INTRODUCTION

Graph labeling is one of the active research areas in Graph Theory. Much research can be found in different types of graph labeling. A recent survey on graph labeling can be found in [3]. The notion of prime labeling was introduced by Roger Entringer in 1980’s and was discussed in [5]. Present work on graph labeling is focused on complete bipartite graphs. In our work, which is a generalization of [2], prime labeling for tripartite graphs \( K_{2,m,n} \) and \( K_{3,m,n} \) of the form \( K_{1,m,n} \) was considered. In [2] conditions for the existence of prime labeling of complete tripartite graphs of the form \( K_{1,m,n} \) were given where \( m, n \in \mathbb{Z}^+ \). In our work, we have proved that prime labeling of \( K_{2,m,n} \) and \( K_{3,m,n} \) exists for some values of \( m \) and \( n \) where \( m, n \in \mathbb{Z}^+ \). Further, few non-existence cases have been discussed.

DEFINITIONS

A simple graph is tripartite if its vertices can be partitioned into three disjoint subsets in such a way that no edge joins two vertices in the same set. Tripartite graph is complete tripartite if each vertex in one partite set is adjacent to all the vertices in the other two sets. If the three partite sets have cardinalities \( l, m, \) and \( n \), then the resulting complete tripartite graph is \( K_{l,m,n} \).

DEFINITION 1.A simple graph is tripartite if its vertices can be partitioned into three disjoint subsets in such a way that no edge joins two vertices in the same set. Tripartite graph is complete tripartite if each vertex in one partite set is adjacent to all the vertices in the other two sets. If the three partite sets have cardinalities \( l, m, \) and \( n \), then the resulting complete tripartite graph is \( K_{l,m,n} \).

THEOREM 1

(i) Let \( m, n \) be positive integers. Then, \( K_{2,m,n} \) has prime labeling if and only if
\[
2 + m + n \leq \left\lceil \frac{2 + m + n}{2} \right\rceil + 1 \text{ where } \left\lceil \frac{2 + m + n}{2} \right\rceil \text{ denotes the set of all primes } p \text{ such that } \frac{2 + m + n}{2} < p \leq 2 + m + n.
\]

(ii) Let \( m, n \) be positive integers. Then, \( K_{3,m,n} \) has prime labeling if and only if
\[
3 + m \leq \left\lceil \frac{3 + m + n}{2} \right\rceil + 1 \text{ where } \left\lceil \frac{3 + m + n}{2} \right\rceil \text{ denotes the set of all primes } p \text{ such that } \frac{3 + m + n}{2} < p \leq 3 + m + n.
\]

Proof s.(i) First, consider the set of primes given by \( X = \left\lceil \frac{2 + m + n}{2} \right\rceil \{p_1\} \), where \( p_1 \) is a prime. Let \( Y = \{1, \ldots, 2 + m + n\}, Z = \{1, p_1\} \) and \( Y' = Y \setminus \{X \cup Z\} \). Further, \( Z \) is relatively prime to \( Y' \) and \( X \). So, \( Z \) and \( X \) are relatively prime to \( Y' \). Consider set of \( 2 + m \) points labeling \( \{1, \ldots, 2 + m\} \). Join each vertex in \( Z \) to all vertices in \( X \). Then, join each vertex in \( Z \) to all vertices in \( Y' \) and then each vertex in \( X \) to all vertices in \( Y' \). The resulting graph is tripartite and it is \( K_{2,m,n} \) and further it is a prime labeling of \( K_{2,m,n} \).
Similar proof as in part (i) can be obtained by defining $$X = p \left( \frac{3+m+n}{2}, 3+m+n \right) / \{p_1, p_2\}$$ where $$p_1, p_2$$ are primes, $$Y = \{1, \cdots, 3 + m + n\}$$ and $$Z = \{1, p_1, p_2\}$$.

Ramanujan primes are used to obtain the next result of our work. It gives a lower bound for possible prime labeling for large $$n$$. As mentioned in [1, p.10]; let $$\pi(x)$$ denote the number of primes less than or equal to $$x$$, then the $$n$$th Ramanujan prime is the least integer $$R_n$$ for which $$\pi(x) - \pi\left(\frac{x}{2}\right) \geq n$$ holds for all $$x \geq R_n$$. The first five Ramanujan primes are $$R_1 = 2, R_2 = 11, R_3 = 17, R_4 = 29$$ and $$R_5 = 41$$.

**Theorem 2**

(i) $$K_{2,m,n}$$ is prime if $$n \geq R_{m+1} - (m + 2)$$.

(ii) $$K_{3,m,n}$$ is prime if $$n \geq R_{m+2} - (m + 3)$$.

