Shehu Transformation for Activity Costing System

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ARTICLE INFO	ABSTRACT
	In this paper, we expanded the application of Shehu transformation to obtained the solution for system of ordinary differential equations that subjected to known or unknown initial conditions, which applies in calculation for the cost of activity.

Introduction

The company's overlapping activities (one provides support to the other) difficult to calculate the correct cost of the activity. The activity must be determined according to its cost and the amount of the benefit cost that it gets from other activities. In addition to subtracting the benefit cost that this activity provides to other activities from their total cost. Given the time factor, things will get more complicated, and system of ordinary differential equations has been used to simplify the calculation of the cost of the activity in the Time-Driven Activity- Based Costing method.

Many researchers have worked to develop the solution of differential equations for their wide applications in other sciences, such as calculations the correct cost of the activity. One of these methods is the integral transformation that were used to solve many different types of differential equations, such as Laplace, Novel, AlTememi, Elzaki…etc[1,2,3,4,5,12]. Recently, anew integral transformation which called Shehu transformation emerged in 2019. Fundamental properties prored by Maitama[8,9]. Later Aggrarwal applied for handling Volterra integral equation[7]. Posteriorly, heat and transport equations solved by Shehu transformation[4,6,10,11].

The Shehu transform of the function $\varphi(t)$ of exponential order is defined over the set of functions,

$$A = \left\{\varphi(t): \exists N, \eta_1, \eta_2 > 0, |\varphi(t)| \, N \exp\left(\frac{|t|}{\eta_i}\right), if \, t \in (-1)^i \times [0, \infty)\right\},$$

$$\mathbb{S}[\varphi(t)] = \psi(\delta, \vartheta) = \int_0^\infty \exp\left(\frac{-\delta t}{\vartheta}\right) \varphi(t) dt \qquad (1)$$

$$= \lim_{\alpha \to \infty} \int_0^\alpha \exp\left(\frac{-\delta t}{\vartheta}\right) \varphi(t) dt; \delta > 0, \vartheta > 0,$$

It converges if the limit of the integral exists. In this study, we reviewed some important definitions and evidences that we need in the following of research and we show the applicability of interesting anew transform and its efficiency in solving, the linear system of ordinary differential equation, when the condition exist or not.

Basic definitions and fundamental properties:

The inverse Shehu transform is defined by

$$\mathbb{S}^{-1}[\psi(\delta, \vartheta)] = \varphi(t), \, for \, t \geq 0 \,,$$

which is rewarded

$$\varphi(t) = \mathbb{S}^{-1}[\psi(\delta, \vartheta)] = \frac{1}{2\pi i} \int_{\alpha - i\infty}^{\alpha + i\infty} \frac{1}{\vartheta} \exp\left(\frac{\delta t}{\vartheta}\right) \psi(\delta, \vartheta) d\delta \qquad (2)$$

the Shehu transform variables are δ and ϑ , and the real constant is α, the integral in equation (1.2) is taken along $\delta = \alpha$ in the complex plane $\delta = x + iy$

Property 1[9]: Let the functions $\alpha\varphi(t)$ and $\beta\omega(t)$ be in set A, then $\left(\alpha\varphi(t) + \beta\omega(t)\right)\epsilon A$, $\alpha, \beta > 0$ arbitrary constants,so

$$\mathbb{S}[\alpha\varphi(t) + \beta\omega(t)] = \alpha\mathbb{S}[\varphi(t)] + \beta\mathbb{S}[\omega(t)] \qquad (3)$$

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Lemma(1) : Derivative of Shehu transform

If $\varphi(t), \varphi'(t), \dots, \varphi^{(n-1)}(t)$ are continuous functions for $t>0$ and of exponential order as $t \to \infty$, also $\varphi^{(n)}(t)$ is continuous functions , it follows that :

$$\mathbb{S}[\varphi^{(n)}(t)\ ] = \frac{\delta^n}{\vartheta^n} \varphi\, \psi(\delta, \vartheta) - \sum_{r=0}^{n-1} \left(\frac{\delta}{\vartheta}\right)^{n-(r+1)} \varphi^{(r)}(0) \qquad (4)$$

