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$\vec{P}_{2k}$ – Factorization of Complete Bipartite Symmetric Digraphs

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ABSTRACT

Min-li Yu [1] gives the necessary and sufficient conditions on path factorization of complete multipartite graphs and earlier for path factorization of complete bipartite graph Kazuhiko Ushio[2] gave the necessary and sufficient conditions for the existence of the $\boldsymbol{P_k}$ –design when $\boldsymbol{k}$ is odd. However, for the odd value of $\boldsymbol{k}$ the path factorization problem of complete bipartite graphs i.e. $\boldsymbol{P_k}$ –factorization of complete bipartite graphs, have been studied by many number researchers [3,4,5,6,7]. For any positive integer $\boldsymbol{p}$, the necessary and sufficient conditions for the existence of the $\boldsymbol{P_{2p}}$ –factorization of a complete bipartite graph were studied by Hong Wang [8]. Beiliang Du [9] extended the work of Hong Wang [8] and gave the necessary and sufficient conditions for the existence of $\boldsymbol{P_{2k}}$ –factorization of the complete bipartite multigraphs. In path factorization of complete bipartite symmetric digraphs B. Du [10] already discussed the necessary and sufficient conditions for the existence of $\vec{\boldsymbol{P}}_{\boldsymbol{3}}$ –factorization of complete bipartite symmetric digraphs. Here in this paper, we will discuss necessary and sufficient conditions for the existence of $\vec{\boldsymbol{P}}_{\boldsymbol{2k}}$ –factorization of complete bipartite symmetric digraphs, and also in this paper, we construct the $\vec{\boldsymbol{P}}_{\boldsymbol{2k}}$ –factorization of complete bipartite symmetric digraphs $\boldsymbol{K^*_{m,n}}$.

1. Introduction

Consider the complete bipartite symmetric digraph $K^*_{m,n}$ with two partite sets V_1 and V_2 where total numbers of points in V_1 are m i.e. $|V_1| = m$ and total numbers of points in V_2 are in n i.e. $|V_2| = n$ points, and for any positive integer k, a directed path on $2k$ points is denoted by $\vec{P}_{2k}$. A spanning subgraph $\vec{F}$ of $K^*_{m,n}$ is called a path factor if each component of $\vec{F}$ is a symmetric path of order atleast two. In particular, a spanning subgraph $\vec{F}$ of $K^*_{m,n}$ is called a $\vec{P}_{2k}$ – factor of $K^*_{m,n}$ if each component of $\vec{F}$ is isomorphic to $\vec{P}_{2k}$. If $K^*_{m,n}$ is expressed as an arc disjoint sum of $\vec{P}_{2k}$ – factors, then this sum is called a $\vec{P}_{2k}$ – factorization of complete bipartite symmetric digraphs $K^*_{m,n}$. It is easy to see that a $\vec{P}_{2k}$ – factorization of complete bipartite symmetric digraphs $K^*_{m,n}$ gives rise to a P_{2k} – factorization of complete bipartite graph $2K_{m,n}$. In this paper we will give the necessary and sufficient conditions for the existence of $\vec{P}_{2k}$ – factorization of complete bipartite symmetric digraphs $K^*_{m,n}$.

2. Mathematical Analysis

In the study of $\vec{P}_{2k}$ – factorization of complete bipartite symmetric digraphs $K^*_{m,n}$, the following theorem i.e. theorem 2.1 is used to gives the necessary conditions for the complete bipartite symmetric digraph $K^*_{m,n}$ to be $\vec{P}_{2k}$ – factorable.

Theorem 2.1: If $K^*_{m,m}$ has $\vec{P}_{2k}$ – factorization then $K^*_{sm,sm}$ has $\vec{P}_{2k}$ – factorization for any positive integer s.

Proof: Let $K_{s,s}$ is a 1-factorable$[11]$, and $\{H_1, H_2, \ldots H_s\}$ be a 1-factorization of it. For each i with $\{1 \le i \le s\}$, replace every edge of H_i with a $K^*_{m,m}$ to get a spanning subgraph G_i of $K^*_{sm,sm}$, such that the $G_i's\{1 \le i \le s\}$ are pair wise edge-disjoint and there sum is $K^*_{sm,sm}$. Since $K^*_{m,m}$ is $\vec{P}_{2k}$ –factorable, therefore G_i is also $\vec{P}_{2k}$ –factorable, and hence, $K^*_{sm,sm}$ is also $\vec{P}_{2k}$ –factorable.

By using the following theorem i.e. theorem 2.2 we prove the necessary conditions for $\vec{P}_{2k}$ – factorization of the complete bipartite symmetric digraph $K^*_{m,n}$.

Theorem 2.2: If a complete bipartite symmetric digraph $K^*_{m,n}$ has $\vec{P}_{2k}$ –factorization then $m = n$ and $m \equiv 0(mod\ k(2k-1))$, where k, m and n are positive integers.

