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Properties of Laplace Transforms With Some of Their Types on Differential Equations

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ABSTRACT

In this manuscript, we will discuss a definition of Laplace transforms with some of their properties. with common applications. This will be according to some types and ranks of differential equations. We will also show how to find the inverse Laplace transform. We also showed what are differential equations of the third order and how the Laplace transform is applied to them. In addition to (Heterogeneous Linear Equations), which is a type of differential equation. The articles of this manuscript were interspersed with a mention of some types of solutions.

Introduction

In a previous article, we discussed, in the first section, about Laplace transforms as a start and definitions

And here in this article, which represents the second section, we will expand a little on the subject of Laplace Transforms. There is a third section on this topic, which we will publish through a third article

Definitions:

1. Laplace Transform: It is an integral transformation when it affects an external function completely different from the original function.
2. The Laplace transform means transforming the independent variable of the original function into another variable, and thus the pronunciation and extent of the original function changes.
3. Converts the seat shape to another, simpler and easier shape.
4. Know the Laplace transform of the function f(t) and denote it by the symbol: f(t) as:
5. From the previous definition, we conclude that the Laplace transform is an indicator of the function f(t) and converts it to another function fs, and it is also an integral indicator and an asymptotic indicator.
6. It is a conversion defined as an integral on the range from the element to infinity.

Properties of Laplace Transforms:

The Laplace transform has the following properties, where we assume that:

F(s)=$[\mathbf{f(t)}]$

1. Linear property:

$$\mathcal{L}[\mathbf{c_1 + f(t) + c_2 g(t)}] = \mathbf{c_1}\mathcal{L}(\mathbf{t}) + \mathbf{c_2}\mathcal{L}[\mathbf{g(t)}] = \mathbf{c_1 f(x) + c_2 G(x)}$$

Application:

Find the Laplace transform of the following function:

The solution :

$$\begin{aligned}\mathcal{L}[\mathbf{3\sin 5t} \quad \mathbf{-2\cos 4t}] &= \mathcal{L}[\mathbf{3\sin 5t}] - \mathcal{L}[\mathbf{2\cos 4t}] \\ &= \mathbf{3L[\sin 5t] - 2L[\cos 2t]} \\ &= -\frac{\mathbf{25}}{\mathbf{s+16}} = \frac{\mathbf{15}}{\mathbf{s^2+25}}\end{aligned}$$

2. The property of the first transition "displacement":

$$\mathcal{L}[e^{at}f(t)] = f(s-a)$$

Application (2-7):

Find the Laplace Transform of the function ekt

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$$= \mathcal{L}[f(t)] = \int_0^\infty e^{-st} f(t)dt$$

Since f(t)=ekt

$$= \mathcal{L}[e^{kt}] = \int_0^\infty (e^{-st})(e^{kt})dt$$

$$\mathcal{L}[e^{kt}] = \int_0^\infty (e^{-(s-k)})dt = \left[\frac{e^{-(s-k)t}}{-(s-k)}\right]$$

$$= \frac{0}{-(s-k)} + \frac{e^0}{s-k} = \frac{1}{s-k}$$

$$\mathcal{L}[e^{-kt}] = \frac{1}{s+k}$$

where k is a constant

3. The second transition feature:

$$g(t) = \begin{bmatrix} f(t-a), & t > a \\ 0, & t < a \end{bmatrix}$$

Van:

$$\mathcal{L}[g(t)] = e^{-as} f(s)$$

$$f(x) = \begin{bmatrix} x & 0 < x < 4 \\ 6 & x > 4 \end{bmatrix}$$

$$C[f(x)] = \int_0^\infty e^{-sx} f(x)dx = \int_0^4 e^{-sx} x dx + \int_4^\infty e^{-sx} 6dx$$

$$= \left[\frac{-xe^{-x}}{s} - \frac{e^{-sx}}{s^2}\right]_0^4 - \left[\frac{6e^{-sx}}{s}\right]_4^\infty$$

$$= \frac{2}{s} e^{-4s} - \frac{e^{-4s}}{s^2} - \frac{1}{s^2}$$

4. Scale change feature:

$$\mathcal{L}[f(at)] = \frac{1}{a} f\left(\frac{s}{a}\right)$$

$$\int (at)[f(t)] = \int_0^\infty e^{-st} f(t)dt = f(s)$$

$$\mathcal{L}[fat] = \int_0^\infty e^{-st} fa(t)dt$$

$$U = at \; t = 0 \to \infty \; du = adt \; u = 0 \to \infty$$

$$\mathcal{L}[f(a,t)] = \int_0^\infty e^{\frac{-s}{a}} uf(u) \frac{du}{a}$$

$$= \frac{1}{a} \int_0^\infty e^{\frac{-s}{a}u} f(u)du$$

Multiplication property of the t_n coefficient:

