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Differentiation and Integration of Shehu Transformation

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ABSTRACT
In 2019, Matiama presented a new integral transformation called Shehu transformation, which used to solve many types of differential equations. In this article, we obtained the derivation and integration of Shehu transformation with its applications.

Introduction

In success for using Integral transformation was for quite much two centuries in finding several transformations in mathematical, physics, and branch of knowledge. The origin of the Integral transformations is Laplace which return to celebrated work of P.S. Laplace (1749 – 1827) on applied mathematical in the 1780[1,2,3,4].

Sumudu transform is similar to Laplace transform, but the first used to solve differential equations with variable coefficient. We note that the result of solutions for initial value problems that used integral transform represent particular solutions[5,6,7].

The Shehu transform is a new transformation emerged in 2019 and used to solve some types of differential equations. Also, it can be applied in some fields like physics, engineering[8,10] .

In this paper, [9] the Shehu transform is denoted by an operator $\mathbb{s}$ and defined as:

The Shehu transform of the function $\mathbf{v(t)}$ of exponential order is over the set of functions

$$\mathbf{A} = \{\mathbf{u(t)}: \exists\, \mathbf{N}, \eta_1, \eta_2 > 0, |\mathbf{u(t)}| < \mathbf{N} \exp(\frac{|\mathbf{t}|}{\eta_i}), \text{if } \mathbf{t} \in (-1)^{\mathbf{i}} \times [0,\infty)\}$$

By the following integral:

$$\mathbb{s}[\mathbf{u(t)}] = \mathbf{U}(\mathfrak{b},\kappa) = \int_0^\infty \exp\left(\frac{-\mathfrak{b}\mathbf{t}}{\kappa}\right)\mathbf{u(t)dt}$$
$$= \lim_{\alpha\to\infty}\int_0^\alpha \exp\left(\frac{-\mathfrak{b}\mathbf{t}}{\kappa}\right)\mathbf{u(t)dt}\,;\quad \mathfrak{b} > 0, \kappa > 0 \qquad (1.1)$$

It converges if the limit of the integral exists, and diverges if not.
The inverse Shehu transform is given by *:*

$$\mathbb{s}^{-1}[\mathbf{U}(\mathfrak{b},\kappa)] = \mathbf{u(t)},\ \text{for } \mathbf{t} \ge 0 \qquad (1.2)$$
$$u(t) = \mathbb{s}^{-1}[U(\mathfrak{b},\kappa)] = \frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty}\frac{1}{u}\exp\left(\frac{\mathfrak{b}t}{\kappa}\right)U(\mathfrak{b},\kappa) \qquad (1.3)$$

Where s and u are the Shehu transform variables, and α is a real constant and the integral in equation (1.3) is taken along $\mathfrak{b} = \alpha$ in the complex plane $\mathfrak{b} = x + iy$.

Theorem (1). Derivative of Shehu transform [9].

If the function $u^{(n)}(t)$ is the $n-th$ derivative of the function $u(t) \in A$ with respect to t then its Shehu transform is defined by:

$$\mathbb{s}\left[u^{(n)}(t)\right] = \frac{\mathfrak{b}^n}{\kappa^n}U(\mathfrak{b},\kappa) - \sum_{k=0}^{n-1}\left(\frac{\mathfrak{b}}{\kappa}\right)^{n-(k+1)} u^k(0). \qquad (1.1)$$

When n=1, 2, and 3 in equation (1.1) above, we obtain the following derivatives with respect to t

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$\mathbb{s}[u'(t)] = \frac{\mathrm{b}}{\kappa} U(\mathrm{b},\kappa) - u(0)$ (1.2)

$\mathbb{s}[u''(t)] = \frac{\mathrm{b}^2}{\kappa^2} U(\mathrm{b},\kappa) - \frac{\mathrm{b}}{\kappa} u(0).$ (1.3)

$\mathbb{s}[u'''(t)] = \frac{\mathrm{b}^3}{\kappa^3} U(\mathrm{b},\kappa) - \frac{\mathrm{b}^2}{\kappa^2} u(0) - \frac{\mathrm{b}}{\kappa} u'(0) - u''(0)$ (1.4)

Proof. Now suppose equation (1.1) is true for $n = k$ then using equation (1.2) and the induction hypothesis, we deduce $\mathbb{s}\left[\left(u^{(k)}(t)\right)'\right] = \frac{\mathrm{b}}{\kappa}\mathbb{s}[u^k(t)] - u^k(0).$

$$= \frac{\mathrm{b}}{\kappa}\left[\frac{\mathrm{b}^k}{\kappa^k}\mathbb{s}[u(t)] - \sum_{i=o}^{k-1}\left(\frac{\mathrm{b}}{\kappa}\right)^{k-(i+1)} u^{(i)}(0)\right] - u(0).$$

$$= \left(\frac{\mathrm{b}}{\kappa}\right)^{k+1}\mathbb{s}[u(t)] - \sum_{i=0}^{k}\left(\frac{\mathrm{b}}{\kappa}\right)^{k-i} u^{i}(0)$$

Which implies that equation (1.1) holds for $n = k + 1$, by induction hypothesis the proof is complete.

The following important properties are obtain using the Leibniz's rule

$$\mathbb{s}\left[\frac{\partial u(x,t)}{\partial x}\right] = \int_0^\infty exp\left(\frac{-\mathrm{b}t}{\kappa}\right)\frac{\partial u(x,t)}{\partial x}dt = \frac{\partial}{\partial x}\int_0^\infty exp\left(\frac{-\mathrm{b}t}{\kappa}\right)u(x,t)dt = \frac{\partial}{\partial x}[U(x,\mathrm{b},\kappa)] = \frac{d}{dx}[U(x,\mathrm{b},\kappa)].$$

$$\mathbb{s}\left[\frac{\partial^2 u(x,t)}{\partial x^2}\right] = \int_0^\infty exp\left(\frac{-\mathrm{b}t}{\kappa}\right)\frac{\partial^2 u(x,t)}{\partial x^2}dt = \frac{\partial^2}{\partial x^2}\int_0^\infty exp\left(\frac{-\mathrm{b}t}{\kappa}\right)u(x,t)dt = \frac{\partial^2}{\partial x^2}[U(x,\mathrm{b},\kappa)] = \frac{d^2}{dx^2}[U(x,\mathrm{b},\kappa)]$$

$$\mathbb{s}\left[\frac{\partial^n u(x,t)}{\partial x^n}\right] = \int_0^\infty exp\left(\frac{-\mathrm{b}t}{\kappa}\right)\frac{\partial^n u(x,t)}{\partial x^n}dt = \frac{\partial^n}{\partial x^n}\int_0^\infty exp\left(\frac{-\mathrm{b}t}{\kappa}\right)u(x,t)dt = \frac{\partial^n}{\partial x^n}[U(x,\mathrm{b},\kappa)] = \frac{d^n}{dx^n}[U(x,\mathrm{b},\kappa)]$$

Table 1. The following table showed the Shehu Transform for some functions such as.

