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Transforming Metric on Knot Types

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ABSTRACT

Knot theory is a branch of topology which is important in studying of the 3 dimensional geometry. In this study, we have proved a theorem which is related to Knot theory. Also, using some known results, one theorem has been well developed and adapted to Knot Theory.

Introduction

Knot theory is a branch of topology. As an American mathematician, J.W. Alexander (1888-1971) has shown, Knot theory is extremely important in studying 3 dimensional topology. It is a geometrical investigation of placements of one topological space in a larger topological space. Here, the smaller space which represents the length of the rope is $\mathbb{R}^1$, the real line , and the bigger space which represent the universe about us is $\mathbb{R}^3$, a real 3 dimensional space.

Definition 1 (Knot) [4]

A closed curve in $\mathbb{R}^3$ which has no self-intersections is called a Knot.

Definition 2 (Unknotting number) [3]

The unknotting number of a regular diagram $\boldsymbol{D}$ is defined as a the minimum of the minimal number of crossing changes that are needed to transform the regular diagram D of the Knot $\boldsymbol{K}$ into the regular diagram of the trivial knot where minimum is taken all over the regular diagrams $\boldsymbol{D}$ of $\boldsymbol{K}$. The unknotting number of a regular diagram $\boldsymbol{D}$ is denoted by $\boldsymbol{u}(\boldsymbol{D})$.

Methodology

In our work, two theorems have been proved.

Definition 3 [3]

Let K be a knot and $\boldsymbol{\mathcal{D}}$ be the set of regular diagrams of $\boldsymbol{K}$. Then, the unknotting number of $\boldsymbol{K}$ is defined as follows,

$$\boldsymbol{u}(\boldsymbol{K}) = \min_{\boldsymbol{D}\in\boldsymbol{\mathcal{D}}}\{\boldsymbol{u}(\boldsymbol{D}); \boldsymbol{D} \textit{ is a regular diagram of } \boldsymbol{K}\}$$

Theorem 1 [3]

The unknotting number of a given knot $\boldsymbol{K}$ is an invariant of $\boldsymbol{K}$.

Proof.

Let $\boldsymbol{D_0}$, $\boldsymbol{K_1}$ and $\boldsymbol{D_1}$ be the minimal regular diagram of the knot $\boldsymbol{K}$, a knot which is equivalent to the knot $\boldsymbol{K}$ and the minimal regular diagram of $\boldsymbol{K_1}$ respectively.

By the definition of the unknotting number, it can be written as $\boldsymbol{u}(\boldsymbol{D_0}) \leq \boldsymbol{u}(\boldsymbol{D_1})$.

Since $\boldsymbol{D_0}$ is also a regular diagram of $\boldsymbol{K_1}$, it can be written as $\boldsymbol{u}(\boldsymbol{D_1}) \leq \boldsymbol{u}(\boldsymbol{D_0})$.

Therefore, $\boldsymbol{u}(\boldsymbol{D_0}) = \boldsymbol{u}(\boldsymbol{D_1})$.

Therefore, $\boldsymbol{u}(\boldsymbol{D_0})$ is the minimal of number of unknotting operations that are needed to transform any regular diagram $\boldsymbol{D}$ of $\boldsymbol{K}$ into the regular diagram of the trivial knot for all knots which are equivalent to $\boldsymbol{K}$.

Theorem 2 (Transforming metric on knot types) [1] [2] [3] [5]

Let X be the set of all the knots in $\mathbb{R}^3$. Then, $\boldsymbol{u}: \boldsymbol{X} \times \boldsymbol{X} \to \mathbb{R}$ is metric on the underlying set $\boldsymbol{X}$ is defined by

$\boldsymbol{u}(\boldsymbol{K}, \boldsymbol{K'}) = \min_{\boldsymbol{D}\in\boldsymbol{\mathcal{D}}, \boldsymbol{D'}\in\boldsymbol{\mathcal{D}'}}\{\boldsymbol{u}(\boldsymbol{D}, \boldsymbol{D'})\}$ = The minimal number of unknotting operations which are needed to transform the knot $\boldsymbol{K}$ into the knot $\boldsymbol{K'}$ where $\boldsymbol{\mathcal{D}}, \boldsymbol{\mathcal{D}'}$ are sets of regular diagrams of $\boldsymbol{K}, \boldsymbol{K'}$ respectively.

Proof.

It should be shown that the following four properties hold for $\boldsymbol{u}(\boldsymbol{K}, \boldsymbol{K'})$ in order to prove the above theorem. i.e.