**Proofs.** (i) There are at least $$m$$ primes in the interval $$\left\lfloor \frac{2 + m + n}{2}, 2 + m + n \right\rfloor$$ for $$n \geq R_{m+1} - (m + 2)$$. Denote the first $$m$$ of these primes by $$p_1, p_2, \cdots, p_m$$, then the sets given by $$A_{2,m,n} = \{1, p_1\}, B_{2,m,n} = \{p_2, p_3, \cdots, p_m\}$$ and $$C_{2,m,n} = \{1, \cdots, 2 + m + n\} \setminus (A_{2,m,n} \cup B_{2,m,n})$$ give a prime labeling of $$K_{2,m,n}$$ for $$n \geq R_{m+1} - (m + 2)$$. (iii) Similar proof as in part (i) can be obtained by defining $$A_{3,m,n} = \{1, p_1, p_2\}, B_{3,m,n} = \{p_3, p_4, \cdots, p_m\}$$ and $$C_{3,m,n} = \{1, \cdots, 3 + m + n\} \setminus (A_{3,m,n} \cup B_{3,m,n})$$.

**Results and Discussion**

In this section, results are proved for $$K_{2,m,n}$$ when $$m = 2$$ and 3 and for $$K_{3,3,n}$$. Also a prime labeling of $$K_{2,2,9}$$ is illustrated.

**Corollary 1**

(i) $$K_{2,2,n}$$ is a prime graph if $$n = 9$$ or $$n \geq 13$$.

(ii) $$K_{2,2,n}$$ is not a prime graph if $$2 \leq n \leq 12$$ excluding 9.

**Proofs.** (i) (By inspection) If $$n = 9$$, then $$A_{2,2,9} = \{1, 7\}, B_{2,2,9} = \{11, 13\}$$ and $$C_{2,2,9} = \{2, 3, 4, 5, 6, 8, 9, 10, 12\}$$ give a prime labeling of $$K_{2,2,9}$$. The prime graph of $$K_{2,2,9}$$ is given below:

![Figure 1. Prime labeling of $$K_{2,2,9}$$](image)

Since $$R_3 = 17$$, there are at least three primes $$p_1, p_2, p_3$$ in the interval $$\left\lfloor \frac{n + 4}{2}, n + 4 \right\rfloor$$ for $$n \geq 13$$. Hence, the sets $$A_{2,2,n} = \{1, p_1\}, B_{2,2,n} = \{p_2, p_3\}$$ and $$C_{2,2,n} = \{1, \cdots, n + 4\} \setminus (A_{2,2,n} \cup B_{2,2,n})$$ give a prime labeling of $$K_{2,2,n}$$ for $$n \geq 13$$.

(iii) When $$2 \leq n \leq 12$$, excluding 9, we cannot obtain two subsets with two vertices in each subset such that the vertices of these subsets are relatively prime with labeling of vertices of the other subset.

**Corollary 2**

(i) $$K_{3,3,n}$$ is a prime graph if $$n = 14, 15, 16, 18, 19, 20$$ or $$n \geq 24$$.

(ii) $$K_{3,3,n}$$ is not a prime graph if $$2 \leq n \leq 13, n = 17, 21, 22, 23$$.

**Proofs.** (i) (By inspection) if $$n = 14, 15$$ or 16 then choose $$A_{3,3,n} = \{1, 11\}$$ and $$B_{3,3,n} = \{13, 17, 19\}$$. If $$n = 18, 19$$ or 20, then choose $$A_{3,3,n} = \{1, 13\}$$ and $$B_{3,3,n} = \{17, 19, 23\}$$. In each case, $$C_{3,3,n} = \{1, \cdots, n + 5\} \setminus (A_{3,3,n} \cup B_{3,3,n})$$ gives a prime labeling of $$K_{3,3,n}$$. Since $$R_4 = 29$$, there are at least four primes $$p_1, p_2, p_3, p_4$$ in the interval $$\left\lfloor \frac{n + 5}{2}, n + 5 \right\rfloor$$ for $$n \geq 24$$. Hence, the sets $$A_{3,3,n} = \{1, p_1\}, B_{3,3,n} = \{p_2, p_3, p_4\}$$ and $$C_{3,3,n} = \{1, \cdots, n + 5\} \setminus (A_{3,3,n} \cup B_{3,3,n})$$ give a prime labeling of $$K_{3,3,n}$$ for $$n \geq 24$$.

(iii) A similar proof as for Corollary 1.(ii) can be given.

**Corollary 3**

(i) $$K_{3,3,n}$$ is a prime graph if $$n = 25, 26, 27, 31$$ or $$n \geq 35$$.