S.No.	$u(t)$	$\mathbb{s}\,[u(t)]$
1	1	$\frac{\kappa}{\mathrm{b}}$
2	t	$\frac{\kappa^2}{\mathrm{b}}$
3	$\exp(\propto (t))$	$\frac{\kappa}{\mathrm{b}-\propto\kappa}$
4	$\sin(\propto t)$	$\frac{\propto\kappa^2}{\mathrm{b}^2+\propto^2\kappa^2}$
5	$\cos(\propto t)$	$\frac{\kappa\,\mathrm{b}}{\mathrm{b}^2+\propto^2\kappa^2}$
6	$t\exp(\propto t)$	$\frac{\kappa^2}{(\mathrm{b}-\propto\kappa)^2}$
7	$\frac{\exp(\beta t)\sin(\propto t)}{\propto}$	$\frac{\kappa^2}{(\mathrm{b}-\beta\kappa)^2+\propto^2\kappa^2}$

In our research, we showed Shehu transform to solve for some functions with variable coefficients and theorems. In section *2*, we will find Shehu transform for some functions with examples. In section *3*, we will give us the ability to find inverse of Shehu transform by new methods for some functions [9] with examples. Finally, section *4*, we will extend the idea that exists in [9] to find the integration of Shehu transform with examples.

2. Theorems of Shehu transform for some functions.

Theorem (2.1). Let the function u (t) be in set A. Then its Shehu transform is given by

$$\mathbb{s}[t^n\,\mathbf{u}(\mathbf{t})] = (-1)^n\,\kappa^n\,\frac{\mathbf{d}^n}{\mathbf{d}\mathrm{b}^n}\,\mathbb{s}[\,\mathbf{u}(\mathbf{t})]$$

Proof. Using the Definition of Shehu transform, we get

Then $\mathbb{s}[t^n\,\mathbf{u}(\mathbf{t})] = \int_0^\infty exp\left(\frac{-\mathrm{b}t}{\kappa}\right)t^n\,u(t)\,dt$

$\therefore\ \mathbf{U}^{(\mathbf{n})}(\mathbf{s},\kappa) = \frac{\mathbf{d}^n}{\mathbf{d}\mathrm{b}^n}\int_0^\infty exp\left(\frac{-\mathrm{b}t}{\kappa}\right)u(t)\,dt$

$= \int_0^\infty \frac{\partial^n}{\partial\mathrm{b}^n}\left(exp\left(\frac{-\mathrm{b}t}{\kappa}\right)u(t)\,dt\right)$

$= \int_0^\infty \frac{\partial^{n-1}}{\partial\mathrm{b}^{n-1}}\left(exp\left(\frac{-\mathrm{b}t}{\kappa}\right)u(t)\left(\frac{-t}{\kappa}\right)dt\right)$

The derivation continues for n-times

$\mathbf{U^{(n)}}(\mathfrak{b},\kappa) = \int_0^\infty exp\left(\frac{-\mathfrak{b}t}{\kappa}\right) u(t)(\frac{-t}{\kappa})^n\, dt$

$= \frac{(-1)^n}{\kappa^n} \int_0^\infty exp\left(\frac{-\mathfrak{b}t}{\kappa}\right) t^n\, u(t)\, dt$

$= \frac{(-1)^n}{\kappa^n}\ \mathbb{s}[t^n\, \mathbf{u}(\mathbf{t})]$

$$\therefore\ \mathbb{s}[t^n\, \mathbf{u}(\mathbf{t})] = (-1)^n\, \kappa^n\, \frac{d^n}{d\mathfrak{b}^n}\, \mathbb{s}[\,\mathbf{u}(\mathbf{t})]$$

Lemma (1). Derivative of Shehu transform. If the function $\mathbf{u}'(t)$ is the derivative of the function $\mathbf{u}(\mathbf{t}) \in \mathbf{A}$ with respect to **t where n is number positive integer**, then its Shehu transform is defined by:

$$\mathbb{s}[t^n\, \mathbf{u}'(t)] = (-1)^n\, \kappa^n\ [\mathfrak{b}\, \frac{d^n}{d\mathfrak{b}^n}\, \mathbb{s}[\mathbf{u}(t)] + \mathbf{n}\, \frac{d^{n-1}}{d\mathfrak{b}^{n-1}}\, \mathbb{s}[\mathbf{u}(t)]]$$

Proof. Using the Definition of Shehu transform, we get

$\mathbb{s}[t^n\, \mathbf{u}'(t)] = \int_0^\infty exp\left(\frac{-\mathfrak{b}t}{\kappa}\right) t^n\, \mathbf{u}'(t)\, dt$

$\therefore\ \mathbf{U^{(n)}}(\mathbf{s},\kappa) = \frac{d^n}{d\mathfrak{b}^n}\int_0^\infty exp\left(\frac{-\mathfrak{b}t}{\kappa}\right)\, \mathbf{u}'(t)\, dt$

$=\int_0^\infty \frac{\partial^n}{\partial \mathfrak{b}^n}(exp\left(\frac{-\mathfrak{b}t}{\kappa}\right) \mathbf{u}'(t)\, dt)$

$=\int_0^\infty \frac{\partial^{n-1}}{\partial \mathfrak{b}^{n-1}}(exp\left(\frac{-\mathfrak{b}t}{\kappa}\right) \mathbf{u}'(t)(\frac{-t}{\kappa})\, dt)$

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$$\mathbf{U^{(n)}}(\mathfrak{b},\kappa) = \int_0^\infty exp\left(\frac{-\mathfrak{b}t}{\kappa}\right)\ (\frac{-t}{\kappa})^n\, \mathbf{u}'(t)\, dt$$

$$= \frac{(-1)^n}{\kappa^n}\int_0^\infty exp\left(\frac{-\mathfrak{b}t}{\kappa}\right)\ t^n\, \mathbf{u}'(t)\, dt$$

$$\mathbf{U^{(n)}}(\mathfrak{b},\kappa) = \frac{d^n}{d\mathfrak{b}^n}\int_0^\infty exp\left(\frac{-\mathfrak{b}t}{\kappa}\right)\ \mathbf{u}'(t)\, dt = \frac{(-1)^n}{\kappa^n}\ \mathbb{s}[t^n\, \mathbf{u}'(t)]$$

$\therefore\ \ \mathbb{s}[t^n\, \mathbf{u}'(t)]=(-1)^n\, \kappa^n\, \frac{d^n}{d\mathfrak{b}^n}\, \mathbb{s}[\,\mathbf{u}'(t)]$

Since $\mathbb{s}[\mathbf{u}'(\mathbf{t})] = \frac{\mathfrak{b}}{\kappa}\mathbf{U}(\mathfrak{b},\kappa) - \mathbf{u}(\mathbf{0})$

$\mathbb{s}[t^n\, \mathbf{u}'(t)]=(-1)^n\, \kappa^n\, \frac{d^n}{d\mathfrak{b}^n}\left[\frac{\mathfrak{b}}{\kappa}\mathbf{U}(\mathfrak{b},\kappa) - \mathbf{u}(\mathbf{0})\right]$

$$\therefore \mathbb{s}[t^n\, \mathbf{u}'(t)] = (-1)^n\, \kappa^n\ [\mathfrak{b}\, \mathbf{U^{(n)}}(\mathfrak{b},\kappa) + \mathbf{n}\, \mathbf{U^{(n-1)}}(\mathfrak{b},\kappa)]$$

Lemma(2*).* Derivative of Shehu transform. If the function $\mathbf{u}''(\mathbf{t})$ is the derivative of the function $\mathbf{u}(\mathbf{t}) \in \mathbf{A}$ with respect to **t where n is number positive integer**, then its Shehu transform is defined by:

$$\mathbb{s}[t^n\, \mathbf{u}''(\mathbf{t})] = (-\kappa)^n\ [\frac{\mathfrak{b}^2}{\kappa^2}\mathbf{U^{(n)}}(\mathfrak{b},\kappa) + \frac{\mathbf{6}\,\mathfrak{b}}{\kappa^2}\ \mathbf{U^{(n-1)}}(\mathfrak{b},\kappa) + \frac{\mathbf{4}}{\kappa^2}\mathbf{U^{(n-2)}}(\mathfrak{b},\kappa)]$$

