Non-wandering Sets in Topological Dynamical Systems
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1.0 Introduction
A discrete-time dynamical system $(X,T)$ is a continuous map $T$ on a non-empty topological space $X$ [10][8]. This dynamics is obtained by iterating the map $T$. The discrete logistic function operates within a range by a control parameter. This function changes in state as the parameter is being altered. If we are to take the set of points in the given space $X$ and upon operation by iterating with an initial point, it comes close or exactly back to the points in the set where there is a change which is either, stationary (fixed points), cycles (periodic) or chaos. This state (set) is what we termed as the non-wandering set.

2.0 Preliminaries
Definition 2.1: DISCRETE-TIME DYNAMICAL SYSTEM (See [6],[8]): Let be a non-empty topological space and $T$ be a continuous map. A discrete-time dynamical system $(X,T)$ is defined as; $T:X \rightarrow X$, where the dynamics are obtained by iterating the map $T$, hence, a dynamical system $(X,T)$ induces an action on $X$ by $\theta = x$ and $T^n(x) = x$ for all $n \in \mathbb{Z}$.

Illustration 2.2: (see[1]) Let $f$ be a continuous function on $X$ such that $x \in X$ i.e. $f:X \rightarrow X$, then $\{\ldots,f^{-2}(x),f^{-1}(x),f^{0}(x),f^{1}(x),f^{2}(x),\ldots\}$ are the orbit sequence of $x$ which are bi infinite sequence and form the discrete-time of the solution [1].

But since we are interested in the set type, we let $f$ be a function from the $Z$ of discrete-time to the state space $X$, with parameter and initial point $x$, then the orbit relation is defined as,

$$\theta f = \bigcup_{n=1}^{\infty} f^n \quad (1.0)$$

Hence, the following is the set of the relation but not a sequence; $\vartheta f(x) = \{f^0(x),f^1(x),f^2(x),\ldots\}$ made up of the states and the initial point $x$ in time [9].

2.3 Non-wandering Set: It is the set of points in the phase space for which all points beginning from a point of this set come arbitrarily close and arbitrarily often to any point of the set. In [3], the following shows the types and the existence of non-wandering set:
1. fixed points (stationary)
2. periodic solutions (limit cycles)
3. quasi-periodic orbits
4. chaotic orbits

2.4 Logistic Function: The logistic function is a difference equation which is non-linear system. It is a function that transform into different state or phenomenon depending on changes of the parameter $\alpha$.

Definition 2.5: if $x_n$ is a state with discrete-time $n$, then the function is defined as;

$$x_{n+1} = \alpha x_n(1 - x_n) \quad (1.1)$$

where $n \in \mathbb{N}$ and for $X_n \in [0, 1]$ and $\alpha \in [1, 4]$

2.6 Using the logistic function to illustrate the fixed points and the periodic solutions as the types of the non-wandering set

2.6.1 fixed points (stationary) (see [2],[9])
If a point $x \in X$, its orbit or trajectory is; $\vartheta(x) = \bigcup_{n=1}^{\infty} X^n \quad (1.2)$

Hence, a point $x$ is said to be fixed or stationary if $\vartheta(x) = x \quad (1.3)$

That is $f(X_n) = x_n \quad (1.4)$

Also for the logistic, let $f: X_n \rightarrow X_n$ be defined as;

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Theorem 2.7: A non-wandering set is a fixed point (stationary) if a point $x$ in a space $X$ comes arbitrarily back to the starting point after iterating it for a number of times. That is if $n \in \mathbb{N}$ and $x \in X$ then $\partial f^n(x) = \{x\}$, is a stationary non-wandering set.

Proof: To show that a stationary (fixed point) is a non-wandering set, we take equations (1.4) and (1.5) and Let $n = 0$. Then equations (1.4) and (1.5) become:

$$\begin{align*}
f(x_0) &= x_0 \\
f(x_0) &= \alpha x_0 (1 - x_0)
\end{align*}$$

Equating, Eqn (1.6) to Eqn (1.7)

$$\alpha x_0 (1 - x_0) = x_0$$

Algebraically, $x_0 = 0$ and $x_0 = \frac{\alpha - 1}{\alpha}$ tends to be the solutions for this logistic function.

Illustration 2.8: Taking $x_0 = 0$ and $x_0 = \frac{\alpha - 1}{\alpha}$ to show the existence of the stationary non-wandering set.

At $x_0 = 0$, Trivial
At $x_0 = \frac{\alpha - 1}{\alpha}$,

Let $\lim_{x_0 \to \frac{\alpha - 1}{\alpha}} f(X_0) = \lim_{x_0 \to \frac{\alpha - 1}{\alpha}} [\alpha x_0 (1 - x_0)]$

$= \alpha \lim_{x_0 \to \frac{\alpha - 1}{\alpha}} [x_0 (1 - x_0)]$

$= \alpha \left[ \frac{\alpha - 1}{\alpha} \left(1 - \frac{\alpha - 1}{\alpha} \right) \right]$

$= \alpha \left[ \frac{\alpha - 1}{\alpha} - \frac{\alpha - 1}{\alpha} \right]$

Clearly, if $x = \{0\} \in X$ and $x = \{\frac{\alpha - 1}{\alpha}\} \in X$, where $X \in [0, 1]$ and $\alpha \in [1, 4]$, a discrete value for $\alpha$ gives a point $x$ in the space $X$, by iterating comes back to that same $x$ in the space $X$, hence a non-wandering set.

