Available online at www.elixirpublishers.com (Elixir International Journal)

Statistics

Elixir Statistics 150 (2021) 55116-55121

ISSN: 2229-712X

Statistical Power Function of Average Control Charts under the Effect of Second Order Auto Regressive Model for Non-Normal Population

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ARTICLE INFO

Article history:

Received: 19 November 2020;
Received in revised form:
27 December 2020;
Accepted: 7 January 2021;

Keywords

Second order Auto-Regressive Model,
Non-Normality,
Edgeworth Series,
Control charts.

ABSTRACT

The effect of Second Order Autoregressive (AR-2) model and non-normality on the power function of the control chart for known σ is studied. The power function is derived by considering the first four terms of an Edgeworth series for AR-2 model. The values of power function for three situation viz. when the roots are (i) real and distinct (ii) real and equal and (iii) complex conjugate along with case when there is no dependency are presented.

1. Introduction

Autoregressive models are utilized by specialized experts to estimate securities costs. For example, moving averages, regression and trends take into consideration past prices in an attempt to make forecasts of future prices. The main difference is that numerous technical indicators endeavor to catch composite nonlinearity of monetary costs to maximize profits, while autoregressive models entirely try to limit the mean squared errors and may yield more exact figures for linear underlying processes. An AR (2) autoregressive process is the second order process, meaning that the current value is based on the previous two values. Statistical process control techniques have found widespread application in industry for process improvement and for estimating process parameters or determining capability. Traditionally, control charts are developed assuming that the sequence of process observations to which they are applied are uncorrelated. Unfortunately, this assumption is frequently violated in practice. The presence of autocorrelation has a profound effect on control charts developed using the assumption of independent observations. The primary impact is to increase frequency with which false action signals are generated, that is, the in-control power function of the control chart is much shorter than advertised. Even very low levels of serial correlation will produce dramatic disturbances in these control charts properties. These disturbances lead to erroneous conclusions about the state of control of the process. Many authors have seen the effect of autocorrelation on the control charts for non-normal population. Chen (2004) used the Burr distribution to find the economic design of VSI control charts under non-normal population. Chen and Chiou (2005) studied the VSI control charts when the process data are correlated. Yang and Hancock (1990) studied the effect of non-normal population on control charts for correlated data. Montegomery and Mastrangelo (1991) presented the methods for applying statistical control charts to auto-correlated data.

2. Power Function of the Average Control Chart under AR (2) Model for Non-normal Population

Consider the equation expressing AR (1) process given by the following model:

$$x_t = \mu + \xi_t,\ t = 1, 2, \dots n \qquad (2.1)$$

where x_t the response at time t is, μ is a population mean, ξ_t is a stationary time series with zero mean and standard deviation σ. Suppose that a correlation test revealed the presence of data dependence and the identification technique suggested autoregressive model of order two AR (2) say, then ξ_t can be expressed as:

$$\xi_t = \rho_1 \xi_{t-1} + \rho_2 \xi_{t-2} + \varepsilon_t$$

$$t = 1, 2, 3, \dots, n \qquad (2.2)$$

where

(i) $\varepsilon_t \sim N(0, \sigma_\varepsilon^2)$,

(ii) $$Cov(\varepsilon_t, \varepsilon_\tau) = \begin{cases} \sigma_\varepsilon^2, & t = \tau \\ 0, & t \neq \tau, \end{cases} \qquad (2.3)$$

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Following Kendall and Stuart (1976) it can be shown that for stationarity, the roots of the characteristic equation of the process in equation (2.2),

$$\phi(B) = 1 - \alpha_1 B - \alpha_2 B^2. \quad (2.4)$$

must lies outside the unit circle, which implies that the parameters α_1 and α_2 must satisfy the following conditions,

$$\alpha_1 + \alpha_2 < 1, \quad \alpha_2 - \alpha_1 < 1 \quad and \; -1 < \alpha_2 < 1. \quad (2.5)$$

Suppose that r_1^{-1} and r_2^{-1} are the roots of the characteristic equation of the process given by equation (2.4) then,

Suppose that r_1^{-1} and r_2^{-1} are the roots of the characteristic equation of the process given by equation (2.4) then,

$$r_1 = \frac{\alpha_1 + \sqrt{\alpha_1^2 + 4\alpha_2}}{2}. \quad (2.6)$$

$$r_2 = \frac{\alpha_1 - \sqrt{\alpha_1^2 + 4\alpha_2}}{2}. \quad (2.7)$$

For stationarity we require $|r_i| < 1, i = 1, 2$ .Thus, there occurs three situations:

(i) Roots r_1 and r_2 are real and distinct $\alpha_1^2 + 4\alpha_2 > 1$.