Proof. Using the Definition of Shehu transform, we get

$$\mathbb{s}[t^n\, \mathbf{u}''(\mathbf{t})] = \int_0^\infty exp\left(\frac{-\mathfrak{b}t}{\kappa}\right) t^n\, \mathbf{u}''(\mathbf{t})\, dt$$

$$\textbf{Since}\ \ \mathbf{U^{(n)}}(\mathfrak{b},\kappa) = \frac{d^n}{d\mathfrak{b}^n}\int_0^\infty exp\left(\frac{-\mathfrak{b}t}{\kappa}\right)\ \mathbf{u}''(\mathbf{t})\, dt$$

$=\int_0^\infty \frac{\partial^n}{\partial \mathfrak{b}^n}(exp\left(\frac{-\mathfrak{b}t}{\kappa}\right) \mathbf{u}''(\mathbf{t})\, dt)$

$=\int_0^\infty \frac{\partial^{n-1}}{\partial \mathfrak{b}^{n-1}}(exp\left(\frac{-\mathfrak{b}t}{\kappa}\right) \mathbf{u}''(\mathbf{t})(\frac{-t}{\kappa})\, dt)$

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$$\mathbf{U^{(n)}}(\mathfrak{b},\kappa) = \int_0^\infty exp\left(\frac{-\mathfrak{b}t}{\kappa}\right)\ (\frac{-t}{\kappa})^n\, \mathbf{u}''(\mathbf{t})\, dt$$

$= \frac{d^n}{d\mathfrak{b}^n}\int_0^\infty exp\left(\frac{-\mathfrak{b}t}{\kappa}\right)\ \mathbf{u}''(\mathbf{t})\, dt$

$= \frac{(-1)^n}{\kappa^n}\int_0^\infty exp\left(\frac{-\mathfrak{b}t}{\kappa}\right)\ t^n\, \mathbf{u}''(\mathbf{t})\, dt$

$$\mathbf{U}^{(\mathbf{n})}(\mathrm{b},\kappa) = \frac{(-\mathbf{1})^{n}}{\kappa^{n}}\ \mathbb{s}[\boldsymbol{t}^{n}\ \mathbf{u}''(\mathbf{t})]$$

$$\therefore\ \mathbb{s}[\boldsymbol{t}^{n}\ \mathbf{u}''(\mathbf{t})] = (-\kappa)^{n}\ \frac{\mathrm{d}^{\mathrm{n}}}{\mathrm{db}^{\mathrm{n}}}\ \mathbb{s}[\ \mathbf{u}''(\mathbf{t})]$$

Since $\mathbb{s}[\ \mathbf{u}''(\mathbf{t})] = \frac{\mathrm{b}^{2}}{\kappa^{2}}\mathbf{U}(\mathrm{b},\kappa) - \frac{\mathrm{b}}{u}\ \mathbf{u}(\mathbf{0}) - \mathbf{u}'(\mathbf{0})$

$$\therefore \mathbb{s}[\boldsymbol{t}^{n}\ \mathbf{u}''(\mathbf{t})] = (-\kappa)^{n}\ \frac{\mathbf{d}^{\mathbf{n}}}{\mathbf{d}\mathrm{b}^{\mathbf{n}}}\ [\frac{\mathrm{b}^{2}}{\kappa^{2}}\mathbf{U}(\mathrm{b},\kappa) - \frac{s}{\kappa}\ \mathbf{u}(\mathbf{0}) - \mathbf{u}'(\mathbf{0})]$$

$$\therefore \mathbb{s}[\boldsymbol{t}^{n}\ \mathbf{u}''(\mathbf{t})] = (-\kappa)^{n}\ \left[\frac{\mathrm{b}^{2}}{\kappa^{2}}\mathbf{U}^{(\mathbf{n})}(\mathrm{b},\kappa) + \frac{\mathbf{6}\ \mathrm{b}}{\kappa^{2}}\ \mathbf{U}^{(\mathbf{n-1})}(\mathrm{b},\kappa) + \frac{\mathbf{4}}{\kappa^{2}}\mathbf{U}^{(\mathbf{n-2})}(\mathrm{b},\kappa)\right]$$

Lemma (3). Derivative of Shehu transform. If the function $\mathbf{u}^{(\mathbf{n})}(\mathbf{t})$ is the $\mathbf{nth}$ derivative of the function $\mathbf{u}(\mathbf{t}) \in \mathbf{A}$.

$$\mathbb{s}\left[\boldsymbol{t}\ \boldsymbol{u}^{(n)}(\boldsymbol{t})\right] = (-\kappa)^{-(n-1)}\ [\mathrm{b}^{n}\ \mathbf{U}'(\mathrm{b},\kappa) + \boldsymbol{n}\ \mathrm{b}^{n-1}\mathbf{U}(\mathrm{b},\kappa)] + \sum_{k=0}^{n-1}(\boldsymbol{n}-\boldsymbol{k}-\mathbf{1})\ \boldsymbol{u}^{n-k-1}\ \ \mathrm{b}^{n-k-2}\ \boldsymbol{u}^{(k)}(\mathbf{0})$$

Proof. Using the Definition of Shehu transform, we get

$$\mathbb{s}\left[\boldsymbol{t}\ \boldsymbol{u}^{(n)}(\boldsymbol{t})\right] = \int_{0}^{\infty} \boldsymbol{exp}\left(\frac{-\mathrm{b}\boldsymbol{t}}{\kappa}\right)\ \boldsymbol{t}\ \boldsymbol{u}^{(n)}(\boldsymbol{t})\ \ \boldsymbol{dt}$$

Since $\frac{\mathbf{d}}{\mathbf{d}\mathrm{b}}\ \mathbb{s}\left[\ \boldsymbol{u}^{(n)}(\boldsymbol{t})\right] = \frac{\mathbf{d}}{\mathbf{d}\mathrm{b}}\int_{0}^{\infty} \boldsymbol{exp}\left(\frac{-\mathrm{b}\boldsymbol{t}}{\kappa}\right)\ \ \boldsymbol{u}^{(n)}(\boldsymbol{t})\boldsymbol{dt}$

$$= \int_{0}^{\infty} \frac{\partial}{\partial \mathrm{b}}\left(\boldsymbol{exp}\left(\frac{-\mathrm{b}\boldsymbol{t}}{\kappa}\right)\ \boldsymbol{u}^{(n)}(\boldsymbol{t})\ \boldsymbol{dt}\right)$$

$$= \int_{0}^{\infty} \boldsymbol{exp}\left(\frac{-\mathrm{b}\boldsymbol{t}}{\kappa}\right)\ \ \boldsymbol{u}^{(n)}(\boldsymbol{t})\left(\frac{-\boldsymbol{t}}{\kappa}\right)\boldsymbol{dt}$$

$$= \frac{-\mathbf{1}}{\boldsymbol{u}}\int_{0}^{\infty} \boldsymbol{exp}\left(\frac{-\mathrm{b}\boldsymbol{t}}{\kappa}\right)\ \ \boldsymbol{u}^{(n)}(\boldsymbol{t})\ \mathbf{t}\ \boldsymbol{dt}$$

$$= \frac{-\mathbf{1}}{\kappa}\ \mathbb{s}\left[\ \boldsymbol{t}\ \boldsymbol{u}^{(n)}(\boldsymbol{t})\right]$$