1. $\boldsymbol{u}(\boldsymbol{K}, \boldsymbol{K'}) \geq \boldsymbol{0}$,
2. $\boldsymbol{u}(\boldsymbol{K}, \boldsymbol{K'}) = 0$ if and only if $\boldsymbol{K} = \boldsymbol{K'}$.
3. $\boldsymbol{u}(\boldsymbol{K}, \boldsymbol{K'}) = \boldsymbol{u}(\boldsymbol{K'}, \boldsymbol{K})$.
4. $\boldsymbol{u}(\boldsymbol{K}, \boldsymbol{K'}) \leq \boldsymbol{u}(\boldsymbol{K}, \boldsymbol{K_0}) + \boldsymbol{u}(\boldsymbol{K_0}, \boldsymbol{K'})$ for any $\boldsymbol{K_0}, \boldsymbol{K}, \boldsymbol{K'} \in \boldsymbol{X}$.

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1. Using the following Murusugi's generalized signature lower bound for transforming metric on knots. i.e. using $\boldsymbol{u(K_1,K_2)} \geq \frac{1}{2}|\boldsymbol{\sigma(K_1) - \sigma(K_2)}|$ [1]

Now, $\boldsymbol{u(K,K')} \geq \frac{1}{2}|\boldsymbol{\sigma(K) - \sigma(K')}|$

Since, $|\boldsymbol{\sigma(K) - \sigma(K')}| \geq \boldsymbol{0}$, we have $\boldsymbol{u(K,K')} \geq \boldsymbol{0}$.

2. $\boldsymbol{u(K,K')} = 0 \Leftrightarrow \boldsymbol{0} \geq \frac{1}{2}|\boldsymbol{\sigma(K) - \sigma(K')}| \geq \boldsymbol{0}$

$\Leftrightarrow |\boldsymbol{\sigma(K) - \sigma(K')}| = \boldsymbol{0}$

$\Leftrightarrow \boldsymbol{\sigma(K) - \sigma(K')} = \boldsymbol{0}$

$\Leftrightarrow \boldsymbol{\sigma(K) = \sigma(K')}$

$\Leftrightarrow \boldsymbol{K = K'}$

3. $\boldsymbol{u(K,K')}$ = minimum number of unknotting operations which are needed to transform $\boldsymbol{K}$ and $\boldsymbol{K'}$ where the minimum is taken over all diagrams.

$$= \min_{D\in\mathcal{D}, D'\in\mathcal{D}'} \{\boldsymbol{u(D,D')}\}$$

$$= \min_{D\in\mathcal{D}, D'\in\mathcal{D}'} \{\boldsymbol{u(D',D)}\}$$

= minimum number of unknotting operations which are needed to transform $\boldsymbol{K'}$ and $\boldsymbol{K}$ where the minimum is taken over all diagrams.

= $\boldsymbol{u(K',K)}$

4. For any three knots $\boldsymbol{K_0, K, K'} \in \boldsymbol{X}$,

$\boldsymbol{u(K,K_0)} \geq \frac{1}{2}|\boldsymbol{\sigma(K) - \sigma(K_0)}|$ and $\boldsymbol{u(K_0,K')} \geq \frac{1}{2}|\boldsymbol{\sigma(K_0) - \sigma(K')}|$.

Then, $\boldsymbol{u(K,K_0)} + \boldsymbol{u(K_0,K')} \geq \frac{1}{2}|\boldsymbol{\sigma(K) - \sigma(K_0)}| + \frac{1}{2}|\boldsymbol{\sigma(K_0) - \sigma(K')}|$

$$= \frac{1}{2}\{|\boldsymbol{\sigma(K) - \sigma(K_0)}| + |\boldsymbol{\sigma(K_0) - \sigma(K')}|\}$$

$$\geq |\boldsymbol{\sigma(K) - \sigma(K_0) + \sigma(K_0) - \sigma(K')}|$$

$$= |\boldsymbol{\sigma(K) - \sigma(K')}|$$

$$= \boldsymbol{u(K,K')}$$

i.e. $\boldsymbol{u(K,K')} \leq \boldsymbol{u(K,K_0)} + \boldsymbol{u(K_0,K')}$ for any $\boldsymbol{K_0, K, K'} \in \boldsymbol{X}$.

Therefore, the four properties of a metric are satisfied.

So, $\boldsymbol{u}$ is a metric on *X* and $(\boldsymbol{u, X})$ is the metric space.

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References

[1] H. Murakami, "Some metrics on classical knots", Math. Ann. 270(1985), pp 35-45

[2] I.K. Darcy, and D.W. Sumners, "Applications of Topology to DNA", Knot Theory, Banach Center Publications, vol. 42, pp. 65-75, 1998.

[3] Kunio Murasugi, "Knot Theory and Its Applications", 1st ed., ISBN 3-7643-3817-2, Boston, 1996, pp 31-68.

[4] Sieradski, Allean J., "An introduction to topology and homotopy", 1st ed., ISBN 0-534-92960-5, PWS-KENT Publishing Company, Boston, 1992, pp 364 -370

[5] Munkres, James, "Topology", 2nd ed., ISBN 978-93-325-4953-1, Pearson India Education Ltd., India, 2015,pp 117-118.