Hamiltonian Mechanics systems with Three Para- Complex Structures on Para-complex Geometry

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ABSTRACT

In this paper we presented an analysis of Hamilton formulas .with Three Almost Complex Structures. We have reached important results in differential geometry that can be applied in theoretical physics.

1. Introduction

The theory of manifolds is very important and branch in differential geometry since it has many important application in physics and some other branches of sciences

The geometric study of dynamical systems is an important chapter of contemporary mathematics due to its applications in Mechanics, Theoretical Physics. If M is a differential manifold that corresponds to the configuration space, a dynamical system can be locally given by a system of ordinary differential equations of the form $\dot{x}^i = f^i(t; x)$,of evolution. Globally, a dynamical system is given by a vector field X on the manifold $M \times R$ whose integral curves, $c(t)$ are given by the equations of evolution, $X \circ c(t) = \dot{c}(t)$.The theory of dynamical systems deals with the integration of such systems .

One of the most important papers on the topic entitled Mechanical Equations with Two Almost Complex Structures on Symplectic Geometry It has been used in this paper using two complex structures, examined mechanical systems on symplectic geometry.

In this paper, we study dynamical systems with Three Almost Complex Structures . After Introduction in Section 1, we consider Historical Background paper basic . Section 2 deals with the study Almost Complex Structures. Section 3 is devoted to study Lagrangian Dynamics .Section 4 is devoted to study Hamiltonian Dynamics .

2. Preliminaries

In this section we introduce the concept of Para-complex structure and study its definitions and Theorems properties. We start by the following definition

Definition 2.1 Almost Para-Complex

Let $\mathcal{M}$ be a Para-complex manifold of Para-complex dimension n and denote by $(\mathcal{M}, I)$ the manifold considered as a real 2n-dimensional manifold with the induced almost Para-complex structure I.

Definition 2.2 [6]

A Para-complex Riemannian metric on $\mathcal{M}$ is a covariant symmetric 2-tensor field $G : A(T\mathcal{M}) \times A(T\mathcal{M}) \to C$ which is non-degenerate at each point of M and satisfies

$$G(\overline{z_1}, \overline{z_2}) = \overline{G(z_1, z_2)} \quad , z_1, z_2 \in A(T\mathcal{M})$$
$$G(z_1, z_2) = 0, \quad z_1 \in A(T\mathcal{M}), \quad z_2 \in A(T\mathcal{M}) \quad (1)$$

the relation (1) is equivalent to

$$G(Iz_1, Iz_2) = G(z_1, z_2) \quad , \quad z_1, z_2 \in A(T\mathcal{M})$$

Definition 2.3

1-Let $\mathcal{M}$ be configuration manifold of real dimension 2m ,A tensor field I on $T\mathcal{M}$ Is called almost Para -complex manifold .such that $I^2 = 1$.

2- She holomorphic structures on cotangent space for Hamilton equations, I^* stands for the dual endomorphism of cotangent space $T^*(T\mathcal{M})$ of manifold $T\mathcal{M}$ satisfying $I^{*2} = I^* \circ I^* = 1$

Definition 2.4 [3]

An almost Para -complex structure on $\mathcal{M}$ $\mathcal{M}$amanifold is a differentiable map$I: T\mathcal{M} \to T\mathcal{M}$ on the tangent bundle $T\mathcal{M}$ of $\mathcal{M}$such that I preserves each fiber ,A manifold with affixed almost para -complex structure is called an almost para -complex manifold

Definition 2.5

Let Para complex manifold . In local holomorphic coordinates$\{x_1, x_2, x_3, x_4, x_5, x_6\}$one can define the tangent space $T(T\mathcal{M})$and$T\mathcal{M}$,and$\left\{\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}, \frac{\partial}{\partial x_4}, \frac{\partial}{\partial x_5}, \frac{\partial}{\partial x_6}V\right\}$and cotangent space$T^*(T\mathcal{M})$and $T\mathcal{M}$and$\{dx_1, dx_2, dx_3, dx_4\}$. respectively .then I ,as denoted

$$I\left(\frac{\partial}{\partial x_1}\right) = \frac{\partial}{\partial x_2} \qquad , \quad I\left(\frac{\partial}{\partial x_2}\right) = \frac{\partial}{\partial x_1}$$
$$I\left(\frac{\partial}{\partial x_3}\right) = \frac{\partial}{\partial x_4} \qquad , \quad I\left(\frac{\partial}{\partial x_4}\right) = \frac{\partial}{\partial x_3}$$
$$I\left(\frac{\partial}{\partial x_5}\right) = \frac{\partial}{\partial x_6} \qquad , \quad I\left(\frac{\partial}{\partial x_6}\right) = \frac{\partial}{\partial x_5}(2)$$