Therefore:

$$\mathbb{s}\left[\ \boldsymbol{t}\ \boldsymbol{u}^{(n)}(\boldsymbol{t})\right] = -\kappa\ \frac{\boldsymbol{d}}{\boldsymbol{d}\mathrm{b}}\ \mathbb{s}\left[\ \boldsymbol{u}^{(n)}(\boldsymbol{t})\right]$$

So $\mathbb{s}\left[\ \boldsymbol{u}^{(n)}(\boldsymbol{t})\right] = \frac{\mathrm{b}^{\mathbf{n}}}{\kappa^{\mathbf{n}}}\mathbf{U}(\mathrm{b},\kappa) - \sum_{\mathbf{k}=\mathbf{0}}^{\mathbf{n}-\mathbf{1}}\left(\frac{\mathrm{b}}{\kappa}\right)^{\mathbf{n}-(\mathbf{k}+\mathbf{1})}\ \mathbf{u}^{\mathbf{k}}(\mathbf{0})$

$$\mathbb{s}\left[\ \boldsymbol{t}\ \boldsymbol{u}^{(n)}(\boldsymbol{t})\right] = -\kappa\ \frac{\mathbf{d}}{\mathbf{d}\mathrm{b}}\left[\frac{\mathrm{b}^{\mathbf{n}}}{\kappa^{\mathbf{n}}}\mathbf{U}(\mathrm{b},\kappa) - \sum_{\mathbf{k}=\mathbf{0}}^{\mathbf{n}-\mathbf{1}}\left(\frac{\mathrm{b}}{\kappa}\right)^{\mathbf{n}-(\mathbf{k}+\mathbf{1})}\ \mathbf{u}^{\mathbf{k}}(\mathbf{0})\right]$$

$$\mathbb{s}\left[\ \boldsymbol{t}\ \boldsymbol{u}^{(n)}(\boldsymbol{t})\right] = -\kappa\ \left[\frac{\mathbf{1}}{\kappa^{\mathbf{n}}}(\ \mathrm{b}^{n}\ \mathbf{U}'(\mathrm{b},\kappa) + \boldsymbol{n}\ \mathrm{b}^{n-1}\mathbf{U}(\mathrm{b},\kappa)) - \sum_{\mathbf{k}=\mathbf{0}}^{\mathbf{n}-\mathbf{1}}\left(\frac{\mathbf{1}}{\kappa}\right)^{\mathbf{n}-(\mathbf{k}+\mathbf{1})}\ \mathbf{u}^{\mathbf{k}}(\mathbf{0})\ \left(\mathbf{n}-(\mathbf{k}+\mathbf{1})\right)\ \mathrm{b}^{n-(k+2)}\right]$$

$$\therefore\ \ \mathbb{s}\left[\boldsymbol{t}\ \boldsymbol{u}^{(n)}(\boldsymbol{t})\right] = (-\kappa)^{-(n-1)}\ [\mathrm{b}^{n}\ \mathbf{U}'(\mathrm{b},\kappa) + \boldsymbol{n}\ \mathrm{b}^{n-1}\mathbf{U}(\mathrm{b},\kappa)] + \sum_{k=0}^{n-1}(\boldsymbol{n}-\boldsymbol{k}-\mathbf{1})\ \kappa^{n-k-1}\ \ \mathrm{b}^{n-k-2}\ \boldsymbol{u}^{(k)}(\mathbf{0})$$

Theorem (2.2) Derivative of Shehu transform. If the function $\mathbf{u}^{(\mathbf{n})}(\mathbf{t})$ is the $\mathbf{nth}$ derivative of the function $\mathbf{u}(\mathbf{t}) \in \mathbf{A}$ with respect to $\mathbf{t}$, then its Shehu transform is defined by:

$$\mathbb{s}\left[\boldsymbol{t}^{n}\ \boldsymbol{u}^{(n)}(\boldsymbol{t})\right] = (-\mathbf{1})^{n}\ \kappa^{n}\ \frac{\mathbf{d}^{\mathbf{n}}}{\mathbf{d}\mathrm{b}^{\mathbf{n}}}\ [\frac{\mathrm{b}^{\mathbf{n}}}{\kappa^{\mathbf{n}}}\ \mathbf{U}(\mathrm{b},\kappa) - \sum_{\mathbf{k}=\mathbf{0}}^{\mathbf{n}-\mathbf{1}}\ \left(\frac{\mathrm{b}}{\kappa}\right)^{\mathbf{n}-(\mathbf{k}+\mathbf{1})}\ \boldsymbol{u}^{(k)}(\mathbf{0})]$$

Proof. By Lemma (1) and Lemma (3), we get

$$\mathbb{s}\left[\boldsymbol{t}^{n}\ \boldsymbol{u}^{(n)}(\boldsymbol{t})\right] = \int_{0}^{\infty} \boldsymbol{exp}\left(\frac{-\mathrm{b}\boldsymbol{t}}{\kappa}\right)\boldsymbol{t}^{n}\ \boldsymbol{u}^{(n)}(\boldsymbol{t})\ \boldsymbol{dt}$$

Since $\mathbf{U}^{(\mathbf{n})}(\mathrm{b},\kappa) = \frac{\mathbf{d}^{\mathbf{n}}}{\mathbf{d}\mathrm{b}^{\mathbf{n}}}\int_{0}^{\infty} \boldsymbol{exp}\left(\frac{-\mathrm{b}\boldsymbol{t}}{\kappa}\right)\ \boldsymbol{u}^{(n)}(\boldsymbol{t})\boldsymbol{dt}$

$$= \int_{0}^{\infty} \frac{\partial^{n}}{\partial \mathrm{b}^{n}}\left(\boldsymbol{exp}\left(\frac{-\mathrm{b}\boldsymbol{t}}{\kappa}\right)\boldsymbol{u}^{(n)}(\boldsymbol{t})\ \boldsymbol{dt}\right)$$

$$= \int_{0}^{\infty} \frac{\partial^{n-1}}{\partial \mathrm{b}^{n-1}}\left(\boldsymbol{exp}\left(\frac{-\mathrm{b}\boldsymbol{t}}{\kappa}\right)\boldsymbol{u}^{(n)}(\boldsymbol{t})\left(\frac{-\boldsymbol{t}}{\kappa}\right)\ \boldsymbol{dt}\right)$$

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$$= \int_{0}^{\infty} \boldsymbol{exp}\left(\frac{-\mathrm{b}\boldsymbol{t}}{\kappa}\right)\left(\frac{-\boldsymbol{t}}{\kappa}\right)^{\mathbf{n}}\ \boldsymbol{u}^{(n)}(\boldsymbol{t})\ \ \boldsymbol{dt}$$

$$= \left(\frac{-\mathbf{1}}{\kappa}\right)^{\mathbf{n}}\int_{0}^{\infty} \boldsymbol{exp}\left(\frac{-\mathrm{b}\boldsymbol{t}}{\kappa}\right)\boldsymbol{u}^{(n)}(\boldsymbol{t})\ \boldsymbol{t}^{n}\ \boldsymbol{dt}$$

$=(\frac{-1}{\kappa})^{n}\ \mathbb{s}[t^{n}\,u^{(n)}(t)]$

$$\therefore\ \mathbb{s}[t^{n}\,u^{(n)}(t)] = (-1)^{n}\ \kappa^{n}\ \frac{d^{n}}{d\text{b}^{n}}\ \mathbb{s}[\,u^{(n)}(t)]$$