If n=1,2 and 3 , we get:

$$\mathbb{S}[\varphi'(t)\ ] = \frac{\delta}{\vartheta}\, \psi(\delta, \vartheta) - \varphi(0) \qquad (5)$$

$$\mathbb{S}[\varphi''(t)] = \frac{\delta^2}{\vartheta^2} \psi(\delta, \vartheta) - \frac{s}{\vartheta}\varphi(0) - \varphi'(0) \qquad (6)$$

$$\mathbb{S}[\varphi'''(t)] = \frac{\delta^3}{\vartheta^3} \psi(\delta, \vartheta) - \frac{\delta^2}{\vartheta^2}\psi(0) - \frac{\delta}{\vartheta}\varphi'(0) - \varphi''(0) \qquad (7)$$

Proof: We let equation(1.5) is true for *n=r*, we get

$$\mathbb{S}[\varphi^{(r)}(t)] = \frac{\delta}{\vartheta}\mathbb{S}[\varphi^{(r)}(t)] - \varphi^{(r)}(0)$$

$$= \frac{\delta}{\vartheta}\left[\frac{\delta^r}{\vartheta^r}\mathbb{S}[\varphi(t)] - \sum_{i=0}^{r-1}\left(\frac{\delta}{\vartheta}\right)^{r-(i+1)}\varphi^{(i)}(0)\right] - \varphi^{(r)}(0)$$

$$= \left(\frac{\delta}{\vartheta}\right)^{r+1}\mathbb{S}[\varphi(t)] - \sum_{i=0}^{r}\left(\frac{\delta}{\vartheta}\right)^{r-i}\varphi^{(i)}(0)$$

By induction, the last equation implies that equation (1.5) holds for *n=r+1* and the proof is complete.

Remark:

To know number of arbitrary constants in general solution of linear system, such as:

$$k_{11}(D)x_{11} + k_{12}(D)x_{13} + \cdots + k_{1n}(D)z_{1n} = b_1(t)\,,$$

$$k_{21}(D)x_{21}x + k_{22}(D)x_{22} + \cdots + k_{2n}(D)x_{2n} = b_2(t),$$

$$k_{n1}(D)x_{n1} + k_{n2}(D)x_{n2} + \cdots, k_{nn}(D)x_{nn} = b_n(t),$$

Where D is an operator which is represent $= \frac{d}{dt}$, $k_{11}(D) \dots k_{nn}(D)$ are functions of D . $x_{ij}\ i\,, j = 1, 2, \dots, n$ and t represent dependent and independent variable in the system, respectively.

Can be equal to the degree of D in the determent

$$\begin{vmatrix} k_{11}(D) & k_{12}(D) & \dots & k_{1n}(D) \\ k_{21}(D) & k_{22}(D) & \dots & k_{2n}(D) \\ \dots & \dots & \dots & \\ k_{1n}(D) & k_{2n}(D) & \dots & k_{3n}(D) \end{vmatrix} \equiv \Delta \qquad (8)$$

If $\Delta \equiv 0$ then the set of the solutions is not independent and it is out of our studying so assume Δ is not zero.