$$f(s) = \mathcal{L}[f(t)] = \int_0^\infty f(t)e^{-st}dt$$

$$\mathcal{L}[t^n f(t)] = (-1)^n \frac{d}{ds^n} f(s)$$

$$n = 1,2,3 \ldots \ldots$$

Application:

Or the Laplace transform of the function:

$$z(t) = t^2 e^{-3t}$$

The solution:

take

$$\mathcal{L}[t^n f(t)] = (-1)^n \frac{d^n}{ds^n}[\mathcal{L}[e^{-3t}]f(t) = e^{-3t}$$

n=2

$$\frac{d^2}{ds^2}\left[\frac{1}{s+3}\right]$$

$$\frac{d}{ds} = \frac{-1}{(s+3)^2} = \frac{2}{(s+3)^3}$$

5. Divisibility property by t

$$\mathcal{L}\left[\frac{f(t)}{t}\right] = \int_f^\infty f(s)ds$$

behavior when s→∞

$$\lim_{s\to\infty} f(s) = 0$$

Application:

Find : $\mathcal{L}\left[\frac{\sin t}{t}\right]$

The solution:

$$\mathcal{L}[\sin at] = \frac{1}{s^2+1}$$

And since the limit of $\mathbf{lim_{t\to 6}\left[\frac{\sin t}{t}\right] = 1}$ the limit is true and present and by application:

$$\mathcal{L}\left[\frac{\sin t}{t}\right] = \int_s^\infty \frac{du}{u^2+1} = [\tan^{-1} u]_S^\infty = \tan^{-1}\frac{1}{s}$$

6. Integral Transformation Property:

$$\mathcal{L}\left[\int_0^1 (f(u)du\right] = F\frac{(s)}{s}$$

Application:

find:

$$\mathcal{L}\left[\int_0^1 \sin 2udu\right]$$

The solution:

as:

$$\mathcal{L}[\sin 2udu] = \frac{2}{s(s^2+4)}$$

will be:

$$\mathcal{L}\left[\int_0^1 \sin 2udu\right] = \frac{2}{s(s^2+4)}$$

7. Derivative Transformation Property:

For the first derivative:

If f(t) is continuous over (0,N) and is of the order of magnitude for all t>n and if f(t) is continuous over (0,N) parts then:

$$\mathcal{L}[f(t)] = sf(s) - f(0)$$

Application:

Find the Laplace transform of the following function using the derivatives

$$f(t) = e^{2t}, f((t) = e^{2t}, f'(x) = 2e^{2t} \Rightarrow a = 2$$

$$\mathcal{L}[f'(t)] = s\mathcal{L}[f(t) - f(0)$$

$$\mathcal{L}[2e^{2t} = s\mathcal{L}[e^{2t}] - f(0)$$

$$2\mathcal{L}[e^{2t}] - s\mathcal{L}[e^{2t}] = -1$$

$$\mathcal{L}[e^{2t}](2-s) = -1$$

$$\mathcal{L}[e^{2t}] = \frac{-1}{2-s} = \frac{1}{s-2}$$

and the second derivative: $\mathcal{L}[f(t)] = s^2 f(s) - sf(0) - f'(0)$

Application:

Find the Laplace transform of the following function:

$$g(t) = \cos at \Rightarrow g'(t) = -a\sin at$$
$$\mathcal{L}[g''(x)] = -a^2 \cos at$$
$$[\mathcal{L}[g''(t)] = s^2 \mathcal{L}[\cos at] - s(\cos 0) - (-a\sin(0))$$
$$-a^2 \mathcal{L}\cos at = s^2 \mathcal{L}[\cos at] - s$$
$$\mathcal{L}[\cos at](s^2 + a^2) = s$$
$$\mathcal{L}[\cos at] = \frac{s}{s^2 + a^2}$$

Inverse Laplace Transform:

Definition of:

The inverse Laplace transform of the function f(s) is known and symbolized by the symbol L-1 as an inverse operator of the integrative operator L. If L affects the function f(t), then convert it to the function f(s), then the inverse operator L-1 affects the function f(s).) then converts it to its original form f(t), meaning that:

$$\mathcal{L}[f(t) = f(t) \leftrightarrow \mathcal{L} - 1f(S)] = f(t)$$

Properties of the inverse Laplace transform:

If there are both the Lft, L(g) transformations of the functions ft ,g(t) and the inverse Laplace transform of ....