Since $\mathbb{s}[\,u^{(n)}(t)] = \frac{s^{n}}{\kappa^{n}}\mathbf{U}(\text{b},\kappa) - \sum_{k=0}^{n-1}\left(\frac{\text{b}}{\kappa}\right)^{n-(k+1)}\mathbf{u}^{k}(\mathbf{0})$

$$\therefore\ \ \mathbb{s}[t^{n}\,u^{(n)}(t)] = (-\kappa)^{n}\ \frac{d^{n}}{d\text{b}^{n}}\left[\frac{\text{b}^{n}}{\kappa^{n}}\ \mathbf{U}(\text{b},\kappa) - \sum_{k=0}^{n-1}\left(\frac{\text{b}}{\kappa}\right)^{n-(k+1)}\ u^{(k)}(\mathbf{0})\right]$$

Example: To find $\mathbb{s}[t^{n}\,u(\,t)]$ when n = 2 if

$\mathbf{1}-\ u(\,t) =$ sin (t)

$\mathbf{2}-\ u(\,t) =$ cos (t)

$\mathbf{3}-\ u(\,t) =$ sinh (t)

$\mathbf{4}-\ u(\,t) =$ cosh (t)

$\mathbf{5}-\ u(\,t) =$ exp (t) sin (t)

$\mathbf{6}-\ u(\,t) =$ exp (t) cos (t)

Sol: Use theory (*2*.1), we get

$\mathbf{1}-\ \mathbb{s}[t^{2}\,sin\ (t)] = (-1)^{2}\ \kappa^{2}\ \frac{d^{2}}{d\text{b}^{2}}\ \ \mathbb{s}[sin\ (t)]$

$= \kappa^{2}\ \frac{d}{d\text{b}}\ [\,-2\ \kappa^{2}(\text{-}2\ \text{b}(\text{b}^{2}+\kappa^{2})^{-3}\ (2\ \text{b}) + \ (\text{b}^{2}+\kappa^{2})^{-2})]$

$= \frac{-2\ \kappa^{4}}{(\text{b}^{2}+\kappa^{2})^{2}} + \frac{8\ \kappa^{4}\ \text{b}^{2}}{(\text{b}^{2}+\kappa^{2})^{3}} = \left[\frac{6\ \kappa^{4}\ \text{b}^{2}-\ 2\ \kappa^{6}}{(\text{b}+\kappa^{2})^{3}}\right]$

$\mathbf{2}-\ \mathbb{s}[t^{2}\,cos\ (t)] = \ (-1)^{2}\ \kappa^{2}\ \frac{d^{2}}{d\text{b}^{2}}\ \ \mathbb{s}[cos\ \ (t)]$

$= \kappa^{2}\ \frac{d}{d\text{b}}\ [\ \kappa\ (-2\ \text{b}^{2}(\text{b}^{2}+\kappa^{2})^{-2} + \ (\text{b}^{2}+\kappa^{2})^{-1})]$

$= \kappa^{2}[\ \frac{8\ \kappa\ \ \text{b}^{3}}{(\text{b}^{2}+\kappa^{2})^{3}} - \ \frac{4\ \kappa\ \ \text{b}}{(\text{b}^{2}+\kappa^{2})^{2}} - \ \frac{2\ \kappa\ \ \text{b}}{(\text{b}^{2}+\kappa^{2})^{2}}\ ]$

$$= 2\kappa^{3}\ \text{b}\ \frac{\text{b}^{2}-3\ \kappa^{2}}{(\text{b}^{2}+\kappa^{2})^{3}}$$

$\mathbf{3}-\ \mathbb{s}[t^{2}\,sinh\ \ (t)] = \ (-1)^{2}\ \kappa^{2}\ \frac{d^{2}}{d\text{b}^{2}}\ \ \mathbb{s}[sinh\ \ (t)]$

$= \kappa^{2}\ \frac{d^{2}}{d\text{b}^{2}}\ [\frac{\kappa^{2}}{\text{b}^{2}-\kappa^{2}}]$

$= \ \kappa^{2}\ \frac{d}{d\text{b}}\ [\,-\ \kappa^{2}\ (\text{b}^{2}-\kappa^{2})^{-2}\ (2\ \text{b})]$

$= \frac{6\ \kappa^{4}\ \text{b}^{2}+2\ \kappa^{6}}{(\text{b}^{2}-\kappa^{2})^{3}}$

$\mathbf{4}-\ \mathbb{s}[t^{2}\,cosh\ (t)] = \ (-1)^{2}\ \kappa^{2}\ \frac{d^{2}}{d\text{b}^{2}}\ \ \mathbb{s}[cosh\ (t)]$

$= \kappa^{2}\ \frac{d^{2}}{d\text{b}^{2}}\ \ \mathbb{s}\left[\frac{\kappa\ \text{b}}{\text{b}^{2}-\kappa^{2}}\right] = \ \kappa^{2}\ \frac{d}{d\text{b}}\left[\frac{-2\ \kappa\ \text{b}^{2}}{(\text{b}^{2}-\kappa^{2})^{2}} + \frac{\kappa}{\text{b}^{2}-\kappa^{2}}\right]$

$= \kappa^{2}\ \left[\frac{8\ \kappa\ \text{b}^{3}}{(\text{b}^{2}-\kappa^{2})^{3}} - \ \frac{6\ \kappa\ \text{b}}{(\text{b}^{2}-\kappa^{2})^{2}}\right]$

$= 2\kappa^{3}\ \text{b}\ \frac{\text{b}^{2}+3\ \kappa^{2}}{(\text{b}^{2}-\kappa^{2})^{3}}$

$\mathbf{5}-\ \mathbb{s}\ [t^{2}\,exp\ (t)\ sin\ (t)] = \ (-1)^{2}\ \kappa^{2}\ \frac{d^{2}}{d\text{b}^{2}}\ \ \mathbb{s}[exp\ (t)\ sin\ (t)]$

$= \kappa^{2}\ \frac{d^{2}}{d\text{b}^{2}}\ \left[\frac{\kappa^{2}}{(\text{b}-\kappa)^{2}+\kappa^{2}}\right]$

$= \kappa^{2}\ \frac{d}{d\text{b}}\ [\,-2\ \kappa^{2}\ (\ \text{b}-\kappa)((\text{b}-\kappa)^{2}+\kappa^{2})^{-2}]$

$= \kappa^{2}\ \frac{6\ \kappa^{2}\ (\text{b}-\kappa)^{2}-\ 2\ \kappa^{4}}{((\text{b}-\kappa)^{2}+\kappa^{2})^{3}}$

$\mathbf{6}-\ \mathbb{s}[t^{2}\,exp\ (t)\ cos\ (t)] = \ (-1)^{2}\ \kappa^{2}\ \frac{d^{2}}{d\text{b}^{2}}\ \mathbb{s}[\ exp\ (t)\ cos\ (t)]$

$= \ \kappa^{2}\ \frac{d^{2}}{d\text{b}^{2}}\ \left[\frac{\kappa\ (\text{b}-\kappa)}{(\text{b}-\kappa)^{2}+\kappa^{2}}\right]$

$= \kappa^{2}\ \frac{d}{d\text{b}}\ [\ \kappa\ (-2\ (\text{b}-\kappa)^{2}\ ((\text{b}-\kappa)^{2}+\kappa^{2})^{-2} + ((\text{b}-\kappa)^{2}+\kappa^{2})^{-1})]$

