An Inventory Model with a Trade Credit as Demand dependent and Time Variant Parameters

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ABSTRACT
In the classic Economic Order Quantity model the purchasing cost of an order is paid at the time of its receipt. In some cases sellers give their retailers a permissible for payments and some others ask buyers to pay all or a fraction of the purchasing cost in advance and may allow them to divide the prepayment into several equal-sized parts. We propose a generalized supplier–retailer inventory model under a given deterministic planning horizons which consists of different periods using a trade credit policy that attracts more retailers. The trade credit policy adopted here is a demand dependent policy in which the supplier offers the retailer a permissible delay period as a non-decreasing function of the buyer demand. In addition, the demand rate is assumed to be a function of time and of the related period. During the given permissible period no interest is charged by the supplier on the retailer, but beyond that period an interest, with the conditions agreed upon, is charged on. However, during the permissible credit period, the retailer can accumulate the revenue of sales and earns an interest on that revenue. Hence, determining such a trade credit period is recognized as an important strategy to increase seller’s profitability and minimize the retailer’s costs. Sufficient conditions on the existence and uniqueness of the optimal replenishment policy are provided.

INTRODUCTION
In reality, sellers frequently offer his/her buyers a permissible delay in payment known as a trade credit period. Some suppliers, however, allow a certain period to settle payment accounts, but in most cases such allowed periods are fixed and independent of the demand rate or of any other parameters of the problem such as the parameters which may motivate the supplier to extend his allowed permissible period according to the retailers demand rate. For example, the supplier would be encouraged to such extension of the trade credit period given to the retailer if the last increases his demand rate accordingly. Therefore, we propose an economic order quantity model (EOQ) from the seller’s prospective to determine his/her optimal trade credit period in which the given trade credit increases not only sales but also minimizes the opportunity cost and default risk. The time horizon is deterministic and is divided into different periods each of which has its own trade credit and its own demand rate as a non-decreasing function the permissible trade credit of that period. Shortages are not allowed in any period. Moreover in most inventory models it is assumed that the parameters of the model do not vary with time. Here, such restriction shall not be considered. That is, the parameters of our model are to be taken as arbitrary functions of time and as dependent on the related period. The mechanism of our system is working as follows. During the related trade credit period given to the retailer, no interest is charged by the supplier on the retailer, but beyond that related period an interest, with the conditions agreed upon, is charged on. Such offers provide the buyers with several advantages. First, the buyers will have enough money to run their jobs. Second, buyers may benefit from the generated sales revenue by depositing it into an interest bearing account. Third, allowing the delay in payment may motivate the retailers to order more quantities which in turn lead to a reduction in the purchasing cost, ordering cost, and shortage cost. Further, we shall consider that all model parameters are period dependent. Such consideration gives the flexibility of applying our model on more than one item(commodity) each of which has its own parameters and its own trade credit as a demand rate dependent. However, ordering large quantities will, in general, increase the holding cost, the cost of deteriorated or decaying of the stored items, and the potential cost due to inflation and/or time value of money. Given that an interest can be earned by the retailer from the revenue that he receives during the given credit period, the retailers are likely to balance between the effects of all above cost components so as to minimize the net total relevant cost.

The effect of supplier’s trade credit policies on the traditional (EOQ) models has received the attention of many researchers. Chia-ueiHo (2011 and 2013) proposed a generalized, integrated, supplier–retailer inventory model using a two-level trade credit policy from both retailer and supplier. Taleizadeh A.A(2014) developed an (EOQ) model for an evaporating item with partial backordering and partial consecutive repayments with a real case study of a gasoline station. Zhang, Yu et al( 2014) investigated the buyer’s inventory policy under advance payment, including all payment in advance partial-advanced, and partial-delayed payments.
Guria, A. et al (2013) presented an inventory policy for an item with inflation and selling price as demand dependent under deterministic and random planning horizons allowing and not allowing shortages under the existence of a provision for an immediate part payment to the wholesaler and some other conditions.