(ii) $$K_{3,3,n}$$ is not a prime graph if $$2 \leq n \leq 24, 28 \leq n \leq 34$$ except 31.

**Proofs.** (i) If $$n = 25, 26$$ or 27 then choose $$A_{3,3,n} = \{1, 17, 19\}$$ and $$B_{3,3,n} = \{23, 29, 31\}$$. If $$n = 31$$, then choose $$A_{3,3,n} = \{1, 19, 23\}$$ and $$B_{3,3,n} = \{29, 31, 37\}$$. In each case, $$C_{3,3,n} = \{1, \cdots, n + 6\} \setminus (A_{3,3,n} \cup B_{3,3,n})$$ gives a prime labeling of $$K_{3,3,n}$$.
Since $R_5 = 41$, there are at least five primes $p_1, p_2, p_3, p_4, p_5$ in the interval $\left(\frac{n+6}{2}, n+6\right)$ for $n \geq 35$. Hence, the sets $A_{3,3,n} = \{1, p_1, p_2\}, B_{2,3,n} = \{p_3, p_4, p_5\}$ and $C_{2,3,n} = \{1, \cdots, n+6\} \setminus (A_{3,3,n} \cup B_{3,3,n})$ give a prime labeling of $K_{3,3,n}$ for $n \geq 35$.

(iii) A similar proof as for Corollary 1.(ii) can be given.

The following table gives the prime labeling of tripartite graphs for some $m$.

<table>
<thead>
<tr>
<th>$K_{2,m,n}$</th>
<th>$K_{3,m,n}$</th>
<th>$n$ values for which prime graphs can be obtained</th>
</tr>
</thead>
<tbody>
<tr>
<td>$K_{2,1,n}$</td>
<td>$K_{3,1,n}$</td>
<td>$n = 4, 5, 6$ and $n \geq 8$</td>
</tr>
<tr>
<td>$K_{2,2,n}$</td>
<td>$K_{3,2,n}$</td>
<td>$n = 9$ and $n \geq 13$</td>
</tr>
<tr>
<td>$K_{2,3,n}$</td>
<td>$K_{3,3,n}$</td>
<td>$n = 14, 15, 16, 18, 19, 20$ and $n \geq 24$</td>
</tr>
<tr>
<td>$K_{2,4,n}$</td>
<td>$K_{3,4,n}$</td>
<td>$n = 25, 26, 27, 31$ and $n \geq 35$</td>
</tr>
<tr>
<td>$K_{2,5,n}$</td>
<td>$K_{3,5,n}$</td>
<td>$n = 36, 37, 38$ and $n \geq 40$</td>
</tr>
<tr>
<td>$K_{2,6,n}$</td>
<td>$K_{3,6,n}$</td>
<td>$n = 45, 46, 47, 48, 49$ and $n \geq 51$</td>
</tr>
<tr>
<td>$K_{2,7,n}$</td>
<td>$K_{3,7,n}$</td>
<td>$n = 52$ and $n \geq 58$</td>
</tr>
<tr>
<td>$K_{2,8,n}$</td>
<td>$K_{3,8,n}$</td>
<td>$n \geq 61$</td>
</tr>
<tr>
<td>$K_{2,9,n}$</td>
<td>$K_{3,9,n}$</td>
<td>$n = 62, 68, 69, 70, 72, 73, 74, 78, 79, 80, 81, 82$ and $n \geq 86$</td>
</tr>
<tr>
<td>$K_{2,10,n}$</td>
<td>$K_{3,10,n}$</td>
<td>$n \geq 89$</td>
</tr>
<tr>
<td>$K_{2,11,n}$</td>
<td>$K_{3,11,n}$</td>
<td>$n = 90, 91, 92$ and $n \geq 94$</td>
</tr>
</tbody>
</table>

**Conclusion**

In our work, it has been shown that prime labeling of complete tripartite graphs of the form $K_{2,m,n}$ and $K_{3,m,n}$ exist for some values of $m$ and $n$ where $m, n \in \mathbb{Z}^+$. Those are given in the above table. Also, some non-existing results have been proved.

For a general complete tripartite graph $K_{l,m,n}$, $l, m, n$ can be permuted in $3! = 6$ ways. Moving partite sets with $l, m$ and $n$ number of vertices six graphs can be obtained and all are isomorphic. Hence, permuting $2, m, n$ in $K_{2,m,n}$ and permuting $3, m, n$ in $K_{3,m,n}$ large number of prime graphs can be constructed.

This work can be further generalized for complete bipartite graphs of the form $K_{l,m,n}$ where $l, m, n \in \mathbb{Z}^+$.

**References**


