonometric deflection functions are shown on Table 1:

Table 1. Stiffness coefficients (k-vakues) for SSSS plate.

	k_x	k_{xy}	k_y	k_{xxy}	k_{xyy}	k_N
Trig	24.3523	24.3523	24.3223	0	0	2.4674

4.0 Results and Discussions

The result for buckling analysis were presented on Table 2 and Table 3 in terms of short and long lengths of the plate respectively. Results from the present study were compared with the results from the works of Reddy [2] and Osman and Suleiman [3]. The work of Reddy was based on Navier theory and Levy's theory; it is regarded as exact method because the shape function he used was an exact shape function for the SSSS plate while the work of Osman and Suleiman was based on shape function and finite element method of analysis which is an approximate method. These comparisons were presented on Table 4 to Table 8.

From Table 2 and Table 3 it was observed that the buckling load increase as the ratio of elastic modulus (E1/E2) increases. It was also seen from Table 2 that the buckling load increases as the aspect ratio (b/a) decreases. This means that the buckling coefficient is higher when the elastic modulus is constant and the ratio of the length in terms of the shorter span is high; The reverse is the case when considering the ratio of the length in terms of the longer span. The percentage differences recorded on Table 4 to Table 8 between the values from the present study and the values from Reddy, one would see consistency and very minimal difference. The observed maximum percentage difference is 0.06. This means that the values obtained herein is almost the same with the values obtained by Reddy (exact method). The recorded minimal difference may be as result of round off errors introduced along the computation line. The case is not the same with the comparison with the values from the work of Osman and Suleiman. A slightly higher percentage differences were recorded. The maximum percentage difference recorded is 1.81. This difference is expected because they used finite element method, which is known as approximate method.

Table 2. Critical buckling load in terms of short length, "a" from the present study, $\left[\frac{N_x a^2}{D_2}\right] \div \pi^2$

$0/90/90/0 \quad G_{12}/E_2 = 0.5; \quad \mu_{12} = 0.25$					
$\propto = b/a$	$E_1/E_2 = 5$	$E_1/E_2 = 10$	$E_1/E_2 = 20$	$E_1/E_2 = 25$	$E_1/E_2 = 40$
2/3	11.769	11.867	11.941	11.960	11.991
1	5.649	6.346	6.960	7.123	7.403
2	3.475	4.532	5.469	5.718	6.147

Table 3. Critical buckling load in terms of long length, "b" from the present study, $\left[\frac{N_x b^2}{D_2}\right] \div \pi^2$

$0/90/90/0 \quad G_{12}/E_2 = 0.5; \quad \mu_{12} = 0.25$					
$\propto^{-1} = a/b$	$E_1/E_2 = 5$	$E_1/E_2 = 10$	$E_1/E_2 = 20$	$E_1/E_2 = 25$	$E_1/E_2 = 40$
1.5	5.231	5.274	5.307	5.315	5.329
1	5.649	6.346	6.960	7.123	7.403
0.5	13.900	18.126	21.878	22.873	24.590

Table 4. Difference between values from the present study those from Reddy for $E_1/E_2 = 5$, $\left[\frac{N_x b^2}{D_2}\right] \div \pi^2$

p = a/b	Present	Reddy	% diff. with Reddy
1.5	5.231	5.233	0.05
1	5.649	5.650	0.02
0.5	13.900	13.900	0.00

Table 5. Difference between values from the present study those from Reddy and Osman for $E_1/E_2 = 10$, $\left[\frac{N_x b^2}{D_2}\right] \div \pi^2$

p = a/b	Present	Reddy	Osman	% diff. with Reddy	% diff. with Osman
1.5	5.274	5.277	5.215	0.06	1.13
1	6.346	6.347	6.274	0.02	1.14
0.5	18.126	18.126	17.958	0.00	0.94

Table 6. Difference between values from the present study those from Reddy for $E_1/E_2 = 20$, $\left[\frac{N_x b^2}{D_2}\right] \div \pi^2$

p = a/b	Present	Reddy	% diff. with Reddy
1.5	5.307	5.310	0.05
1	6.960	6.961	0.02
0.5	21.878	21.878	0.00

Table 7. Difference between values from the present study those from Reddy and Osman for $E_1/E_2 = 25$, $\left[\frac{N_x b^2}{D_2}\right] \div \pi^2$

p = a/b	Present	Reddy	Osman	% diff. with Reddy	% diff. with Osman
1.5	5.315	5.318	5.221	0.05	1.81
1	7.123	7.124	7.003	0.02	1.71
0.5	22.873	22.874	22.566	0.00	1.36

5.0 Results and Discussions

The Buckling Analysis of Thin Laminated Composite Plates with Exact Displacement Functions is carried out considering the total potential energy functional. The important thing about thee method is that it can be apply no matter the type of boundary condition under consideration.

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