Van : $\mathcal{L}^{-1}(f(s))\mathcal{L}^{-1}G(s)$

(1) $\mathcal{L}((f(t) + g(t)) = \mathcal{L}(f(t)) + \mathcal{L}(g(t)) = f(s) + G(s)$
(2) $\mathcal{L}(af(t)) = a\mathcal{L}(f(s)) = af(s)$
(3) $\mathcal{L}^{-1}(f(s) + G(s)) = \mathcal{L}^{-1}(f(s)) + \mathcal{L}^{-1}(G(s)) = f(t) + g(t)$
(4) $\mathcal{L}^{-1}\big(af(s) = a\mathcal{L}^{-1}\big(f(s)\big) = af(t)$

The sufficient condition for the existence of the Laplace transform:

If the function f(t) is continuous over paragraphs or discontinuous over a finite period:

$$[\mathbf{0}, \mathbf{b}]; \mathbf{b} > \mathbf{0}$$

And it was:

$$|\mathbf{f}(\mathbf{t})| \leq \mathbf{C}\mathbf{e}^{\mathbf{Bt}} \forall \mathbf{t} \geq \mathbf{t_0}$$

Since c,b,t0 are constants, the function f(t) has the Laplace transform L(ft), so for all s>b .

But the function does not have to be continuous over a fixed interval 0,b, because this function gives an infinite value at t = 0 .

Application:

$$\mathbf{f}(\mathbf{t}) = \frac{\mathbf{1}}{\sqrt{\mathbf{t}}}$$

The solution:

$$\int_{\mathbf{0}}^{\mathbf{b}} \frac{\mathbf{1}}{\sqrt{\mathbf{t}}} \mathbf{dt} = \mathbf{lim}_{\mathbf{a}\to\mathbf{0}} \int_{\mathbf{a}}^{\mathbf{b}} \mathbf{t}^{-\mathbf{2}} = \mathbf{2}\sqrt{\mathbf{t}}_{\mathbf{a}}^{\mathbf{b}}$$
$$\mathbf{lim}_{\mathbf{a}\to\mathbf{0}} [\mathbf{2}\sqrt{\mathbf{b}} - \mathbf{2}\sqrt{\mathbf{a}}] = \mathbf{2}\sqrt{\mathbf{b}}$$

The Laplace transform of this function is:

$$\boldsymbol{\mathcal{L}}\left(\frac{\mathbf{1}}{\sqrt{\mathbf{t}}}\right) = \int_{\mathbf{0}}^{\infty} \frac{\mathbf{1}}{\sqrt{\mathbf{t}}} \mathbf{e}^{-\mathbf{st}} \mathbf{dt} = \int_{\mathbf{0}}^{\infty} \mathbf{t}^{-\frac{1}{2}} \mathbf{e}^{-\mathbf{st}} \mathbf{dt}$$

To perform this integration we use integration by substitution:

$$\mathbf{st} = \mathbf{y} \to \mathbf{t} = \frac{\mathbf{y}}{\mathbf{s}}, \mathbf{dt} = \frac{\mathbf{dy}}{\mathbf{ds}}$$
$$\boldsymbol{\mathcal{L}}\left(\frac{\mathbf{1}}{\sqrt{\mathbf{t}}}\right) = \mathbf{s}^{-\frac{1}{2}} \int_{\mathbf{0}}^{\infty} \mathbf{y}^{-\frac{1}{2}} \mathbf{e}^{-\mathbf{y}} = \frac{\mathbf{1}}{\sqrt{\mathbf{s}}} \boldsymbol{\Gamma}\left(\frac{\mathbf{1}}{\mathbf{2}}\right)$$