Proof: Since $K^*_{m,n}$ be a complete bipartite symmetric digraph with two partite sets V_1 and V_2 where the total number of points in V_1 i.e. $|V_1| = m$ and a total number of points in V_2 i.e. $|V_2| = n$ points. Let $\{\vec{F}_1, \vec{F}_2, \dots \vec{F}_r\}$ be the factors of a $\vec{P}_{2k} -$ factorization of the complete bipartite symmetric digraph $K^*_{m,n}$. Let $\vec{F}_i$ for each $1 \le i \le r$ have t components in each factor since $\vec{F}_i$ is a spanning subgraph of complete bipartite symmetric digraph $K^*_{m,n}$ hence we have $= n = \frac{kt}{2}$. The total numbers of $\vec{P}_{2k}$ factors in each $\vec{P}_{2k} -$ factorization of the complete bipartite symmetric digraph $K^*_{m,n}$ are $|\vec{F}_i| = \frac{m(2k-1)}{k}$, which is an integer that does not depend on the individual P_{2k} –factors. Hence $m = n$ and

$$m \equiv 0 (mod\ k) \qquad \dots (1)$$

Let d be the total number of components of $\vec{P}_{2k} -$ factors in $\vec{P}_{2k}$ –factorization of the complete bipartite symmetric digraph $K^*_{m,n}$ then, $d = \frac{2m^2}{(2k-1)}$ and hence total number of $\vec{P}_{2k}$ factors in $\vec{P}_{2k} -$ factorization of $K^*_{m,n}$ are given by

$$r = \frac{d}{t}$$

$$r = \frac{\frac{2m^2}{(2k-1)}}{\frac{2m}{k}} = \frac{km}{(2k-1)}.$$

Where r is a positive integer. Since gcd$(k, (2k-1)) = 1$ therefore

$$m \equiv 0 (mod\ (2k-1)) \qquad \dots (2)$$

Hence by combining these two results from equations (1) and (2), when $m = n$ then we have

$$m \equiv 0 (mod\ k(2k-1)).$$

To show $\vec{P}_{2k} -$ factorization of the complete bipartite symmetric digraph $K^*_{m,n}$. We consider a particular case for $k = 1, m = 2$ and $n = 2$ i.e. here we construct $\vec{P}_2 -$ factorization of the complete bipartite symmetric digraph $K^*_{2,2}$,

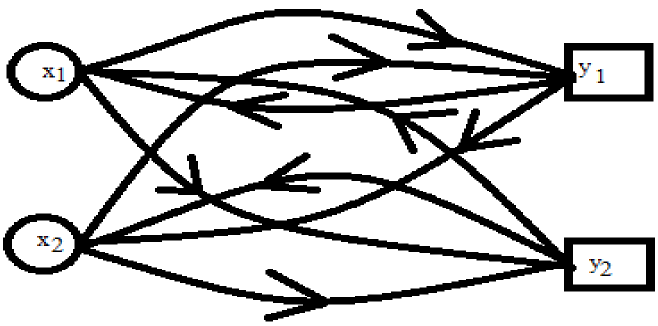

Fig. 2.1 Complete bipartite symmetric digraph $K^*_{2,2}$.

Fig. 2.2 to Fig. 2.3 shown the $\vec{P}_2 -$ factors of complete bipartite symmetric digraph $K^*_{2,2}$.

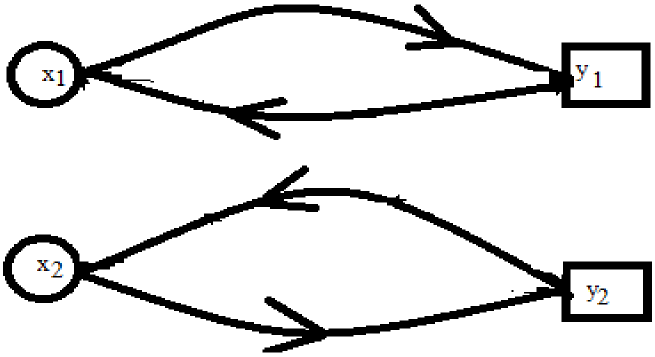

Fig.2.2 $\vec{P}_2 -$ **factor** $x_1y_1, x_2y_2 ; y_2x_2, y_1x_1$.

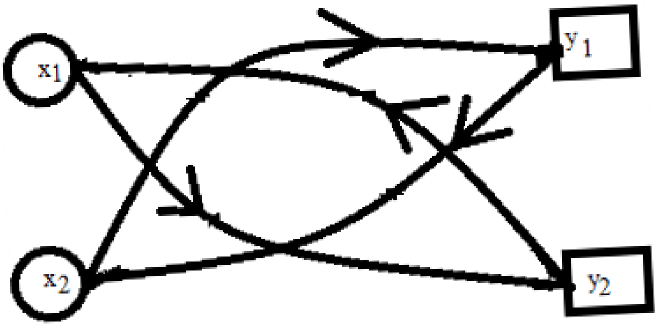

Fig. 2.3 $\vec{P}_2 -$ **factor** $x_1y_2, x_2y_1 ; y_1x_2, y_2x_1$.