And

$$I(dx_1) = dx_2 \qquad , \qquad I(dx_2) = dx_1$$
$$I(dx_3) = dx_4 \qquad , \qquad I(dx_4) = dx_3$$
$$I(dx_5) = dx_6 \qquad , \qquad I(dx_6) = dx_5(3)$$

Theorem 2.6

Suppose that $\{x_1, x_2, x_3, x_4, x_5, x_6\}$,be a real coordinate system on$(\mathcal{M}, J)$. Then we denote by

$$\left\{\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}, \frac{\partial}{\partial x_4}, \frac{\partial}{\partial x_5}, \frac{\partial}{\partial x_6}\right\}$$
$$\{dx_1, dx_2, dx_3, dx_4, dx_5, dx_6\}$$
$$J\left(\frac{\partial}{\partial x_1}\right) = \frac{\partial}{\partial x_2} = j\left(\frac{\partial}{\partial x_1}\right) , \qquad J\left(\frac{\partial}{\partial x_2}\right) = \frac{\partial}{\partial x_1} = j\left(\frac{\partial}{\partial x_2}\right)$$
$$J\left(\frac{\partial}{\partial x_3}\right) = \frac{\partial}{\partial x_4} = j\left(\frac{\partial}{\partial x_3}\right) , \qquad J\left(\frac{\partial}{\partial x_4}\right) = \frac{\partial}{\partial x_3} = j\left(\frac{\partial}{\partial x_4}\right)$$
$$J\left(\frac{\partial}{\partial x_5}\right) = \frac{\partial}{\partial x_6} = j\left(\frac{\partial}{\partial x_5}\right) , \quad J\left(\frac{\partial}{\partial x_6}\right) = \frac{\partial}{\partial x_5} = j\left(\frac{\partial}{\partial x_6}\right)(4)$$

And the dual convector fields

$$I(dx_1) = dx_2 = j(dx_1) \qquad , \qquad I(dx_2) = dx_1 = j(dx_2)$$
$$I(dx_3) = dx_4 = j(dx_3) \qquad , \qquad I(dx_4) = dx_3 = j(dx_4)$$
$$I(dx_5) = dx_6 = j(dx_5) \qquad , \qquad I(dx_6) = dx_5 = j(dx_6)$$

Definition 2.7

Let$Z_1 = x_1 + ix_2$, $Z_2 = x_3 + ix_4$,$Z_3 = x_5 + ix_6$,$i^2 = -1$,be Para complex manifold .In local coordinates system on a neighborhood V of $T\mathcal{M}$.We define the vector fields by

$$I\left(\frac{\partial}{\partial Z_1}\right) = \frac{1}{2}\left(\frac{\partial}{\partial x_1} - i\frac{\partial}{\partial x_2}\right) \qquad , \quad I\left(\frac{\partial}{\partial Z_2}\right) = \frac{1}{2}\left(\frac{\partial}{\partial x_3} - i\frac{\partial}{\partial x_4}\right)$$
$$I\left(\frac{\partial}{\partial \bar{Z}_1}\right) = \frac{1}{2}\left(\frac{\partial}{\partial x_1} + i\frac{\partial}{\partial x_2}\right) \qquad , \quad I\left(\frac{\partial}{\partial \bar{Z}_2}\right) = \frac{1}{2}\left(\frac{\partial}{\partial x_3} + i\frac{\partial}{\partial x_4}\right)$$
$$I\left(\frac{\partial}{\partial Z_3}\right) = \frac{1}{2}\left(\frac{\partial}{\partial x_5} - i\frac{\partial}{\partial x_6}\right) \qquad , I\left(\frac{\partial}{\partial \bar{Z}_3}\right) = \frac{1}{2}\left(\frac{\partial}{\partial x_5} + i\frac{\partial}{\partial x_6}\right)$$