Example 2.9: Given $f(x_0) = \alpha x_0 (1 - x_0)$. Then at $\alpha = [1,4]$, and $X \ni \{x_0 = \frac{\alpha - 1}{\alpha}\}$. Then a non-wandering set is stationary or fixed point if:

$f\left(x_0 = \frac{\alpha - 1}{\alpha}\right) = \left(x_0 = \frac{\alpha - 1}{\alpha}\right) \in X$

Solution: Let $\alpha = 2$, implies $\left\{(x_0 = 2(1 - \frac{1}{2}) )\right\} n = N$

Then $f(x_0) = 2 x_0 (1 - x_0)$, implies that at $x_0 = \frac{1}{2}$

$$\begin{align*}
f\left(\frac{1}{2}\right) &= 2 \left(\frac{1}{2}\right) \left(1 - \frac{1}{2}\right) = \frac{1}{2} = f\left(f\left(\frac{1}{2}\right)\right) = \frac{1}{2}
\end{align*}$$

Then $f\left(\frac{1}{2}\right) = \left\{\frac{1}{2}\right\} \in X$

Let $\alpha = 3$, implies $\left\{(x_0 = 3(1 - \frac{2}{3}) \right\}$

Then $f(x_0) = 3 x_0 (1 - x_0)$, implies that at $x_0 = \frac{2}{3}$

$$\begin{align*}
f\left(\frac{2}{3}\right) &= 3 \left(\frac{2}{3}\right) \left(1 - \frac{2}{3}\right) = \frac{2}{3} = f\left(f\left(\frac{2}{3}\right)\right) = \frac{2}{3}
\end{align*}$$

Then $f\left(\frac{2}{3}\right) = \left\{\frac{2}{3}\right\} \in X$

Let $\alpha = 4$, implies $\left\{(x_0 = 4(1 - \frac{3}{4}) \right\}$

Then $f(x_0) = 4 x_0 (1 - x_0)$, implies that at $x_0 = \frac{3}{4}$

$$\begin{align*}
f\left(\frac{3}{4}\right) &= 4 \left(\frac{3}{4}\right) \left(1 - \frac{3}{4}\right) = \frac{3}{4} = f\left(f\left(\frac{3}{4}\right)\right) = \frac{3}{4}
\end{align*}$$

Then $f\left(\frac{3}{4}\right) = \left\{\frac{3}{4}\right\} \in X$

Let $\alpha = 1$, implies $\left\{(x_0 = 1 - \frac{1}{1} = 0\right\}$

Then $f(x_0) = x_0 (1 - x_0)$, implies that at $x_0 = 0$

$\therefore f(0) = \{0\} \in X$

Thus, for discrete $\alpha$ value within $[1, 4]$ all $X \ni \{x_0 = \frac{\alpha - 1}{\alpha}\}$ tends to be a fixed point irrespective of the number of iteration, therefore forming their own constant orbit $\partial f^n(x) = \{x\}$ where change in this parameter affect the $X \ni \{x_0 = \frac{\alpha - 1}{\alpha}\}$ and the behavior making it stable or unstable. □

Theorem 2.10: A non-wandering set can be either stable or unstable. Let $X_n \in [0, 1]$ and $\alpha$ be a parameter of the system. Then a change in $\alpha$ of the system can change the stability of a non-wandering set.
Proof: let \( \alpha \in [0, 1] \) and \( X \ni \{ \alpha \} \) where \( \alpha \) is the parameter and define the logistic, let \( f: X_n \rightarrow X_n \) as:
\[
f(X_n) = \alpha X_n (1 - X_n).
\]
Then for stable non-wandering set, the fixed point must be stable or attracting that is \( |f'(x_0)| < 1 \) that is absolute derivative of the function is less than one [6], where \( f'(x_0) = \alpha - 2\alpha x_0 \)
\[
|f'(x_0)| = |\alpha - 2\alpha x_0| < 1, \\
-1 < \alpha - 2\alpha x_0 < 1 \\
-1 < \alpha - 2\alpha x_0 < 1 \\
\alpha < \frac{1}{\alpha} - 1 \quad \alpha < \frac{1}{\alpha} - 1 \\
\alpha < \frac{1}{\alpha} - 1 \quad \alpha < \frac{1}{\alpha} - 1
\]
Thus \( \alpha \in (1, 3) \) is where the function is asymptotically stable. That is attracting fixed point where there is convergence and stability of the state. Hence the non-wandering set is stable at \( \alpha \in (1, 3) \) and attracting since it is true for the fixed point/stationary point. □

Also, for unstable non-wandering set, the fixed point/stationary point must be unstable as the control parameter is altered, at a repelling fixed point. And in [6] the way to this is \( |f'(x_0)| > 1 \quad f'(x_0) = \alpha - 2\alpha x_0 \)
\[
|f'(x_0)| = |\alpha - 2\alpha x_0| > 1, \\
\alpha - 2\alpha x_0 > 1 \quad \alpha - 2\alpha x_0 < -1 \\
\alpha - 2\alpha \left( \frac{\alpha - 1}{\alpha} \right) > 1 \quad \alpha - 2\alpha \left( \frac{\alpha - 1}{\alpha} \right) < -1 \\
\alpha < 1 \quad \alpha > 3 \\
\alpha < 1 \quad \alpha > 3 \\
\therefore \{ \alpha < 1 \} \text{ or } \{ \alpha > 3 \}
\]
Thus, for unstable non-wandering set \( \alpha < 1 \) or \( \alpha > 3 \).

References