And this symbol (**Γ**) means the gamma function that we studied previously, and since the gamma function is defined as the defective or anomalous integral

$$\boldsymbol{\Gamma}(\mathbf{x}) = \int_{\mathbf{o}}^{\infty} \mathbf{y}^{\mathbf{x}-\mathbf{1}} \mathbf{e}^{-\mathbf{y}} \mathbf{dy}; \boldsymbol{\Gamma}\left(\frac{\mathbf{1}}{\mathbf{2}}\right) = \sqrt{\boldsymbol{\Pi}}$$

$$\mathcal{L}\left(\frac{1}{\sqrt{t}}\right) = \frac{1}{\sqrt{s}}\sqrt{\Pi} = \sqrt{\frac{\Pi}{s}}; s > 0$$

Application:

Find the inverse Laplace transform of the following function:

$$\mathcal{L}^{-1}\left[\frac{e^{-7s}}{5s+6}\right]$$

The solution:

$$\mathcal{L}^{-1}\left[\frac{e^{-7s}}{5s+6}\right] = e^{-7s}f(s)\,; f(s) = \frac{1}{5s+6}$$

$$a = 7$$

$$\mathcal{L}^{-1}\left[\frac{e^{-2s}}{5s+6}\right] = w(t-7)f(t-7)$$

$$f(t) = \mathcal{L}^{-1}(f(s)) = \mathcal{L}^{-1}\left[\frac{1}{5s+6}\right]$$

$$= \frac{1}{5}\mathcal{L}^{-1}\left[\frac{1}{s+\frac{6}{5}}\right] = \frac{1}{5}\mathcal{L}^{-1}\left[\frac{1}{s-\left(\frac{-6}{5}\right)}\right] = \frac{1}{5}e^{-\frac{6}{5}t}$$

$$f(t-7) = \frac{1}{5}e^{-\frac{6}{5}(t-7)}$$

$$\mathcal{L}^{-1}\left[\frac{e^{-7s}}{5s+6}\right] = w(t-7)e^{-\frac{6}{5}(t-7)}$$

Application:

Find the inverse Laplace transform of the following function:

$$\mathcal{L}^{-1}\left[\frac{(-3s+2)e^{-2s}}{s^2-2s+6}\right]$$

Solution:

$$\mathcal{L}^{-1}[e^{-2s}f(s)] = w(t-2)f(t-2)$$

where:

$$f(t) = \mathcal{L}^{-1}\left[\frac{(-3s+2)}{s^2-2s+6}\right] = \mathcal{L}^{-1}\left[\frac{-3(s-1)-1}{(s-1)^2+5}\right]$$

$$= -3e^t\mathcal{L}^{-1}\left[\frac{s}{s^2+5}\right] - \frac{1}{\sqrt{5}}e^t\mathcal{L}^{-1}\left[\frac{\sqrt{5}}{s^2+5}\right]$$

$$= \left[-3e^t\cos\sqrt{5}t - \frac{e^t}{\sqrt{5}}\sin\sqrt{5}t(t-2)\right]$$

$$\mathcal{L}^{-1}\left[\frac{(-3s+2)e^{-25}}{s^2-2s+6}\right] = e^{t-2}w(t-2)$$

$$= \left[-3\cos(\sqrt{5}(t-2)) - \frac{1}{\sqrt{5}}\sin(\sqrt{5}(t-2))\right]$$

We updated the ordinary differential equations of the first order and the Laplace transform and its inverses, and that the Laplace transform has a large beginning and is special in solving differential equations with fixed equations with the attached initial conditions and this leads to an algebraic equation in the Laplace transformation of the required equation, and by solving we get the Laplace transform and then the procedure Reverse transformation and thus we get the desired solution.

In this chapter, we will talk about differential equations of higher order, with some methods to help solve them, such as the operator method, the parameter method, and some theories.

Third-order differential equations:

Definitions:

If yn denotes the nth object, then the nth order differential equation can be put in the form F(x,y,y`,y″…… yn=0 where y(n) is the solution we are looking for.

Definition of the general solution:

If $y1, y2, y'' + a1y` + a2y''$ … … (1) are two independent solutions to equation (1), then

y=c1y1+c2y2 represents the general solution to equation (1) where c1c2 are optional constants.

Linear differential equations of higher order "nth order":

The equation is linear of the nth order, if it can be put in the following form:

y(n) +p n-1(x)y(n-1) +p n-2(x)y(n-2) +………+p1y`+p0(xy)=f(x);x∈I

Likewise, if the function f(x) is nonzero, at least one of the values of the independent variable x is called a heterogeneous equation.