(ii) Roots r_1 and r_2 are real and equal $\alpha_1^2 + 4\alpha_2 = 0$.

(iii) Roots r_1 and r_2 are complex conjugate $\alpha_1^2 + 4\alpha_2 < 1$.

When the correlation is present in the data we have for the distribution of sample mean $\bar{x}$, its mean and variance is as follows

$$E(\bar{x}) = \mu, \quad (2.8)$$

$$V(\bar{x}) = \frac{\sigma^2}{n}\lambda(\alpha_1, \alpha_2, n) = \frac{\sigma^2}{n}k^2, \quad (2.9)$$

where $k^2 = \lambda(\alpha_1, \alpha_2, n)$, depends on the nature of the roots r_1 and r_2, and for different situations is given as follows:

(i) If the roots r_1 and r_2 are real and distinct:

$$\lambda(\alpha_1, \alpha_2, n) = \left[\frac{r_1(1 - r_2^2)}{(r_1 - r_2)(1 + r_1 r_2)}\lambda(r_1, n) - \frac{r_2(1 - r_1^2)}{(r_1 - r_2)(1 + r_1 r_2)}\lambda(r_2, n)\right]$$

$$= \lambda_{rd}(\alpha_1, \alpha_2, n) \quad (2.10)$$

(ii) If the roots r_1 and r_2 are real and equal:

$$\lambda(\alpha_1, \alpha_2, n) = \left[\frac{1 + r}{1 - r} - \frac{2r(1 - r^n)}{n(1 - r)^2}\right]\left[1 + \frac{(1 + r)^2(1 - r^n) - n(1 - r^2)(1 + r^n)}{(1 + r^2)(1 - r^n)}\right],$$

$$= \lambda_{re}(\alpha_1, \alpha_2, n) \quad (2.11)$$

(iii) If the roots r_1 and r_2 are complex conjugate:

$$\lambda(\alpha_1, \alpha_2, n) = \gamma(d, u) + \frac{2d}{n}W(d, u, n) + z(d, u, n), \quad (2.12)$$

where,

$$\gamma(d,u) = \frac{1 - d^4 + 2d(1 - d^2)\cos u}{(1 + d^2)(1 + d^2 - 2d\cos u)},$$

$$W(d,u,n) = \frac{2d(1 + d^2)\sin u - (1 + d^4)\sin 2u - d^{n+4}\sin(n-2)u}{(1 + d^2)(1 + d^2 - 2d\cos u)^2 \sin u},$$

$$Z(d,u,n) = \frac{2d^{n+3}\sin(n-1)u - 2d^{n+1}\sin(n+1)u + d^n\sin(n+2)u}{(1 + d^2)(1 + d^2 - 2d\cos u)^2 \sin u} \qquad d^2 = -\alpha_2$$

and

$$u = \cos^{-1}\left(\frac{\alpha_1}{2d}\right).$$

For non-normal population, the density function is given by the first four terms of an Edgeworth series:

$$f(x) = \frac{1}{\sigma}\left\{\phi\left(\frac{x-\mu}{\sigma}\right) - \frac{\lambda_3}{6}\phi^{(3)}\left(\frac{x-\mu}{\sigma}\right) + \frac{\lambda_4}{24}\phi^{(4)}\left(\frac{x-\mu}{\sigma}\right) + \frac{\lambda_3^2}{72}\phi^{(6)}\left(\frac{x-\mu}{\sigma}\right)\right\}. \quad (2.13)$$