And the dual convector fields

$$I^*(dZ_1) = \frac{1}{2}(dx_1 - i dx_2) \qquad , \quad I^*(dZ_2) = \frac{1}{2}(dx_3 - i dx_4)$$
$$I^*(d\bar{Z}_1) = \frac{1}{2}(dx_1 + i dx_2) \qquad , \quad I^*(d\bar{Z}_2) = \frac{1}{2}(dx_3 + i dx_4)$$
$$I^*(dZ_3) = \frac{1}{2}(dx_5 - i dx_6) \qquad , \quad I^*(d\bar{Z}_3) = \frac{1}{2}(dx_5 + i dx_6)$$

Theorem 2.8

the dual endomorphism of cotangent space$T^*(T\mathcal{M})$of manifold $T\mathcal{M}$satisfying$I^{*2} = 1$andbases cotangent space$T^*(T\mathcal{M})\{x_1, x_2, x_3, x_4, x_5, x_6\}$,and is defined by

$$I^{*2}(dx_1) = dx_2 \quad , \quad I^{*2}(dx_2) = dx_1$$
$$I^{*2}(dx_3) = dx_4 \quad , \quad I^{*2}(dx_4) = dx_3$$
$$I^{*2}(dx_5) = dx_6 \quad , \quad I^{*2}(dx_6) = dx_5 \quad (5)$$

Proof

$$I^{*2}(dx_1) = I^*(I^* dx_1) = I^*(dx_2) = dx_1$$
$$I^{*2}(dx_2) = I^*(I^* dx_2) = I^*(dx_1) = dx_2$$
$$I^{*2}(dx_3) = I^*(I^* dx_3) = I^*(dx_4) = dx_3$$
$$I^{*2}(dx_4) = I^*(I^* dx_4) = I^*(dx_3) = dx_4$$
$$I^{*2}(dx_5) = I^*(I^* dx_5) = I^*(dx_6) = dx_5$$

$$I^{*2}(dx_6) = I^*(I^* dx_6) = I^*(dx_5) = dx_6$$

3. Hamiltonian Dynamical Systems

In this section we introduce the concept of Hamiltonian Dynamical Systems. We start by the following definition.

Definition 3.1 [5]

A Hamiltonian function for a Hamiltonian vector field X on $\mathcal{M}$ is a smooth function $H : \mathcal{M} \rightarrow R$ such that

$$i_{X_H}\omega = dH \quad (6)$$

Definition 3.2[4]

A Hamiltonian system is a triple $(M; \omega; H)$, where $(\omega; H)$ is a Symplectic manifold and $H \in C^\infty(M)$ is a function, called the Hamiltonian function.

Suppose that an almost real structure, a Liouville form and 1-form on $T^*\mathcal{M}$ are shown by Φ^*, λ and ω respectively. Then we have

$$\omega = \frac{1}{2}(x_1 dx_1 - x_2 dx_2 + x_3 dx_3 - x_4 dx_4 + x_5 dx_5 - x_6 dx_6)^{(7)}$$

And

$$\lambda = \frac{1}{2}\left(x_1 J^*(dx_1) + x_2 J^*(dx_2) + x_3 J^*(dx_3) + x_4 J^*(dx_4) + x_5 J^*(dx_5) + x_6 J^*(dx_6) + x_7 J^*(dx_7) + x_8 J^*(dx_8)\right) \quad (13)$$

We substitute equation (6) in equation (7) we get

$$\lambda = \Phi^*(\omega) = \frac{1}{2}[-x_1 dx_2 + x_2 dx_1 - x_3 dx_4 + x_4 dx_3 - x_5 dx_6 + x_6 dx_5]$$

differential of λ

$$\phi = -d\lambda =$$

$$= -d\frac{1}{2}[-x_1 dx_2 + x_2 dx_1 - x_3 dx_4 + x_4 dx_3 - x_5 dx_6 + x_6 dx_5]$$

It is known that if ϕ is a closed 2- form on $T^*\mathcal{M}$ then ϕ_H is also a symplectic structure on $T^*\mathcal{M}$.

$$X_H = jX^1\frac{\partial}{\partial x_1} + jX^2\frac{\partial}{\partial x_2} + jX^3\frac{\partial}{\partial x_3} + jX^4\frac{\partial}{\partial x_4} + jX^5\frac{\partial}{\partial x_5} + jX^6\frac{\partial}{\partial x_6}(8)$$