But if the function fx is zero for all values of the independent variable x in the period I is called a homogeneous equation, solve the linear homogeneous differential equation of higher order and be as follows:

$$\mathbf{a_0y^n + a_1y^{n-1} + a_2y^{n-2} + \cdots \dots \dots + a_{n-1}y' + a_n = 0}$$

Assuming that y= $\mathbf{e^{\lambda x}}$is a solution to the given equation, then the auxiliary equation is:

From which we get the roots

$$\mathbf{a_0\lambda^n + a_1\lambda^{n-1} + \cdots \dots \dots a_{n-1}\lambda + a_n = 0} \dots$$

$$\boldsymbol{\lambda_1, \lambda_2, \lambda_3, \dots \dots \dots, \lambda_n}$$

And we get different solutions according to the relationship between those roots:

1-If $\lambda_1 \neq \lambda_2 \neq \lambda_3 \neq \cdots \dots \dots . = \lambda n$ (real numbers).

The general solution is:-

2-If all roots are real and one m roots is k times

The general solution is

$$\lambda_1 = \lambda_2 = \cdots \dots \dots = \lambda_k, \lambda_{k+1} \neq \cdots \dots \dots \neq \lambda_n \dots$$

$$y = [c_1 + c_2x + c_3x^2 + \cdots \dots \dots + c_kx^{k-1}]e^{\lambda x} + c_ke^{\lambda(k+1)x} + \cdots \dots \dots + c_nx^{\lambda nx}$$

3-If the roots are imaginary numbers:

$$y = \lambda_2 = \lambda_3 = \alpha + Ib$$

There is $\lambda_1 = \lambda_2 = \lambda_3 = \alpha + iB$, and the general solution looking at these roots is:

$$\mathbf{y = e^{\alpha x}[(c_1 + c_2x + c_3x^2)\cos Bx + (c_4 + c_5x + c_6x^2)\sin Bx]}$$

(2-3) Homogeneous Linear Equations:

Application (3-1):-

Find the solution to the equation:-

y″+9y=0

The solution :-

In this case, we find that A = 0, B = 9, and therefore the characteristic equation is:

$$\mathbf{\lambda 2 + 9 = 0, \lambda\,1, \lambda\,2 = 0 \pm 3i}$$

The two roots are imaginations

$$\mathbf{y_c(x) = e^{0(x)}(c_1\cos(3x) + c_2\sin(3x))}$$

$$\mathbf{= c_1\cos(3x) + c_2\sin(3x)}$$

But if the roots are real $\boldsymbol{\lambda}$ 2=+3. $\boldsymbol{\lambda = -3}$

Then:-

$$\mathbf{y_c(x) = c_1e^{-3x} + c_2e^{3x}}$$

Application:-

Find the general solution to the equation:

$$\mathbf{y^{(4)} + 3y^{(3)} - 16y'' + 12y' = 0}$$

The solution :-

The characteristic equation is:

$$\boldsymbol{\lambda(\lambda^3 + 3\lambda^2 - 16\lambda + 12) = 0}$$

$$\boldsymbol{\lambda(\lambda - 1)(\lambda - 2)(\lambda + 6) = 0}$$

Analyzing, we find that

$$\boldsymbol{\lambda(\lambda - 1)(\lambda - 2)(\lambda + 6) = 0}$$

The four roots are $\boldsymbol{\lambda_1 = 0, \lambda_2 = 1, \lambda_3 = 2, \lambda_4 = -6}$

The general solution is:

$$\mathbf{y_g(x) = c_1 + c_2e^{x} + c_3e^{2x} + c_4e^{-6x}}$$

Heterogeneous Linear Equations:

It will be in the following image:-

$$\mathbf{n_0y^n + a_1y^{n-1} + \cdots \ldots \ldots a_{n-1}y + a_n = f(x)a_0 \neq 0}$$

Application:-

Find the solution to the equation:

$$\mathbf{y^{(3)} + 2y'' - 3y' = 4e^{x} - 3\cos(2x)}$$

Solution:

The characteristic equation is:

$$\boldsymbol{\lambda^3 + 2\lambda^2 - \lambda = 0}$$

$$\boldsymbol{\lambda(\lambda^2 + 2\lambda - 1) = 0}$$

The roots are: $\boldsymbol{\lambda_1 = 0, \lambda_2 = -1 + \sqrt{2}, \lambda_3 = -1 - \sqrt{2}}$

The general solution is :- yc=c1+$\mathbf{c_2e(- + \sqrt{2})^{x} + c_3e^{(-1-\sqrt{2})x}}$