$=\kappa^{2}\ [\ \kappa\ (\ 8\ \ (\text{b}-\kappa)^{3}\ ((\text{b}-\kappa)^{2}+\kappa^{2})^{-3} - 4\ (\text{b}-\kappa)((\text{b}-\kappa)^{2}+\kappa^{2})^{-2} - 2\ (\text{b}-\kappa)((\text{b}-\kappa)^{2}+\kappa^{2})^{-2}\ )]$

$= \frac{2\kappa^{3}(\text{b}-\kappa)[4\ (\text{b}-\kappa)^{2}-3(\text{b}-\kappa)-3\kappa^{2}]}{((\text{b}-\kappa)^{2}+\kappa^{2})^{3}}$

Theorem (2.3) Let $(\mathbf{a_0}\, t^n + \mathbf{a_1}\, t^{n-1} + \cdots + a_n)$ *is polynomial of degree n*, let u (t) be the function in set A. Then $\mathbb{s}[\,(\mathbf{a_0}\, t^n + \mathbf{a_1}\, t^{n-1} + \cdots + a_n)\, \mathbf{u}\,(\mathbf{t})] = \mathbf{a_0} \kappa^{n}\, (-1)^n\, \mathbf{U}^{(n)}(\mathrm{b}, \kappa) + \mathbf{a_1}\, \kappa^{n}\, (-1)^{n-1}\, \mathbf{U}^{(n-1)}(\mathrm{b}, \kappa) + \cdots + a_n\, \mathbf{U}(\mathrm{b}, \kappa)$

Proof. By linearly property to Shehu transform. We get

$$\mathbb{s}[\,(\mathbf{a_0}\, t^n + \mathbf{a_1}\, t^{n-1} + \cdots + a_n)\, \mathbf{u}\,(\mathbf{t})] = \mathbb{s}[\,\mathbf{a_0}\, t^n\, \mathbf{u}\,(\mathbf{t})] + \mathbb{s}[\,\mathbf{a_1}\, t^{n-1}\, \mathbf{u}\,(\mathbf{t})] + \cdots + \mathbb{s}[\,a_n\, \mathbf{u}\,(\mathbf{t})]$$

$= \mathbf{a_0}\, \mathbb{s}[\, t^n\, \mathbf{u}\,(\mathbf{t})] + \mathbf{a_1}\, \mathbb{s}[\, t^{n-1}\, \mathbf{u}\,(\mathbf{t})] + \cdots + a_n\, \mathbb{s}[\, \mathbf{u}\,(\mathbf{t})]$

From theorem (3.1.1), yields:-

Then $\mathbb{s}[\,(\mathbf{a_0}\, t^n + \mathbf{a_1}\, t^{n-1} + \cdots + a_n)\, \mathbf{u}\,(\mathbf{t})] = \mathbf{a_0}\, \kappa^{n}$
$(-1)^n\, \mathbf{U}^{(n)}(\mathrm{b}, \kappa) + \mathbf{a_1}\, \kappa^{n}\, (-1)^{n-1}\, \mathbf{U}^{(n-1)}(\mathrm{b}, \kappa) + \cdots + a_n\, \mathbf{U}(\mathrm{b}, \kappa)$ (3.6)

Example (1): Find $\mathbb{s}[(\, t^2 + 3\, t - 1)\cos t]$

Sol:

$\therefore \quad \mathbb{s}[(\, t^2 + 3\, t - 1)\cos t] = \mathbb{s}[(\, t^2 \cos t + 3\, t\, \cos t - \cos t)]$

$= \mathbb{s}[t^2 \cos t] + 3\, \mathbb{s}[t\, \cos t] - \mathbb{s}[\cos t]$

$= 2\, \kappa^3\, \mathrm{b}\, \frac{\mathrm{b}^2 - 3\kappa^2}{(\mathrm{b}^2+\kappa^2)^3} + 3\, \frac{\kappa^2(\mathrm{b}^2-\kappa^2)}{(\mathrm{b}^2+\kappa^2)^2} - \frac{\kappa\, \mathrm{b}}{\mathrm{b}^2+\kappa^2}$

$= \frac{-7\, \kappa^5\, \mathrm{b} + 3\kappa^2\, \mathrm{b}^4 - 3\, \kappa^6 - \kappa\, \mathrm{b}^5}{(\mathrm{b}^2+\kappa^2)^3}$

Example (2): Find $\mathbb{s}[(\, t^2 + 3\, t + 2)\, \sinh\, (\mathbf{t})]$

Sol:

$\therefore \quad \mathbb{s}[(\, t^2 + 3\, t + 2)\, \sinh\, (\mathbf{t})] = \mathbb{s}[t^2 \sinh\, (\mathbf{t})] + 3\, \mathbb{s}[t\, \sinh\, (\mathbf{t})] + 2\, \mathbb{s}[\sinh\, (\mathbf{t})]$

$$= \frac{6\, \kappa^4\, \mathrm{b}^2 + 2\, \kappa^6}{(\mathrm{b}^2 - \kappa^2)^3} + 3\, \frac{2\kappa^3\, \mathrm{b}}{(\mathrm{b}^2 - \kappa^2)^2} + 2\, \frac{\kappa^2}{(\mathrm{b}^2 - \kappa^2)}$$

$= \frac{6\, \kappa^4\, \mathrm{b}^2 + 2\, \kappa^6 + 6\, \kappa^3\, \mathrm{b}\, (\mathrm{b}^2-\kappa^2) + 2\, \kappa^2 (\mathrm{b}^2-\kappa^2)^2}{(\mathrm{b}^2-\kappa^2)^3}$

$= \frac{2\, \kappa^4\, \mathrm{b}^2 + 4\, \kappa^6 + 6\, \kappa^3\, \mathrm{b}^3 - 6\, \kappa^5 \mathrm{b} + 2\, \kappa^2 \mathrm{b}^4}{(\mathrm{b}^2-\kappa^2)^3}$

3. Inverse of Shehu Transform

Utilizing the previous theorem; we can obtained the inverse of Shehu transform for some terms, such as:

Example (1): To find $\mathbb{s}^{-1}\left[\frac{(\kappa\, \mathrm{b} + \sqrt{2}\, \kappa^2)(\kappa\, \mathrm{b} - \sqrt{2}\, \kappa^2)}{3\, \mathrm{b}^4 - 12\, \kappa\, \mathrm{b}^3 + 24\, \kappa^2\, \mathrm{b}^2 - 24\, \kappa^3\, \mathrm{b} + 12\, \kappa^4}\right]$

Sol:

Then $\mathbb{s}^{-1}\left[\frac{(\kappa\, \mathrm{b} + \sqrt{2}\, \kappa^2)(\kappa\, \mathrm{b} - \sqrt{2}\, \kappa^2)}{3\, \mathrm{b}^4 - 12\, \kappa\, \mathrm{b}^3 + 24\, \kappa^2\, \mathrm{b}^2 - 24\, \kappa^3\, \mathrm{b} + 12\, \kappa^4}\right]$

$= \mathbb{s}^{-1}\left[\frac{\kappa^2\, s^2 - 2\, \kappa^4}{3\, \mathrm{b}^4 - 12\, \kappa\, \mathrm{b}^3 + 24\, \kappa^2\, \mathrm{b}^2 - 24\, \kappa^3\, \mathrm{b} + 12\, \kappa^4}\right]$