The distribution of the sample mean is given by Singh (1983), as:

$$g(\bar{x}) = \frac{\sqrt{n}}{\sigma}\left\{\phi\left(\frac{\bar{x}-\mu}{k\sigma/\sqrt{n}}\right) - \frac{\lambda_3 k}{6\sqrt{n}}\phi^{(3)}\left(\frac{\bar{x}-\mu}{k\sigma/\sqrt{n}}\right) + \frac{\lambda_4 k^2}{24n}\phi^{(4)}\left(\frac{\bar{x}-\mu}{k\sigma/\sqrt{n}}\right) + \frac{\lambda_3^2 k^2}{72n}\phi^{(6)}\left(\frac{\bar{x}-\mu}{k\sigma/\sqrt{n}}\right)\right\}. \quad (2.14)$$

Now integrating equation (2.14) afterreplacing μ by μ', we have:

$$\gamma(\bar{x}) = \left\{\Phi\left(\frac{\bar{x}-\mu'}{k\sigma/\sqrt{n}}\right) - \frac{\lambda_3 k}{6\sqrt{n}}\phi^{(2)}\left(\frac{\bar{x}-\mu'}{k\sigma/\sqrt{n}}\right) + \frac{\lambda_4 k^2}{24n}\phi^{(3)}\left(\frac{\bar{x}-\mu'}{k\sigma/\sqrt{n}}\right) + \frac{\lambda_3^2 k^2}{72n}\phi^{(5)}\left(\frac{\bar{x}-\mu'}{k\sigma/\sqrt{n}}\right)\right\}, \quad (2.15)$$

where

$$\phi(x) = \frac{1}{\sqrt{2\pi}}e^{-X^2/2}$$

and

$\phi^{(r)}(X) = \frac{d^r}{dX}\phi(\mathbf{X})$.

If the samples of size n are taken from the population $N(\mu', \sigma^2/n)$ and the value of the mean is plotted with the control limits $\mu \pm 3\,\sigma/\sqrt{n}$, then the power of detecting the change of process is given by the following formula:

$$P_{\bar{X}} = P_r\{\bar{X} \geq \mu + 3\,\sigma/\sqrt{n}\} + P_r\{\bar{X} \leq \mu - 3\,\sigma/\sqrt{n}\} \quad (2.16)$$

Standardizing the above equation (2.16),

$$Z = \frac{\bar{x}-\mu'}{k\sigma/\sqrt{n}} \quad (2.17)$$

The power function for normal distribution is obtained by converting equation (2.16) into the standardized form, we have:

$$P_{\bar{X}} = \left\{P_r\left(Z \geq (\mu-\mu')\sqrt{\frac{n}{k^2\sigma^2}} + \frac{3}{k}\right) + P_r\left(Z \leq (\mu-\mu')\sqrt{\frac{n}{k^2\sigma^2}} - \frac{3}{k}\right)\right\}, \quad (2.18)$$

$$P_{\bar{X}} = \left\{P_r\left(Z \geq \frac{-d\sqrt{n}}{k} + \frac{3}{k}\right) + P_r\left(Z \leq \frac{-d\sqrt{n}}{k} - \frac{3}{k}\right)\right\},$$

$$P_{\bar{X}} = \left\{P_r\left(Z \leq \frac{1}{k}(d\sqrt{n}-3)\right) + P_r\left(Z \leq \frac{1}{k}(-d\sqrt{n}-3)\right)\right\},$$

$$P_{\bar{X}} = \left\{\Phi\left(\frac{1}{k}(d\sqrt{n}-3)\right) + \Phi\left(\frac{1}{k}(-d\sqrt{n}-3)\right)\right\}, \quad (2.19)$$

where $d = \frac{\mu-\mu'}{\sigma}$.