The special solution can be obtained by using the comparison method of coefficients and supposing the special solution in the form

$$\mathbf{y_p(x) = Ae^{x} + Bcos(2x) + Csin(2x)}$$

In detail, we get:

$$\mathbf{y_p'(x) = Ae^{x} - 2B\sin(2x) + 2C\cos(2x)}$$

$$\mathbf{y_p''(x) = Ae^{x} - 4B\cos(2x) - 4\sin(2x)}$$

$$\mathbf{y_p'''(x) = Ae^{x} + 8B\sin(2x) - 8C\cos(2x)}$$

Substituting $\mathbf{y_p', y_p'', y_p'''}$p into the original equation, we find that:

A=2 $\mathbf{B = \frac{6}{41}}$, $\mathbf{C = \frac{15}{82}}$,

The private solution is $\mathbf{y_p = 2e^{x} + \frac{6}{41}\cos(2x) + \frac{15}{82}\sin(2x)}$ so the general solution is:

$$\mathbf{y_c(x) = c_1 + c_2e^{(-1+\sqrt{2})} + c_3e^{(-1-\sqrt{2})} + 2e^{x} + \frac{6}{41}\cos(2x) + \frac{15}{82}\sin(2x)}$$

Parameters method (arguments_constant):- (4-3)

Definition:-

This method is generally used to find the special solution yp of the differential equation, given the solution to the homogeneous equation yn we consider the optional constants as functions in the variable x

Now, we will explain this method to litigation equations of the second order, noting that it can be applied to higher order differential equations.

y″+a1$\mathbf{y'}$ +a2y=F(x)

Where a1,a2 are constant and F(x) is a function of the independent variable x, and the homogeneous equation is:

Assuming that the solution to the homogeneous equation on the form yn=Ay1+By2

where y1, y2 are two solutions of the homogeneous equation (2)

Now to find the special solution yp of the differential equation, we consider that each of A, B functions in the variable x and the special solution is in the form:

$\mathbf{y_n = A(x)y_1 + B(x)y_2 ------(3)}$

Differentiating (3) with respect to x, we get:

$\mathbf{y_p' = Ay_1' + By_2 + A'y_1 + B'y_2}$

We choose A,B so that:

$\mathbf{A'y_1 + B'y_2 = 0 ---------- (4)}$

From this, $\mathbf{y_p'p = Ay_1' + By_2'}$ and again differentiating with respect to x, we get:

$\mathbf{y_1'' = A_1'' + A'y_1' + By_2'' + B'y_2'}$

Substituting y″p, ẏp, yp into equation (1), we get:

$\mathbf{Ay_1'' + A'y_2 + By_2'' + B'y_2' + a_1(Ay_1' + By_2') + a_2(Ay_1 + By_2) = F(x)}$

Such as :-

$\mathbf{A(y_1'' + a_1y' + a_2y_1)A'y_1 + B(y_2'' + a_1y_2' + a_2y_2' + a_2y_2) + A'y_1' + B'y_2' = \mathit{F(x)}}$

Where y2, y2 are two solutions of the homogeneous equation (2), then:

$\mathbf{= F(x)y_1' + a_1y_1' + a_2y_1 = 0}$

$\mathbf{y_2'' + a_1y_2' + a_2y_2 = 0}$

So: $\mathbf{A'y_1' + B'y_2'' = F(x) --------(5)}$

By solving equations (4) and (5) in functions A, B, we obtain each of them, and by knowing them, we have obtained the specific solution (3), and thus the general solution can be unified.

yc=ynyp for the differential equation (1), noting that this method is used as a function if the function F(x) is on one of the images: $\mathbf{\frac{e^x}{x}, \sec x, \cot x, \tan x, \ln x, \sin^{-1} x}, \dots \dots \dots$

Now we will apply this method in the following application:

Application (3-5):

Find the general solution to the differential equation:

$$\mathbf{y'' - y = \frac{2}{1 + e^x}}$$

Solution:-

The homogeneous equation is: y″-y=0

Assuming that its solution is y=eλx by differential, we find that the solution to the auxiliary equation is: $\boldsymbol{\lambda^2 - 1 = 0}$. By analyzing, we find that the roots of the equation are: $\boldsymbol{\lambda_1 = 1, \lambda_2 = -1}$