$= \frac{1}{3}\, \mathbb{s}^{-1}\left[\frac{\kappa^2\, \mathrm{b}^2 - 2\, \kappa^4 - 2\kappa^3\, s + 2\kappa^3\, \mathrm{b}}{\mathrm{b}^4 - 4\, \kappa\, \mathrm{b}^3 + 8\, \kappa^2\, \mathrm{b}^2 - 8\, \kappa^3\, \mathrm{b} + 4\, \kappa^4}\right]$

$= \frac{1}{3}\, \mathbb{s}^{-1}\left[\frac{(\, \kappa^2\, \mathrm{b}^2 - 2\kappa^3\, \mathrm{b}) + (2\kappa^3\, \mathrm{b} - 2\, \kappa^4)}{\mathrm{b}^4 - 2\, \kappa\, \mathrm{b}^3 + 2\, \kappa^2\, \mathrm{b}^2 - 2\kappa\, \mathrm{b}^3 + 4\, \kappa^2\, \mathrm{b}^2 - 4\, \kappa^3\, \mathrm{b} + 2\, \kappa^2\, \mathrm{b}^2 - 4\kappa^3\, \mathrm{b} + 4\, \kappa^4}\right]$

$= \frac{1}{3}\, \mathbb{s}^{-1}\left[\frac{(\, \kappa^2\, \mathrm{b}^2 - 2\kappa^3\, \mathrm{b}) + (2\kappa^3\, \mathrm{b} - 2\, \kappa^4)}{(\, \mathrm{b}^2 - 2\, \kappa\, \mathrm{b} + 2\, \kappa^2)^{\ 2}}\right]$

$= \frac{1}{3}\left(\mathbb{s}^{-1}\left[\frac{\kappa^2\, \mathrm{b}^2 - 2\kappa^3\, \mathrm{b}}{((\mathrm{b}-\kappa)^2 + \kappa^2)^2}\right] + \mathbb{s}^{-1}\left[\frac{2\kappa^3\, \mathrm{b} - 2\, \kappa^4}{((\mathrm{b}-\kappa)^2 + \kappa^2)^2}\right]\right)$

$= \frac{1}{3}\, \mathbf{t}\, \exp(\mathbf{t})(\sin\, (\mathbf{t}) + \cos\, (\mathbf{t}\,))$

Example (2): Find $\mathbb{s}^{-1}\left[\frac{6\kappa^4\, \mathrm{b}\, (\kappa+\mathrm{b}) + 2\, \kappa^3\, (\, \kappa^3 + \mathrm{b}^3)}{(\mathrm{b}^3-\kappa^3\,)(\, \mathrm{b}^3+\kappa^3\,) + 3(\, \kappa^2\, \mathrm{b} - \kappa\, \mathrm{b}^2)(\, \kappa^2\, \mathrm{b} + \kappa\, \mathrm{b}^2)}\right]$

Sol:

$$\therefore\ \mathbb{s}^{-1}\left[\frac{6\, \kappa^4\, \mathrm{b}\, (\kappa + \mathrm{b}) + 2\, \kappa^3\, (\, \kappa^3\, +\, \mathrm{b}^3)}{(\mathrm{b}^3 - \kappa^3\,)(\, \mathrm{b}^3 + \kappa^3\,) + 3(\, \kappa^2\, \mathrm{b} - \kappa\, \mathrm{b}^2)(\, \kappa^2\, \mathrm{b} + \kappa\, \mathrm{b}^2)}\right]$$

$$= \mathbb{s}^{-1}\left[\frac{6\, \kappa^5 \mathrm{b} + 6\, \kappa^4 \mathrm{b}^2 + 2\, \kappa^6 + 2\, \kappa^3\, \mathrm{b}^3}{\mathrm{b}^6 - \kappa^6 + 3\, (\kappa^4 \mathrm{b}^2 - \kappa^2\, \mathrm{b}^4)}\right]$$

$$= \mathbb{s}^{-1}\left[\frac{(6\, \kappa^4 \mathrm{b}^2 + 2\, \kappa^6) + (2\, \kappa^3\, \mathrm{b}^3 + 6\, \kappa^5\, \mathrm{b})}{\mathrm{b}^6 - 2\, \kappa^2\, \mathrm{b}^4 + \kappa^4 \mathrm{b}^2 - \kappa^2\, \mathrm{b}^4 + 2\, \kappa^4 \mathrm{b}^2 - \kappa^6}\right]$$

$$= \mathbb{s}^{-1}\left[\frac{(6\, \kappa^4 \mathrm{b}^2 + 2\, \kappa^6) + (2\, \kappa^3\, \mathrm{b}^3 + 6\, \kappa^5\, \mathrm{b})}{(\mathrm{b}^2 - \kappa^2)^3}\right]$$

$$= \mathbb{s}^{-1}\left[\frac{6\,\kappa^4 \mathfrak{b}^2 + 2\,\kappa^6}{(\mathfrak{b}^2-\kappa^2)^3}\right] + \mathbb{s}^{-1}\left[\frac{2\,\kappa^3 \mathfrak{b}^3 + 6\,\kappa^5 \mathfrak{b}}{(\mathfrak{b}^2-\kappa^2)^3}\right]$$

$= t^2 \sinh(t) + t^2 \cosh(t)$

4. Integrations of Shehu Transform

Theorem (4.1): If $\mathbb{s}[u(t)] = \mathbf{U}(\mathfrak{b},\kappa)$, *then* $\mathbb{s}\left[\int_0^t u(\tilde{s})\,d\tilde{s}\right] = \frac{\kappa}{\mathfrak{b}}\,\mathbf{U}(\mathfrak{b},\kappa)$

Proof: Let g (t) = $\int_0^t u(\tilde{s})\,d\tilde{s}$, then:

$g'(t) = u(t)$ *and* $g(0) = 0$ *by taking Shehu transform, we get* $\mathbb{s}[g'(t)] = \mathbb{s}[u(t)]$

Since $\mathbb{s}[g'(t)] = \frac{\mathfrak{b}}{\kappa}\,\mathbb{s}[g(t)] - g(0)$ and $\mathbb{s}[u(t)] = \mathbf{U}(\mathfrak{b},\kappa)$, then

$\frac{\mathfrak{b}}{\kappa}\,\mathbb{s}[g(t)] - g(0) = \mathbf{U}(\mathfrak{b},\kappa)$

$\mathbb{s}[g(t)] = \frac{\kappa}{\mathfrak{b}}\,\mathbf{U}(\mathfrak{b},\kappa)$

$$\mathbb{s}\left[\int_0^t u(\tilde{s})\,d\tilde{s}\right] = \frac{\kappa}{\mathfrak{b}}\,\mathbf{U}(\mathfrak{b},\kappa)$$

Theorem (4.2): If $\mathbb{s}[u(t)] = \mathbf{U}(\mathfrak{b},\kappa)$, *then* $\mathbb{s}\left[\frac{u(t)}{t}\right] = \frac{1}{\kappa}\int_{\mathfrak{b}}^{\infty} \mathbf{U}(\tilde{s},\kappa)\,d\tilde{s}$

Proof: $\int_{\mathfrak{b}}^{\infty}\left(\int_0^{\infty} e^{\frac{-\mathfrak{b}t}{\kappa}}\,u(t)\,dt\right) d\tilde{s} = \int_0^{\infty}\left(\int_{\mathfrak{b}}^{\infty} e^{\frac{-\tilde{s}t}{\kappa}}\,d\tilde{s}\right) u(t)\,dt$, $\mathfrak{b} > 0$