System of Linear Ordinary Differential Equations of n-th order:

If ψ is a vector where elements are functions $\varphi_1(t), \varphi_2(t), \dots, \varphi_n(t)$, and is a vector valued of φ and it is derivatives, so $\varphi^{(n)} = F(t, \varphi, \varphi', \varphi'', \dots, \varphi^{(n-1)})$ is an explicit system of ordinary differential equations of order n and dimensions m, such as equation column vector:

$$\begin{pmatrix} \varphi_1^{(n)} \\ \varphi_n^{(n)} \\ \vdots \\ \varphi_n^{(n)} \end{pmatrix} = \begin{pmatrix} F_1(t), \varphi, \varphi', \varphi'', \dots, \varphi^{(n-1)} \\ F_2(t), \varphi, \varphi', \varphi'', \dots, \varphi^{(n-1)} \\ \vdots \\ F_m(t), \varphi, \varphi', \varphi'', \dots, \varphi^{(n-1)} \end{pmatrix} \qquad (9)$$

There for $F(t, \varphi, \varphi', \varphi'', \dots, \varphi^{(n)}) = \overline{0}$, where $\overline{0} = (0, 0, \dots, 0)$ is zero vector. $\varphi_1, \varphi_2, \dots, \varphi_m$ are integrable function to solve this system by take Shehu transform, we have

$$\begin{pmatrix} a_{11}(\delta, \vartheta) & a_{12}(\delta, \vartheta) & a_{1n}(\delta, \vartheta) \\ & & \\ a_{m1}(\delta, \vartheta) & a_{m2}(\delta, \vartheta) & a_{mn}(\delta, \vartheta) \end{pmatrix} \begin{pmatrix} \varphi_1(\delta, \vartheta) \\ \varphi_2(\delta, \vartheta) \\ \vdots \\ \varphi_n(\delta, \vartheta) \end{pmatrix} = \begin{pmatrix} h_1(\delta, \vartheta) \\ h_2(\delta, \vartheta) \\ \vdots \\ h_m(\delta, \vartheta) \end{pmatrix}$$

Where $a_{11} \dots a_{nm}$ and $h_1(\delta, \vartheta) \dots h_m(\delta, \vartheta)$ polynomial of Shehu has degree less than n. $\varphi_1(\delta, \vartheta) \dots \varphi_n(\delta, \vartheta)$ are Shehu transform of the function $\varphi_1(\delta, \vartheta), \dots \varphi_m(\delta, \vartheta)$.

The solution of the above system give us solution of (9). Taking in to (8) and account the number of constants according to (8) and wheth conditions are defined or not

Example 1:

Consider the first order differential equation:

$\frac{d\varphi_1(t)}{dt} - \varphi_2(t) = 1$

$\frac{d\varphi_2(t)}{dt} + \varphi_1(t) = 2cost$ (10)

$\varphi_1(0) = 0, \varphi_2(0) = -1$, with initial condition.

Taking Shehu transform of the system with initial conditions, we get:

$\frac{\delta}{\vartheta}\Psi_1(\delta,\vartheta) - \varphi_1(0) - \Psi_2(\delta,\vartheta) = \frac{\delta}{\vartheta}$

$\frac{\delta}{\vartheta}\Psi_2(\delta,\vartheta) - \varphi_2(0) - \Psi_1(\delta,\vartheta) = \frac{2\delta\vartheta}{\delta^2+\vartheta^2}$

Solving the equation for $\Psi_1(\delta,\vartheta)$ and $\Psi_2(\delta,\vartheta)$, we have:

$\Psi_1(\delta,\vartheta) = \frac{\vartheta\delta^3+3\vartheta^3\delta}{(\delta^3+\vartheta^3)(\delta^3+\vartheta^3)}$

By using partition fraction of the equation above, we obtain:

$\Psi_1(\delta,\vartheta) = \frac{\delta\vartheta}{\delta^2+\vartheta^2} + \frac{2\vartheta^3}{(\delta^2+\vartheta^2)^2}$ (11)

Now, take invers of Shehu transform to both side, we have:

$\varphi_1(t) = cost + tsint$

We substitute of equation (13) in equation (12) by the simple calculation, we can get $\varphi_2(t)$ after taking Shehu transform to both sides, yield:

$\varphi_2(t) = -1 + tcost$

Example 2:

Consider the following system of their depended variables:

$\frac{d\varphi_1(t)}{dt} = \varphi_3(t) - cost$

$\frac{d\varphi_2(t)}{dt} = \varphi_3(t) - e^t$ (12)

$\frac{d\varphi_3(t)}{dt} = \varphi_1(t) - \varphi_2(t),$

with the initial conditions $\varphi_1(0) = 1 , \varphi_2(0) = 0 , \varphi_3(0) = 2$

Take the Shehu transform of system with initial conditions, we get:

$\frac{\delta}{\vartheta}\Psi_1(\delta,\vartheta) - \Psi_3(\delta,\vartheta) = \frac{\delta-\delta\vartheta+\vartheta^2}{\delta^2+\vartheta^2}$

$\frac{\delta}{\vartheta}\Psi_2(\delta,\vartheta) - \Psi_3(\delta,\vartheta) = \frac{-\vartheta}{\delta-\vartheta}$

$\frac{\delta}{\vartheta}\Psi_3(\delta,\vartheta) - \Psi_1(\delta,\vartheta) + \Psi_2(\delta,\vartheta) = 2$

The solutions of these algebraic equation are:

$\Psi_1(\delta,\vartheta) = \frac{\vartheta\delta^2+\vartheta^3}{(\delta-\vartheta)(\delta^2+\vartheta^2)}$, $\Psi_2(\delta,\vartheta) = \frac{\vartheta^2\delta+\vartheta^3}{(\delta-\vartheta)(\delta^2+\vartheta^2)}$, $\Psi_3(\delta,\vartheta) = \frac{2\vartheta\delta^2+\vartheta^3-\vartheta^2\delta}{(\delta-\vartheta)(\delta^2+\vartheta^2)}$

By similar way in pervious example to get

$\varphi_1(t) = e^t , \varphi_2(t) = sint , \varphi_3(t) = e^t + cost$

Example 3:

For solving the system of second order

$\varphi_1''(t) + 2\varphi_1'(t) + \varphi_1(t) + 2\varphi_2(t) + 3\varphi_3(t) = 1$

$\varphi_1'(t) + \varphi_2(t) = 1$ (17)

$\varphi_1(t) - \varphi_2'(t) - \varphi_3'(t) = 0$

with initial conditions:

$\varphi_1(0) = \varphi_3(0) = 1, \varphi_1'(0) = -1 , \varphi_2(0) = 0$

Applied Shehu transform to this system and by substituting these values of initial condition in (17), so:

$\frac{\delta^2}{\vartheta^2}\Psi_1(\delta,\vartheta) - \frac{\delta}{\vartheta}\varphi_1(0) - \varphi_1'(0) + 2\frac{\delta}{\vartheta}\Psi_1(\delta,\vartheta) - 2\varphi_1(0) + \Psi_1(\delta,\vartheta) + 2\Psi_2(\delta,\vartheta) - 2\varphi_2(0) + 3\frac{\delta}{\vartheta}\Psi_3(\delta,\vartheta) - 3\varphi_3(0) = \frac{\vartheta}{\delta}$

$\frac{\delta}{\vartheta}\Psi_1(\delta,\vartheta) - \varphi_1(0) + \Psi_3(\delta,\vartheta) = 0$

$\Psi_1(\delta,\vartheta) - \frac{\delta}{\vartheta}\Psi_2(\delta,\vartheta) - \varphi_2(0) + \frac{\delta}{\vartheta}\Psi_3(\delta,\vartheta) - \varphi_3(0) = 0$

The solution of these algebraic equation are:

$$\Psi_1(\delta,\vartheta) = \frac{\vartheta\delta+\frac{1}{2}\vartheta}{\delta(\delta+\frac{3}{2}\vartheta)} \ , \Psi_2(\delta,\vartheta) = \frac{\frac{1}{2}(5\vartheta^2\delta+\vartheta^3)}{\delta^2(\delta+\frac{3}{2}\vartheta)} \ , \Psi_3(\delta,\vartheta) = \frac{\delta}{\delta+\frac{3}{2}\vartheta}$$