And the solution to the equation is

Homogeneous is: $\mathbf{y_n = Ae^x + Be^{-x}}$

Let us suppose that the specific solution of yp is of the form:

$$\mathbf{y_p = A(x)e^x + B(x)e^{-x}}$$

where Ax,B(x) are two functions of x differentiable with respect to x

We choose both A and B so that: $\mathbf{Ae^x + Be^{-x} = 0}$

Hence: $\mathbf{y_p' = Ae^x - Be^{-x}}$

We differentiate again with respect to

$\mathbf{x}$: $\mathbf{y_p'' = Ae^x + A'e^x + Be^{-x} - B'e^{-x}}$

Substituting y″p, ẏp, yp into the given equation, we find that:

$$\mathbf{Ae^x + A'e^x + Be^{-x} - B'e^{-x} - Ae^x - Be^{-x} = \frac{2}{1 + e^x}}$$

From it we find that:

$\mathbf{A'e^x + B'e^{-x} = \frac{2}{1+e^x} ------(2)}$

Adding equations (1) and (2), we find that: $\mathbf{-2A'e^x = \frac{2}{1+e^x}}$

Separating the variables: $\mathbf{dA = \frac{dx}{e^x(1+e^x)}}$

Integrate:- $\int \mathbf{dA} = \int \mathbf{\frac{dx}{e^x(1+e^x)}} = \int \mathbf{\frac{dx}{e^x}} - \int \mathbf{\frac{dx}{1+e^x}}$

Such as: $= -\mathbf{e}^{-\mathbf{x}} - \int \frac{\mathbf{e}^{-\mathbf{x}}\mathbf{dx}}{\mathbf{e}^{-\mathbf{x}}+\mathbf{1}}$

By subtracting (2) from (1): $\mathbf{B}'\mathbf{e}^{-\mathbf{x}} = \frac{-\mathbf{1}}{\mathbf{1}+\mathbf{e}^{\mathbf{x}}}$

Separating the variables: $\int \mathbf{dB} = -\mathbf{ln}(\mathbf{1}+\mathbf{e}^{\mathbf{x}})$

B=-ln(1+ex)

$$\mathbf{y_p} = \mathbf{e}^{\mathbf{x}}(-\mathbf{e}^{\mathbf{x}} + \mathbf{ln}(\mathbf{1}+\mathbf{e}^{\mathbf{x}})) - \mathbf{e}^{-\mathbf{x}}\,\mathbf{ln}(\mathbf{1}+\mathbf{e}^{\mathbf{x}})$$

$$= -\mathbf{1} + \mathbf{e}^{\mathbf{x}}\,\mathbf{ln}(\mathbf{1}+\mathbf{e}^{-\mathbf{x}}) - \mathbf{e}^{-\mathbf{x}}\,\mathbf{ln}(\mathbf{1}+\mathbf{e}^{\mathbf{x}})$$

The general solution is:

$$\mathbf{y_c} = \mathbf{Ae}^{\mathbf{x}} + \mathbf{Be}^{-\mathbf{x}} - \mathbf{1} + \mathbf{e}^{\mathbf{x}}\mathbf{ln}(\mathbf{1}+\mathbf{e}^{-\mathbf{x}}) - \mathbf{e}^{-\mathbf{x}}\mathbf{ln}(\mathbf{1}+\mathbf{e}^{\mathbf{x}})$$

where B and A are two constant choices.

Theorem:-

If y=eλx is a solution to the homogeneous equation, then λ is a solution to the algebraic equation:

$$\mathbf{a_0\lambda^n e^{\lambda x} + a_1\lambda^{n-1}e^{\lambda x} + a_2\lambda^{n-2}e^{\lambda x}} + \cdots \ldots\ldots. + \mathbf{a_n e^{\lambda x} = 0}$$

the proof :-

$$\mathbf{y' = \lambda e^{\lambda x}, y'' = \lambda^2 e^{\lambda x}, y''' = \lambda^3 e^{\lambda x}}, \ldots\ldots\ldots, \mathbf{y^n = \lambda^n e^{\lambda x}}$$

From these differentials, by direct substitution into the equation, we get the algebraic equation

$$\mathbf{a_0\lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2}e^{\lambda x}} + \cdots \ldots\ldots + \mathbf{a_n e^{\lambda x} = 0}$$

$$\Rightarrow (\mathbf{a_0\lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2}} + \cdots \ldots\ldots + \mathbf{a_n}) = \mathbf{o}$$

This is the name of the algebraic equation for the complementary functions, the auxiliary equation, or the characteristic equation

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