Calculates a value X_H and ϕ

$$i_{X_H}\phi = \phi(X_H) = (dx_2 \wedge dx_1 + dx_4 \wedge dx_3 + dx_6 \wedge dx_5)\left(jX^1\frac{\partial}{\partial x_1} + jX^2\frac{\partial}{\partial x_2} + jX^3\frac{\partial}{\partial x_3} + jX^4\frac{\partial}{\partial x_4} + jX^5\frac{\partial}{\partial x_5} + jX^6\frac{\partial}{\partial x_6}\right)$$

$$i_{X_H}\phi = -X^1 dx_2 + X^2 dx_1 - X^3 dx_4 + X^4 dx_3 - X^5 dx_6 + X^6 dx_5(9)$$

So we find that

$$X^1 = \frac{\partial}{\partial x_2} \quad , \quad X^2 = -\frac{\partial}{\partial x_1} \quad , X^3 = \frac{\partial}{\partial x_4} \quad , X^4 = -\frac{\partial}{\partial x_3} \quad , X^5 = \frac{\partial}{\partial x_6} \quad , X^6 = -\frac{\partial}{\partial x_5}$$

Moreover, the differential of Hamiltonian energy is written as follows:

$$dH = -\frac{\partial H}{\partial x_2}\frac{\partial}{\partial x_1} + \frac{\partial H}{\partial x_1}\frac{\partial}{\partial x_2} - \frac{\partial H}{\partial x_4}\frac{\partial}{\partial x_3} + \frac{\partial H}{\partial x_3}\frac{\partial}{\partial x_4} - \frac{\partial H}{\partial x_6}\frac{\partial}{\partial x_5} + \frac{\partial H}{\partial x_5}\frac{\partial}{\partial x_6}(10)$$

Suppose that a curve

$$\alpha: I \subset R \rightarrow T^*\mathcal{M} = R^{2n}$$

is an integral curve of the Hamiltonian vector field X_H, i.e.,

$$X_H(\alpha(t)) = \frac{d\alpha(t)}{dt} \quad , \quad t \in I.$$

In the local coordinates, if it is considered to be

$$\alpha(t) = (x_1(t), x_2(t), x_2(t), x_4(t), x_5(t), x_6(t))$$

we obtain

$$\frac{d\alpha(t)}{dt} = \frac{dx_1}{dt}\frac{\partial}{\partial x_1} + \frac{dx_2}{dt}\frac{\partial}{\partial x_2} + \frac{dx_3}{dt}\frac{\partial}{\partial x_3} + \frac{dx_4}{dt}\frac{\partial}{\partial x_4} + \frac{dx_5}{dt}\frac{\partial}{\partial x_5} + \frac{dx_6}{dt}\frac{\partial}{\partial x_6}(11)$$

Taking the equation(9) = the equation (11)

$$j\frac{\partial H}{\partial x_2}\frac{\partial}{\partial x_1} - j\frac{\partial H}{\partial x_1}\frac{\partial}{\partial x_2} + j\frac{\partial H}{\partial x_4}\frac{\partial}{\partial x_3} - j\frac{\partial H}{\partial x_3}\frac{\partial}{\partial x_4} + j\frac{\partial H}{\partial x_6}\frac{\partial}{\partial x_5} - j\frac{\partial H}{\partial x_5}\frac{\partial}{\partial x_6}$$

$$= \frac{dx_1}{dt}\frac{\partial}{\partial x_1} + \frac{dx_2}{dt}\frac{\partial}{\partial x_2} + \frac{dx_3}{dt}\frac{\partial}{\partial x_3} + \frac{dx_4}{dt}\frac{\partial}{\partial x_4} + \frac{dx_5}{dt}\frac{\partial}{\partial x_5} + \frac{dx_6}{dt}\frac{\partial}{\partial x_6}$$