The Power Function of the average control charts when the underlying population is non-normal is obtained by putting above value of equation (2.19) in equation (2.15):

$$P_{\bar{X}} = \left\{\Phi\left(\frac{1}{k}(d\sqrt{n}-3)\right) + \Phi\left(\frac{1}{T}(-d\sqrt{n}-3)\right)\right\} - \frac{\lambda_3 k}{6\sqrt{n}}\left\{\phi^{(2)}\left(\frac{1}{k}(d\sqrt{n}-3)\right) + \phi^{(2)}\left(\frac{1}{k}(-d\sqrt{n}-3)\right)\right\} + \frac{\lambda_4 k^2}{24n}\left\{\phi^{(3)}\left(\frac{1}{k}(d\sqrt{n}-3)\right) + \phi^{(3)}\left(\frac{1}{k}(-d\sqrt{n}-3)\right)\right\} + \frac{\lambda_3^2 k^2}{72n}\left\{\phi^{(5)}\left(\frac{1}{k}(d\sqrt{n}-3)\right) + \phi^{(5)}\left(\frac{1}{k}(-d\sqrt{n}-3)\right)\right\}. \quad (2.20)$$

The values of the power curve are obtained by using the equation (2.20) is given in Table-1 to Table-6 and its diagrammatical representation is given in Fig-1 to Fig-4.

3. Numerical Illustration

The values of power functions for some chosen values of $(\lambda_3, \lambda_4) = (0,0), (0,0.5), (0.5,0), (0.5,0.5), (-0.5,0), (0,-0.5)$ and three different roots (i) real and equal viz. $(\alpha_1, \alpha_2) = (0.8, -0.16)$ (ii) real and distinct viz. $(\alpha_1, \alpha_2) = (0.3, 0.6)$ (iii) complex conjugate $(\alpha_1, \alpha_2) = (0.8, -0.6)$ along with the independent case under non-normality have been worked out using equation (2.20) and given in the Table-1 to 6 for $n = 5, 10, 15.$ To give a visual comparison, the power curves have been drawn in Fig-1 to 4 for different values of λ_3 and λ_4. A comparison of various curves for AR(2) process with classical control chart under non-normality shows that complex conjugate roots have the tendency to bring the power curve for independent observation. However, there is a marked difference in the power curve for the other two situations (i) and (ii). In both the situations there is a large deviation from the power curve in independent case under non-normality but from the visual representation it is clearly seen that at one point all the curves are intersect each other, which implies that, a large autocorrelation results in a reduction of power function when $d \geq 1.5$ and vice versa. In practical situations, though, the case of real and equal roots hardly arises. It is seen from the table that possibility of shifts not only in the mean parameters but also in the auto-regression parameters α_1 and α_2 can take place. A shift in the auto-regressive parameters may result from assignable causes occurring over production time, but also from the

Table 1. Values of Power Function for Average Control Chart under non-normal population when $n = 5$.

$\alpha_1 = 0.3, \alpha_2 = 0.6$						
(λ_3, λ_4) → / d↓	$(0,0)$	$(0,0.5)$	$(0.5,0)$	$(0.5,0.5)$	$(-0.5,0)$	$(0,-0.5)$
0.0	0.19436	0.19625	0.18378	0.18567	0.20248	0.19247
0.3	0.28167	0.28151	0.28795	0.28779	0.27257	0.28183
0.5	0.35631	0.35555	0.37326	0.37250	0.33731	0.35708
0.8	0.48313	0.48337	0.51015	0.51039	0.45666	0.48289
1.0	0.57078	0.57279	0.59878	0.60078	0.54542	0.56878
1.3	0.69541	0.69997	0.71645	0.72101	0.67918	0.69085
1.5	0.76843	0.77288	0.78114	0.78560	0.76081	0.76397
1.8	0.85716	0.85504	0.85654	0.85443	0.86151	0.85928
2.0	0.90154	0.88856	0.89391	0.88094	0.91129	0.91451
2.5	0.94787	0.94897	0.93464	0.93574	0.96083	0.94676
3.0	0.96765	0.96677	0.95388	0.95301	0.98005	0.96852

Table2. Values of Power Function for Average Control Chart under non-normal population when $n = 10$.

