A Recurrence Relation to Construct 1- Factors of Complete Graphs

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1. Introduction

A factor of a graph \( G \) is a spanning subgraph of \( G \) which is not totally disconnected. The union of edge disjoint factors which form \( G \) is called factorization of graph \( G \) [3]. An \( n \)-factor is regular of degree \( n \). If \( G \) is the sum of \( n \)-factors, their union is called an \( n \)-factorization [4]. The graph which admits \( n \)-factorization is called an \( n \)-factorable graph.

A 1-factor is a set of pairwise disjoint edges of \( G \) that between them contain every vertex. The necessary conditions to be a 1-factorable graph are that the graph must have an even number of vertices and it should be regular [5]. So, it is conjectured that a regular graph with \( 2n \) vertices and degree greater than \( n \) will always have a 1-factorization [6].

Complete Graphs \( K_n \) is a simple undirected graph such that every pair of distinct vertices is connected by a unique edge and total number of edges is \( n(n-1)/2 \).

Theorem 1: The complete graph \( K_{2n} \) is 1-factorable.

We need to prove a partition of the set \( Y \) of lines of \( K_{2n} \) into \( (2n-1) \) 1-factors. Label the points of \( G \) by \( v_1, v_2, ..., v_{2n} \), and define, for \( i = 1, 2, ..., (2n-1) \), the sets of lines \( Y_i = \{v_i, v_{2n}\} \cup \{v_j, v_{j+i}\} \) for \( j = 1, 2, ..., (n-1) \), where \( i + j \) and \( i - j \) is expressed as one of the numbers \( 1, 2, ..., (2n-1) \) modulo \( (2n-1) \). The collection \( \{Y_i\} \) is displayed to give a suitable partition of \( Y \), and the union of the subgraphs \( G_i \) induced by \( Y_i \) is a 1-factorization of \( K_{2n} \).

The study of 1-factorization is used in various combinatorial applications. An instantaneous application of 1-factorization is that of edge coloring [7]. Also, in scheduling tournament, especially round-robin tournaments [8], study of 1-factorization is used. Other applications of 1-factorization include block designs, 3-designs, and Room square and Steiner system [9], [10].

2. Methodology

In this paper, we produce a recursive method of constructing at 1-factors of \( K_{2n} \) by presenting an algorithm.

Steps of the proposed algorithm

2.1. When \( n = 1 \); Complete graph of 2 vertices. Clearly, it has one 1-factor.
2.2. When \( n = 2 \); Complete graph of 4 vertices.
Label 4 vertices as \( v_1, v_2, v_3 \) and \( v_4 \). Take any vertex (say) \( v_1 \) and join it to any other vertex (say) \( v_2 \). Then join the remaining two vertices.

There are 3 ways to join the vertex \( v_1 \) to other vertices. So, we can construct 3 types of 1-factors.

2.3. When \( n = 3 \); Complete graph of 6 vertices.
Label 6 vertices as \( v_1, v_2, v_3, v_4, v_5 \) and \( v_6 \). Taking any vertex (say) \( v_1 \) and join it to any other vertex (say) \( v_2 \). The remaining 4 vertices could be constructed connected as in \( K_4 \).

There are 5 ways to join the vertex \( v_1 \) to other vertices. So, we can construct 15 types of 1-factors. \( (3 \times 5) \)

By repeating this algorithm, 1-factors corresponding to the complete graph \( K_{2n} \) can be constructed.

3. Results and Discussion

Table 01 illustrate the relationship between number of 1-factors \( K_2, K_4 \) and \( K_6 \).

<table>
<thead>
<tr>
<th>Table 1. Tabulation of the results.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Value of ( n )</td>
</tr>
<tr>
<td>-----------------</td>
</tr>
<tr>
<td>1</td>
</tr>
<tr>
<td>2</td>
</tr>
<tr>
<td>3</td>
</tr>
</tbody>
</table>

Consider the complete graph of \( 2n \) vertices which has \( x_n \) number of 1-factors. Fix one vertex and connect with another vertex. Then there are \( (2n-2) \) remaining vertices. There are \( x_{n-1} \) number of 1-factors corresponding to \( (2n-2) \) vertices. Also, there are \( (2n-1) \) ways of connecting fixed vertex with other vertices.

Using this algorithm a recurrence relation \( x_n = (2n-1)x_{n-1} \) with \( x_1 = 1 \), where \( x_n \) is the number of 1-factors corresponding to the complete graph \( K_{2n} \) can be obtained.

Solving the recurrence relation recursively we can obtain \( x_n \).
\[ x_2 = 3 \times x_1 \]
\[ x_3 = 5 \times x_2 \]
\[ x_4 = 7 \times x_3 \]
\[
\vdots
\]
\[ x_{n-2} = (2n-5) \times x_{n-3} \]
\[ x_{n-1} = (2n-3) \times x_{n-2} \]
\[ x_n = (2n-1) \times x_{n-1} \]

\[ \Rightarrow x_n = (3.5.7\ldots(2n-3)(2n-1)x_1 \]

\[ \Rightarrow x_n = \frac{(2n)!}{2^n.n!} \]

Thus \( K_8 \) has 105 \( I \)-factors and \( K_{10} \) has 945.

Alternative proof is given by the Principle of Mathematical Induction.

When \( n = 1 \), number of 1-factors in \( K_2 = 1 \)

\[ = x_1 = \frac{2!}{2.1!} \]

Thus the result is true for \( n = 1 \).

Assume that the result is true for \( n = p \).

Number of 1-factors in \( K_{2p} = x_p = \frac{(2p)!}{2^p.p!} \)

We must prove that the result is true for \( n = p + 1 \)

Number of 1-factors in \( K_{2(p+1)} = [2(p+1) - 1] \times \) (number of 1-factors of \( K_{2p} \))

\[
= (2p + 1) \frac{(2p)!}{2^p.p!} \\
= 2(p+1)(2p+1)(2p)! \\
= \frac{(2p+2)!}{2^p.p!(p+1)!} \\
= \frac{2(p+2)!}{2^{p+1}(p+1)!} 
\]

The result is true for \( n = p + 1 \)

By the Principle of Mathematical Induction the result is true for all \( n \in \mathbb{Z}^+ \).

In addition, Java program is used to implement our results.
4. Conclusion

The $I$-factors of complete graphs have been constructed using the above generalized algorithm. Recurrence relation of $I$-factors of complete graphs has been proved using the Principle of Mathematical Induction. Further, the complete graphs can be constructed using line disjoint $I$-factors. This construction is illustrated using $K_6$.

References