Now in following theorem (2.3), we have proved the sufficient condition for the existence of the $\vec{P}_{2k}$ –factorization of the complete bipartite symmetric digraph $K^*_{m,n}$.

Theorem 2.3: Let k, m and n are positive integers, if $m = n$ and $m \equiv 0(mod k(2k-1))$, then $K^*_{m,n}$ has a $\vec{P}_{2k} -$ factorization.

Proof: Let $m = n$ and $m \equiv 0(mod k(2k-1))$, hence we can suppose that $m = k(2k-1)s$ where s is any positive integer. Hence from the previous theorem i.e. theorem2.1, to show $K^*_{m,n}$ has a $\vec{P}_{2k} -$ factorization it is only needs to show that $K^*_{k(2k-1),k(2k-1)}$ have a $\vec{P}_{2k} -$ factorization. Let V_1 and V_2 be two partite sets of $K^*_{k(2k-1),k(2k-1)}$ and the set

$$V_1 = \{x_{i,j}: 1 \le i \le k, 1 \le j \le (2k-1)\},$$
$$V_2 = \{y_{i,j}: 1 \le i \le k, 1 \le j \le (2k-1)\},$$

Now we are in position to construct $\vec{P}_{2k} -$ factors of the complete bipartite symmetric digraph $K^*_{k(2k-1),k(2k-1)}$. It is already mention that the addition in the first and second subscripts of the $x_{i,j}$'s and $y_{i,j}$'s are taken in the addition modulo $\{1, 2, \dots, k\}$ and $\{1, 2, \dots, (2k-1)\}$ respectively.

Now for each i with $1 \le i \le k$ and for each j with $1 \le j \le (2k-1)$, let

$$\boldsymbol{E_{2i-1} = \{x_{i,j}y_{i,j+2i-2}: 1 \le i \le k, 1 \le j \le (2k-1)\}}$$

and

$$\boldsymbol{E'_{2i-1} = \{y_{2i+j-2}x_{i,j}: 1 \le i \le k, 1 \le j \le (2k-1)\}}$$

For each i with $2 \le i \le k$ and for each j with $1 \le j \le (2k-1)$, let

$$E_{2i-2} = \{x_{i,j}y_{i-1,(j+2i-3)}: 2 \le i \le k, 1 \le j \le (2k-1)\}$$

and

$$\boldsymbol{E'_{2i-2}} = \{\boldsymbol{y_{(i-1),(2i+j-3)}x_{i,j}}: \boldsymbol{2 \le i \le k, 1 \le j \le (2k-1)}\}$$

Now we construct the digraph $\vec{F}$ which is a union of two vertex sets of directed paths $E_i \cup E'_i$ for each $1 \le i \le 2k-1$. Let $\vec{F} = \bigcup_{i=1}^{2k-1}\{E_i, E'_i\}$ be a $\vec{P}_{2k}-$ factor of the complete bipartite symmetric digraph $K^*_{k(2k-1),k(2k-1)}$. Now define a bijection σ from $V_1 \cup V_2$ onto $V_1 \cup V_2$,

$$\sigma: V_1 \cup V_2 \xrightarrow{onto} V_1 \cup V_2$$

such that

$\sigma(x_{i,j}) = x_{i+1,j}$ and $\sigma(y_{i,j}) = y_{i+1,j}$ for all i,j with $1 \le i \le k$ and $1 \le j \le (2k-1)$. Now we construct the diagraph for each i and j with the condition, $1 \le i,j \le k$, let

$$\boldsymbol{\vec{F}_{i,j}} = \{\boldsymbol{\sigma^i(x)\sigma^j(y)}: \boldsymbol{x \in V_1, y \in V_2, xy \in \vec{F}}\}$$

It is shown that the digraph $\vec{F}_{i,j}$ is a line disjoint $\vec{P}_{2k}-$ factors of $K^*_{k(2k-1),k(2k-1)}$ and their union is also $K^*_{k(2k-1),k(2k-1)}$. Hence we see that $K^*_{m,m}$ has a $\vec{P}_{2k}-$ factorization.

Now for the main result, we will combine two theorems i.e. theorem 2.2 and theorem 2.3.

Theorem 2.4: The complete bipartite graph $K^*_{m,n}$ has a $\vec{P}_{2k}-$factorization if and only if $m = n$ and $m \equiv 0(mod\ k(2k-1))$.

Proof: By applying theorem 2.2 and theorem 2.3 along with theorem 2.1, it can be seen that when $m = n$ and $m \equiv 0(mod\ k(2k-1))$ the graph $K^*_{m,n}$ has a $\vec{P}_{2k}-$ factorization, where k, m and n are positive integers.

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