$\alpha_1 = 0.3, \alpha_2 = 0.6$						
$(\lambda_3, \lambda_4) \rightarrow$ / $d\downarrow$	$(0,0)$	$(0, 0.5)$	$(0.5, 0)$	$(0.5, 0.5)$	$(-0.5, 0)$	$(0, -0.5)$
0.0	0.34380	0.34424	0.34727	0.34772	0.33687	0.34336
0.3	0.45081	0.45171	0.46680	0.46770	0.43485	0.44992
0.5	0.53511	0.53721	0.55574	0.55784	0.51692	0.53301
0.8	0.66409	0.66850	0.68300	0.68741	0.65019	0.65969
1.0	0.74321	0.74807	0.75599	0.76086	0.73590	0.73836
1.3	0.84174	0.84206	0.84242	0.84274	0.84530	0.84144
1.5	0.89162	0.88309	0.88523	0.87670	0.90062	0.90016
1.8	0.94368	0.94511	0.93125	0.93269	0.95621	0.94225
2.0	0.96568	0.96505	0.95252	0.95190	0.97778	0.96632
2.5	0.98510	0.98300	0.97421	0.97211	0.99418	0.98720
3.0	0.99196	0.98972	0.98358	0.98134	0.99860	0.99421

Table 3. Values of Power Function for Average Control Chart under non-normal population when $n = 15$.

$\alpha_1 = 0.3, \alpha_2 = 0.6$						
$(\lambda_3, \lambda_4) \rightarrow$ / $d\downarrow$	$(0,0)$	$(0, 0.5)$	$(0.5, 0)$	$(0.5, 0.5)$	$(-0.5, 0)$	$(0, -0.5)$
0.0	0.42624	0.42656	0.43590	0.43622	0.41372	0.42591
0.3	0.53885	0.54101	0.55527	0.55742	0.52408	0.53669
0.5	0.62373	0.62752	0.64037	0.64416	0.61113	0.61994
0.8	0.74630	0.75118	0.75606	0.76094	0.74199	0.74143
1.0	0.81645	0.81916	0.81888	0.82159	0.81885	0.81373
1.3	0.89717	0.88827	0.88957	0.88068	0.90723	0.90606
1.5	0.93445	0.93662	0.92293	0.92510	0.94672	0.93228
1.8	0.96977	0.96888	0.95723	0.95634	0.98123	0.97066
2.0	0.98312	0.98131	0.97224	0.97043	0.99242	0.98494
2.5	0.99364	0.99169	0.98654	0.98458	0.99924	0.99560
3.0	0.99690	0.99532	0.99216	0.99058	1.00000	0.99849

Table 4. Values of Power Function for Average Control Chart under non-normal population when n=5.

$\alpha_1 = 0.8, \alpha_2 = -0.6$						
$(\lambda_3, \lambda_4) \rightarrow$ / $d\downarrow$	$(0,0)$	$(0, 0.5)$	$(0.5, 0)$	$(0.5, 0.5)$	$(-0.5, 0)$	$(0, -0.5)$
0.0	0.05651	0.05736	0.04851	0.04936	0.06465	0.05566
0.3	0.15357	0.15362	0.15237	0.15242	0.15343	0.15353
0.5	0.25973	0.25915	0.26724	0.26667	0.25051	0.26030
0.8	0.46871	0.46849	0.48603	0.48581	0.45109	0.46892
1.0	0.61738	0.61836	0.63328	0.63426	0.60255	0.61640
1.3	0.80629	0.80732	0.81017	0.81120	0.80403	0.80525
1.5	0.89278	0.88926	0.88879	0.88527	0.89762	0.89629
1.8	0.96463	0.96443	0.95661	0.95641	0.97222	0.96483
2.0	0.98553	0.98470	0.97905	0.97822	0.99129	0.98636
2.5	0.99702	0.99634	0.99413	0.99346	0.99941	0.99770
3.0	0.99912	0.99874	0.99782	0.99744	1.00000	0.99950

Table 5. Values of Power Function for Average Control Chart under non-normal population when $n = 10$.