By using partition fraction and simplification of $\Psi_1(\delta,\vartheta)$, $\Psi_2(\delta,\vartheta)$ **and** $\Psi_3(\delta,\vartheta)$ and take inverse of Shehu transform, we obtain solution:

$$\varphi_1(t) = \frac{1}{3} + \frac{2}{3}e^{\frac{2}{3}t}$$

$$\varphi_2(t) = \frac{13}{9} + \frac{1}{3}t - \frac{13}{9}e^{\frac{-2}{3}t}$$

$$\varphi_3(t) = \frac{-3}{2}t$$

Example 4:

To solve the system:

$$2\varphi_1'(t) + 2\,\varphi_1(t) - 4\varphi_2(t) = -4t - 2 \qquad\qquad (13)$$

$$2\,\varphi_1(t) - \varphi_2'(t) = -4t$$

which is subject to unknown condition,

Taking Shehu transform to equation (18) without using any initial conditions, we have:

$$\left(\frac{\delta}{\vartheta} + 2\right)\Psi_1(\delta,\vartheta) - 4\Psi_2(\delta,\vartheta) = \frac{h_1(\delta,\vartheta)}{\delta^2} \qquad\qquad (14)$$

$$2\Psi_1(\delta,\vartheta) - \frac{\delta}{\vartheta}\Psi_2(\delta,\vartheta) = \frac{h_2(\delta,\vartheta)}{\delta^2}$$

Where $\Psi_1(\delta,\vartheta)$, $\Psi_2(\delta,\vartheta)$ are Shehu transform of $\varphi_1(t)$, $\varphi_2(t)$ respectively and $h_1(\delta,\vartheta)$, $h_2(\delta,\vartheta)$ is polynomial of δ **and** ϑ.

By the simple calculation to equation (19), we have:

$$\Psi_1(\delta,\vartheta) = \frac{k_1(\delta,\vartheta)}{\delta^2(\delta+4\vartheta)(\delta-2\vartheta)} \qquad\qquad (15)$$

$$\Psi_2(\delta,\vartheta) = \frac{k_2(\delta,\vartheta)}{\delta^2(\delta+4\vartheta)(\delta-2\vartheta)}$$

Where $k_1(\delta,\vartheta), k_2(\delta,\vartheta)$ is polynomial of δ **and** ϑ.has degree less than the degree of denominator. By using partition fraction of equation (20) and taking inverse of Shehu transformation, we obtain:

$$\varphi_1(t) = A_1 + B_1t + C_1e^{-4t} + D_1e^{2t}$$

$$\varphi_2(t) = A_2 + B_2t + C_2e^{-4t} + D_2e^{2t}$$

Where $\varphi_1(t)$ **and** $\varphi_2(t)$ represent the general solution of deferential equation has eight constants, therefore we should eliminate extra constants since $\Delta = 2$ in (8), for this we have $\varphi_1(t)$, $\varphi_2(t)$ from general solution and substitute $\varphi_1(t)$, $\varphi_1'(t)$, $\varphi_2(t)$ **and** $\varphi_2'(t)$ in equations(18), we get:

$$A_1 = A_2 = B_2 = 0 \quad ,B_1 = -2 \ , \ C_1 = 2C_1 \ , D_1 = D_2$$

Now, the set solution of the system has the form:

$$\varphi_1(t) = -2t - 4C_1e^{-4t} + 4C_1e^{-4t} + 2D_1e^{2t}$$

$$\varphi_2(t) = -4C_1e^{-4t} + 2D_1e^{2t}$$

Which has two arbitrary constants equal to the degree of D in Δ.