$= \int_0^{\infty}\left(\frac{-\kappa}{t}\int_{\mathfrak{b}}^{\infty} e^{\frac{-\tilde{s}t}{\kappa}}\,d\tilde{s}\right) u(t)\,dt$

$= \kappa \int_0^{\infty} e^{\frac{-\mathfrak{b}t}{\kappa}}\;\frac{u(t)}{t}\;dt$

$= \kappa\,\mathbb{s}\left[\frac{u(t)}{t}\right]$

$= \int_{\mathfrak{b}}^{\infty} \mathbf{U}(\tilde{s},\kappa)\,d\tilde{s}$

$$\textit{then}\quad \mathbb{s}\left[\frac{u(t)}{t}\right] = \frac{1}{\kappa}\int_{\mathfrak{b}}^{\infty} \mathbf{U}(\tilde{s},\kappa)\,d\tilde{s}$$

As a result of the above theorem $\mathbb{s}^{-1}\left[\frac{1}{\kappa}\int_{\mathfrak{b}}^{\infty} \mathbf{U}(\tilde{s},\kappa)\,d\tilde{s}\right] = \frac{u(t)}{t}$

Example (1): To solve u (t) = $\frac{e^{4t} - e^{-3t} + \sin t}{t}$

Sol: Since $\mathbb{s}[e^{4t}] = \frac{\kappa}{\mathfrak{b}-4\kappa}$ and $\mathbb{s}[e^{-3t}] = \frac{\kappa}{\mathfrak{b}+3\kappa}$, $\mathbb{s}[\sin t] = \frac{\kappa^2}{\mathfrak{b}^2+\kappa^2}$

, then

$$\therefore\ \mathbb{s}\left[\frac{v(t)}{t}\right] = \frac{1}{\kappa}\int_{\mathfrak{b}}^{\infty}\frac{\kappa}{\tilde{s}-4\kappa}\,d\tilde{s} - \frac{1}{\kappa}\int_{\mathfrak{b}}^{\infty}\frac{\kappa}{\tilde{s}+3\kappa}\,d\tilde{s} + \frac{1}{\kappa}\int_{\mathfrak{b}}^{\infty}\frac{\kappa^2}{\tilde{s}^2+\kappa^2}\,d\tilde{s}$$

$= [\ln(\tilde{s}-4\kappa)]_{\mathfrak{b}}^{\infty} - [\ln(\tilde{s}+3\kappa)]_{\mathfrak{b}}^{\infty} + [\frac{1}{\kappa}\tan^{-1}\tilde{s}]_{\mathfrak{b}}^{\infty}$

$= [\ln(\tilde{s}-4\kappa) - \ln(\tilde{s}+3\kappa)]_{\mathfrak{b}}^{\infty} + [\frac{1}{\kappa}\tan^{-1}\tilde{s}]_{\mathfrak{b}}^{\infty}$

$= [\ln\frac{\frac{\tilde{s}}{\tilde{s}}-\frac{4\kappa}{\tilde{s}}}{\frac{\tilde{s}}{\tilde{s}}+\frac{3\kappa}{\tilde{s}}}]_{\mathfrak{b}}^{\infty} + [\frac{1}{\kappa}\tan^{-1}\tilde{s}]_{\mathfrak{b}}^{\infty}$

$= [\ln\frac{1-\frac{4\kappa}{\tilde{s}}}{1+\frac{3\kappa}{\tilde{s}}}]_{\mathfrak{b}}^{\infty} + [\frac{1}{\kappa}\tan^{-1}\tilde{s}]_{\mathfrak{b}}^{\infty}$

$= -\ln\frac{1-\frac{4\kappa}{\mathfrak{b}}}{1+\frac{3\kappa}{\mathfrak{b}}} - \frac{1}{\kappa}\tan^{-1}\mathfrak{b}$

$= -\ln\frac{\mathfrak{b}-4\kappa}{\mathfrak{b}+3\kappa} - \frac{1}{\kappa}\tan^{-1}\mathfrak{b}$

$= \ln\frac{\mathfrak{b}+3\kappa}{\mathfrak{b}-4\kappa} - \frac{1}{\kappa}\tan^{-1}\mathfrak{b}$

Example (2): To find $\mathbb{s}^{-1}\left[\ln\left(1+\frac{(\propto\kappa)^2}{\mathfrak{b}^2}\right)\right]$

Sol: This take a little intuition from differentiation the expression we get:-

$-\frac{d}{d\mathfrak{b}}\ln\left(1+\frac{(\propto\kappa)^2}{\mathfrak{b}^2}\right) = \frac{-1}{1+\frac{(\propto\kappa)^2}{\mathfrak{b}^2}}\left(\frac{-2\propto^2\kappa^2}{\mathfrak{b}^3}\right) = \frac{2\propto^2\kappa^2}{\mathfrak{b}(\mathfrak{b}^2+\propto^2\kappa^2)}$ (by partion fractions)

$$\frac{A}{\mathfrak{b}} + \frac{B\mathfrak{b}+C}{(\mathfrak{b}^2+\propto^2\kappa^2)} = \frac{2\propto^2\kappa^2}{\mathfrak{b}(\mathfrak{b}^2+\propto^2\kappa^2)}$$

$2\propto^2\kappa^2 = A(\mathfrak{b}^2+\propto^2\kappa^2) + B\mathfrak{b}^2 + C\mathfrak{b}$

A + B = 0
C = 0
$\mathbf{2}\ \propto^2\ \kappa^2 = \boldsymbol{A} \propto^2\ \kappa^2$
∴ A = 2 and B = -2 and C = 0

$$\therefore\quad \frac{\mathbf{2}\ \propto^2\ \kappa^2}{\mathrm{b}\,(\mathrm{b}^2 + \propto^2\ \kappa^2)} = \frac{\mathbf{2}}{\mathrm{b}} + \frac{-\mathbf{2}\ \mathrm{b}}{(\mathrm{b}^2 + \propto^2\ \kappa^2)}$$

Look if $(\frac{2}{\mathrm{b}} - \frac{2\ \mathrm{b}}{(\mathrm{b}^2 + \propto^2 \kappa^2)})$ is a derivative of $\boldsymbol{ln}(\mathbf{1} + \frac{(\propto\kappa)^2}{\mathrm{b}^2})$, then $\boldsymbol{ln}(\mathbf{1} + \frac{(\propto\kappa)^2}{\mathrm{b}^2})\ \boldsymbol{is}$ integral of $(\frac{2}{\mathrm{b}} - \frac{2\ \mathrm{b}}{(\mathrm{b}^2 + \propto^2 \kappa^2)})$, this exactly what we are going to us: by put

$$\frac{1}{\kappa} \int_{\mathrm{b}}^{\infty} \boldsymbol{V}(\tilde{\boldsymbol{s}}\ , \kappa)\ \boldsymbol{d\tilde{s}} = \boldsymbol{ln}(\mathbf{1} + \frac{(\propto\kappa)^2}{\mathrm{b}^2})$$

$$\therefore\quad \mathbb{s}^{-1}\left[\kappa\ \boldsymbol{ln}(\mathbf{1} + \frac{(\propto\kappa)^2}{\mathrm{b}^2})\right] = \mathbb{s}^{-1}\left[\frac{2\ \kappa}{\mathrm{b}} - \frac{2\ \kappa\ \mathrm{b}}{(\mathrm{b}^2 + \propto^2 \kappa^2)}\right]$$

$= 2 - 2\cos(\propto \boldsymbol{t})$

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