$\alpha_1 = 0.8, \alpha_2 = -0.6$						
$(\lambda_3, \lambda_4) \rightarrow$ / $d\downarrow$	$(0,0)$	$(0, 0.5)$	$(0.5, 0)$	$(0.5, 0.5)$	$(-0.5, 0)$	$(0, -0.5)$
0.0	0.09610	0.09632	0.09251	0.09273	0.09941	0.09588
0.3	0.33281	0.33252	0.34099	0.34070	0.32404	0.33310
0.5	0.55929	0.55949	0.57047	0.57067	0.54834	0.55908
0.8	0.84641	0.84632	0.84641	0.84632	0.84699	0.84650
1.0	0.94547	0.94567	0.94050	0.94069	0.95043	0.94528
1.3	0.99333	0.99349	0.99045	0.99060	0.99593	0.99317
1.5	0.99888	0.99870	0.99792	0.99774	0.99971	0.99906
1.8	0.99996	0.99994	0.99988	0.99985	1.00002	0.99998
2.0	1.00000	0.99999	0.99999	0.99998	1.00000	1.00000
2.5	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
3.0	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000

Table 6. Values of Power Function for Average Control Chart under non-normal population when $n = 15$.

	$\alpha_1 = 0.3, \alpha_2 = 0.6$					
$(\lambda_3, \lambda_4) \rightarrow$ $d \downarrow$	$(0,0)$	$(0,0.5)$	$(0.5,0)$	$(0.5,0.5)$	$(-0.5,0)$	$(0,-0.5)$
0.0	0.15112	0.15114	0.15061	0.15063	0.15129	0.15111
0.3	0.53802	0.53809	0.54679	0.54687	0.52933	0.53794
0.5	0.80145	0.80174	0.80341	0.80369	0.79991	0.80117
0.8	0.97580	0.97565	0.97207	0.97192	0.97938	0.97595
1.0	0.99679	0.99661	0.99534	0.99516	0.99811	0.99696
1.3	0.99994	0.99993	0.99986	0.99985	1.00001	0.99996
1.5	1.00000	1.00000	0.99999	0.99999	1.00000	1.00000
1.8	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
2.0	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
2.5	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
3.0	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000

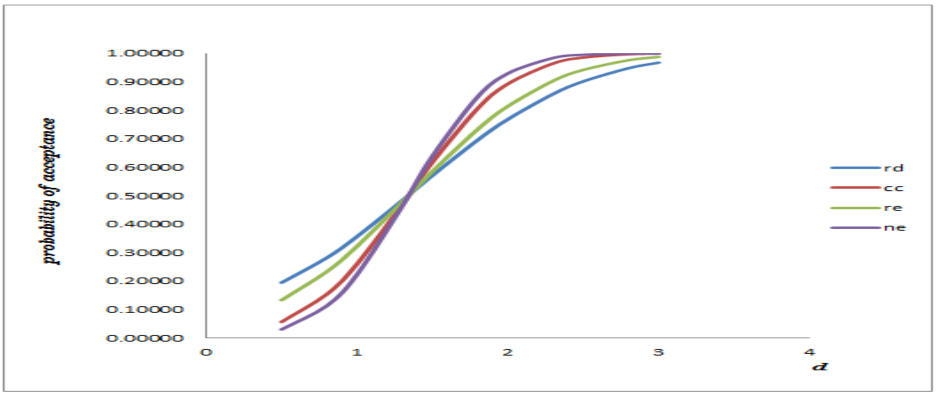

Fig 1. Power Curve for average control chart when $(\lambda_3, \lambda_4) = (0,0)$