Example 5:

To solve the system of second order:

$$\frac{d^2\varphi_2(t)}{dt^2} + \frac{d\varphi_1(t)}{dt} + \frac{d\varphi_2(t)}{dt} = 0$$

$$\frac{2d^2\varphi_1(t)}{dt^2} + \frac{d^2\varphi_2(t)}{dt^2} = 6 + 2e^{-t}$$

Applied Shehu transform to this equation without any initial condition, we get:

$$\frac{\delta^2}{\vartheta^2}\Psi_1(\delta,\vartheta) + \frac{\delta}{\vartheta}\Psi_1(\delta,\vartheta) - \frac{\delta}{\vartheta}\Psi_2(\delta,\vartheta) = h_1(\delta,\vartheta)$$

$$\frac{\delta^2}{\vartheta^2}\Psi_1(\delta,\vartheta) + \frac{\delta^2}{\vartheta^2}\Psi_2(\delta,\vartheta) = \frac{h_2(\delta,\vartheta)}{\delta(\delta+\vartheta)}$$

Where are $h_1(\delta,\vartheta)$, $h_2(\delta,\vartheta)$ polynomial of **and** ϑ, the solution of these algebraic equation are:

$$\Psi_1(\delta,\vartheta) = \frac{k_1(\delta,\vartheta)}{\delta^3(\delta+\vartheta)(\delta+3\vartheta)} \quad , \quad \Psi_2(\delta,\vartheta) = \frac{k_2(\delta,\vartheta)}{\delta^3(\delta+\vartheta)(\delta+3\vartheta)}$$

Where are $k_1(\delta, \vartheta)$, $k_2(\delta, \vartheta)$ are polynomial of δ and ϑ. has degree less than denominator. Now, by similar way for solving the previous example, we get:

$$\varphi_1(t) = A_1 + B_1 t + C_1 t^2 + D_1 e^{-t} + E_1 e^{-3t}$$
$$\varphi_2(t) = A_2 + B_2 t + C_2 t^2 + D_2 e^{-t} + E_2 e^{-3t}$$

Where $\varphi_1(t)$ and $\varphi_1(t)$ represent the general solution of deferential equation which should have three constants, therefore, substitute $\varphi_1(t)$, $\varphi_1'(t)$, $\varphi_2(t)$, $\varphi_2'(t)$ in original equations, we get:

$$D_2 = E_2 = E_1 = 0 \quad , \quad C_1 = C_2 = D_1 = 1$$

Therefore

$$\varphi_1(t) = A_1 + B_1 t + t^2 + e^{-t}$$
$$\varphi_2(t) = A_2 + (2 + B_1)t + t^2,$$

which has three arbitrary constants equal to the degree of D in Δ.

Appendix: The following table showed list of Shehu transform special function such as:

S.No	$\varphi(t)$	$\mathbb{S}[\varphi(t)]$
1	1	$\dfrac{\delta}{\vartheta}$
2	T	$\dfrac{\delta^2}{\vartheta^2}$
3	$exp\alpha(t)$	$\dfrac{\vartheta}{\delta - \alpha\vartheta}$
4	$sin(\alpha t)$	$\dfrac{\alpha\vartheta^2}{\delta^2 + \alpha^2\vartheta^2}$
5	$cos(\alpha t)$	$\dfrac{\vartheta\delta}{\delta^2 + \alpha^2\vartheta^2}$
6	$t\, exp\alpha(t)$	$\dfrac{\vartheta^2}{(\delta - \alpha\vartheta)^2}$
7	$\dfrac{exp(\beta t)\sin(\alpha t)}{\alpha}$	$\dfrac{\vartheta^2}{(\delta - \beta\vartheta)^2 + \alpha^2\vartheta^2}$
8	$\dfrac{exp(\alpha t)}{\beta - \alpha}$	$\dfrac{\vartheta^2}{(\delta - \alpha\vartheta)(\delta - \beta\vartheta)}$
9	$\dfrac{t\sin(\alpha t)}{2\alpha}$	$\dfrac{\vartheta^3\delta}{(\delta^2 - \alpha^2\vartheta^2)^2}$

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