By comparing the two sides of the equation we get the

$$j\frac{\partial H}{\partial x_2}\frac{\partial}{\partial x_1} = \frac{dx_1}{dt}\frac{\partial}{\partial x_1} \quad \Rightarrow \quad j\frac{\partial H}{\partial x_2} = \frac{dx_1}{dt}$$

$$-j\frac{\partial H}{\partial x_1}\frac{\partial}{\partial x_2} = \frac{dx_2}{dt}\frac{\partial}{\partial x_2} \quad \Rightarrow \quad -j\frac{\partial H}{\partial x_1} = \frac{dx_2}{dt}$$

$$j\frac{\partial H}{\partial x_4}\frac{\partial}{\partial x_3} = \frac{dx_3}{dt}\frac{\partial}{\partial x_3} \quad \Rightarrow \quad j\frac{\partial H}{\partial x_4} = \frac{dx_3}{dt}$$

$$-j\frac{\partial H}{\partial x_3}\frac{\partial}{\partial x_4} = \frac{dx_4}{dt}\frac{\partial}{\partial x_4} \quad \Rightarrow \quad -j\frac{\partial H}{\partial x_3} = \frac{dx_4}{dt}$$

$$j\frac{\partial H}{\partial x_6}\frac{\partial}{\partial x_5} = \frac{dx_5}{dt}\frac{\partial}{\partial x_5} \quad \Rightarrow \quad j\frac{\partial H}{\partial x_6} = \frac{dx_5}{dt}$$

$$-\mathbf{j}\frac{\partial \mathbf{H}}{\partial \mathbf{x_5}}\frac{\partial}{\partial \mathbf{x_6}}=\frac{\mathbf{dx_6}}{\mathbf{dt}}\frac{\partial}{\partial \mathbf{x_6}} \quad \Rightarrow \quad -\mathbf{j}\frac{\partial \mathbf{H}}{\partial \mathbf{x_5}}=\frac{\mathbf{dx_6}}{\mathbf{dt}}$$

Thus Hamilton's equations are

$$\mathbf{j}\frac{\partial \mathbf{H}}{\partial \mathbf{x_2}}=\frac{\mathbf{dx_1}}{\mathbf{dt}} \quad , \quad -\mathbf{j}\frac{\partial \mathbf{H}}{\partial \mathbf{x_1}}=\frac{\mathbf{dx_2}}{\mathbf{dt}} \quad ,\mathbf{j}\frac{\partial \mathbf{H}}{\partial \mathbf{x_4}}=\frac{\mathbf{dx_3}}{\mathbf{dt}}$$

$$-\mathbf{j}\frac{\partial \mathbf{H}}{\partial \mathbf{x_3}}=\frac{\mathbf{dx_4}}{\mathbf{dt}} \quad , \quad \mathbf{j}\frac{\partial \mathbf{H}}{\partial \mathbf{x_6}}=\frac{\mathbf{dx_5}}{\mathbf{dt}} \quad , \quad -\mathbf{j}\frac{\partial \mathbf{H}}{\partial \mathbf{x_5}}=\frac{\mathbf{dx_6}}{\mathbf{dt}}(12)$$

Hence the triple$(\boldsymbol{\mathcal{M}}, \boldsymbol{\phi}, \mathbf{X_H})$is shown to be a Hamiltonian mechanical system which are deduced by means of Para complex structure$j^*$and using of basis$\left\{\frac{\partial}{\partial \mathbf{x_i}}: \mathbf{i}=\mathbf{1,2,3,4,5,6}\right\}$on the distributions$\mathbf{T}^*\boldsymbol{\mathcal{M}}$.

4. Conclusions

Thus, equations of Hamiltonian equations (12) with Three Para-Complex Structures.

References

[1] OguzhanCelik and ZekiKasap, Mechanical Equations with Two Almost
Complex Structures on Symplectic Geometry, April 28, 2016.

[2]Mehmet Mekkoyun and Murat Sari- Bi-Para Mechanical Equations on Lagrangian Distributions -arXivo:0902. v1 [math .Ph] 8 Jan 2009 .

[3] Mehmet Mekkoyun and Murat Sari – Constrained paracomplex Mechanical Equations -arXivo:0902-41210. v1 [math.DS] 24 Feb 2009.

[4] ZekiKasap and Mehmet Mekkoyun , Mechanical Equations on Bi-Para Conformal –arXivo1209,3101. V2 [math. GM] 22 Sep 2010.

[5] Mehmet Tekkoyun , On Para –Euler Lagrange and Para Hamiltonian Equations , physics letters A0 340 (2005) pp7-11.

[6] http//en. Wikipedia.org / wiki / almost complex structure.

[7] Cristian Ida, AlexandruIonescu and AdelinaManea, A note on para-holomorphic Riemannian Einstein manifolds, arXiv:1507.01114v2 [math.DG] 19 May 2016.