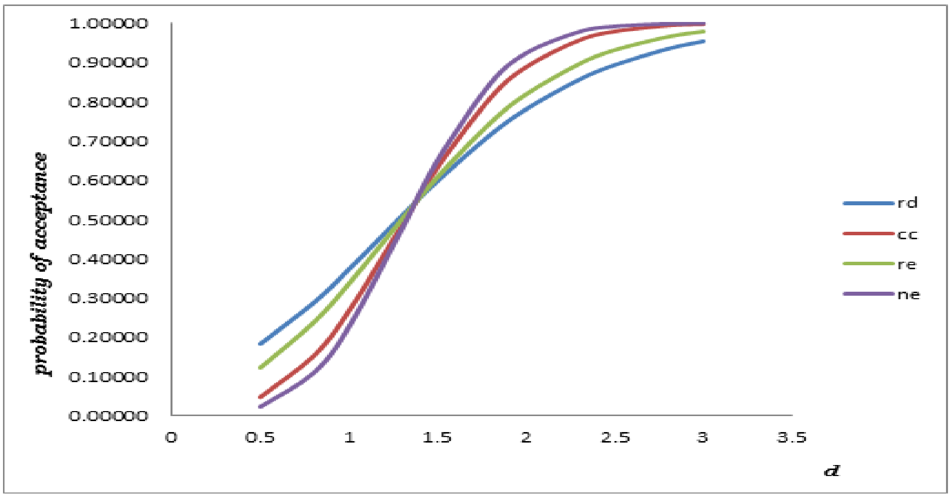

Fig 2. Power Curve for average control chart when $(\lambda_3, \lambda_4) = (0.5,0)$

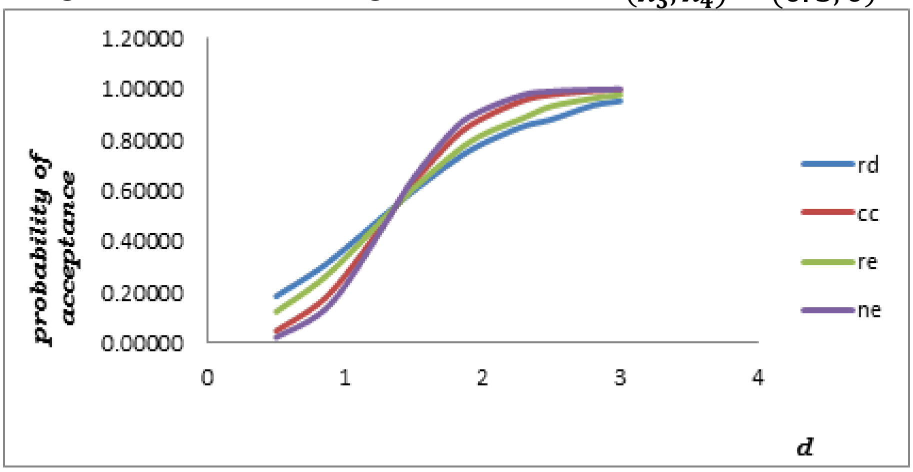

Fig 3. Power Curve for average control chart when $(\lambda_3, \lambda_4) = (0.5,0.5)$

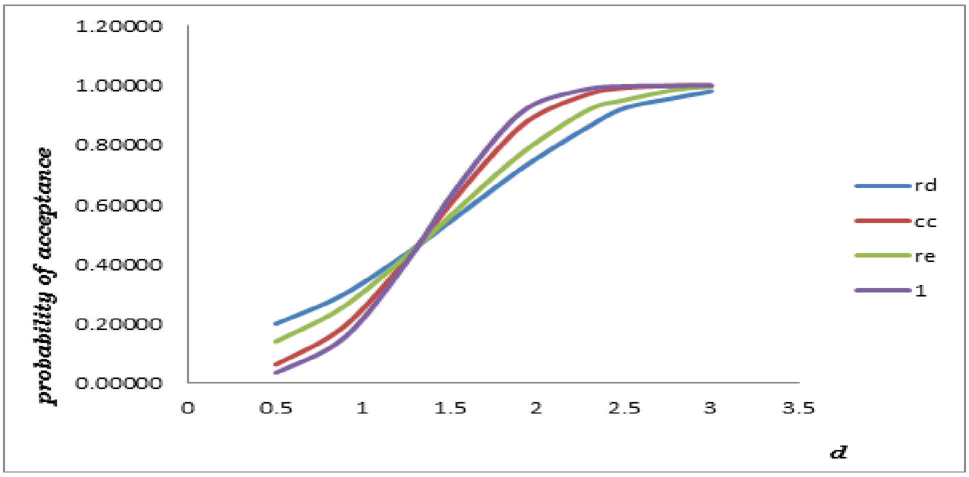

Fig 4. Power Curve for average control chart when $(\lambda_3, \lambda_4) = (-0.5, -0.5)$

misidentification of the auto-regression model ex. a biased estimate of the auto-regression parameters α_1**and**α_2. Hence, we recommend using the power curves for average control chart under the process model where the shifts in the mean and the auto-regression parameters are possible.

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