Jiang Wu et al (2014) have dealt with the trade credit period from the seller’s perspective in which they try to answer how to determine credit period to increase seller’s profitability. Chih-Tsung Yang et al (2015) presented a generalized model to determine the optimal trade credit, in which preservation technology investment and replenishment strategies that maximize the retailer’s total profit and to minimize the default risk occurs over a finite planning horizon. Xu Chen et al (2015) considered a firm facing stochastic demand for two products with a downward, supplier-driven substitution and customer service objectives for a single period problem in which the fundamental challenge is to determine in advance the profit for maximizing inventory levels of both products that will meet given service level objectives. Liang-Yuh Ou Yang and Chun-Tao Chang (2013) explored the effects of reworking imperfect quality items and trade credit on the Economic Production Quantity (EPQ) model with imperfect production processes and complete backlogging. Jin-Tsair Teng et al (2014) proposed an (EPQ) model from the seller’s prospective to determine his/her optimal trade credit period and production lot size simultaneously in which trade credit increases not only sales but also opportunity cost and default risk, and production cost which declines and obeys a learning curve phenomenon. Goyal (1985) has studied the effects of the permissible delay in payment on the standard (EOQ) model for non-deteriorating items where shortages are not allowed. He showed that such delay in payment leads to an increase in both the order quantity and the replenishment interval and to a sharp decrease in the total annual cost. Aggarwal and Jaggi (1995) presented a model similar to that of Goyal (1985) but with constant deterioration rate where they presented a sensitivity analysis that reveal the effects of such deterioration rate on several factors of the (EOQ) policies. Jamal et al. (1994) and Shah et al. (1998) extended the models of Aggarwal and Jaggi and Goyal, respectively, by allowing for shortages where it is further shown that the total cost is less than that in the non-shortage case. Jaggi and Aggarwal (1994) and Chung (1989) considered a similar model as that of Aggarwal and Jaggi (1995) but with a discount cash-flow (DCF) approach, and approximately reflected the effect of delay in payment on the optimal inventory policy. Kun et al. (2009) considered the optimal ordering policy of the (EOQ) model under trade credit depending on the order quantity from the (DCF) approach. Some useful theoretical results on the subject of permissible delay in payment have been also achieved. Chung (1998) provided conditions under which the total cost introduced by Goyal (1985) is convex. Chu et al. (1998) showed that the total cost of the system introduced by Aggarwal and Jaggi is piecewise convex which in turn lead to an improved solution procedure of the considered system. Some other researchers have dealt with the subject from different point of view. Hwang and Shin (1997) studied the case when the demand rate is a function of the retailer’s price where it is, then, shown that the retailer’s optimal price and the optimal lot size can be determined simultaneously. In Khouja and Mehrez (1996) two main types of supplier credit policies, where the credit policy may be independent or linked to the order quantity, are addressed. Kim et al. (1995) showed that the net profit of both the retailer and the suppliers can be increased through a wise selection of the credit period. Balkhi (2004 and 2011) introduced a trade credit inventory model that generalizes several of the previously introduced models so that most of these models result as special case of Balkhi (2011) model. Other types of inventory control models with trade credits are proposed by many researchers. Luo (2007) treated a single-vendor, a single-buyer supply chain for a single product, and a model to study and analyze the benefit of coordinating supply chain inventories through the use of credit period is proposed. Jinn-Tsair Teng (2009) established an (EOQ) model for good retailers who receives a full trade credit by his supplier, and offers either a partial or a full trade credit to his bad customers. Sana. And Chaudhuri (2008) modeled the retailer’s profit-maximizing strategy when confronted with supplier’s trade offer of credit and price-discount on the purchase of merchandise when retailers are facing many scenarios of time dependent demands for different kinds of goods. All above mentioned papers are of infinite time horizon. A finite horizon inventory model with constant rate of deterioration and without shortages and with equal periods, when a delay in payment is permissible, has been investigated by Liao et al. (2000). Balkhi (2011) generalized the model of Liao et al. by considering that the given time horizon consists of several periods with different lengths each of which has its own demand rate and its own trade credit period. Most of the above mentioned papers considered that the cost parameters of the underlying inventory model as well as the demand and deterioration rates are all known and constant. Also, in most of those papers, neither inflation nor time value of money are taken in to accounts. Balkhi and Tadj (2008) have studied a general EOQ model in which the cost parameters and both demand and deterioration rates are arbitrary functions of time. However, the effect of trade credit policies is not considered in this last model.

As it can be noted from the above literature review, there are several limitations while developing mathematical models in inventory control under trade credit policies. One of these limitations is that the assumption of constant demand and/or deterioration rates may not always be appropriate for many inventory items as, for example, it is the case for products whose demand is affected by seasons or occasions. Another limitation is the case of a given finite horizon divided it into equal inventory cycles. A third one is that the permissible delay in payment (i.e., the trade credit period) is assumed to be the same for all cycles. A fourth limitation is concerned with constant cost parameters which is not realistic for most practical inventory control systems. The fifth limitation is that the trade credit is independent of the demand rate of the retailer as well as the related period. Thus, there is a need to drop all above limitations and develop a more general inventory model in which the demand and deterioration rates and all cost parameters are general functions of time, each of the n different periods has its own demand rate, its own trade credit, and its own cost parameters so that it can be specified to on or more item. In this case many of the previously introduced models can be obtained as special cases of such a general model.

The purpose of this paper is to generalize the paper of Balkhi [2011] and most of the above introduced models in the following fronts.

The given time horizon consists of n different periods each of which has its own trade credit as a non-decreasing function of its demand rate, has its own parameters, and has its own ordered quantity which is equal to the inventory level at the start of that period. Also, both demand and deterioration rates and all cost parameters shall be considered as general and continuous functions of time, and period dependent. These assumptions give the flexibility of specifying one or more of the n different periods to one or
more items. These assumptions lead also to the possibility of choosing the demand not only to be a stock dependent, but also to make a possibility of ordering the required item for one or more periods. Therefore, our order quantity is period and time dependent. Further, the effects of both inflation and time value of money shall be incorporated in all cost components. Our main concern will be on the theoretical results. So, the proposed model with the above mentioned general features is developed, so that the above unit cost to be period and time dependent gives the flexibility of specifying each period of the demand rate for period \( j \) as a non-decreasing function of the demand rate, offered from the supplier to his retailer. As examples of such trade credit periods which can be considered as period dependent and as a non-decreasing functions of the demand rate are linear and/or exponent functions. Viz; either the case,

\[
M_j(D_j(t)) = a_j D_j(t) + b_j, \quad \text{with } a_j \geq 0
\]  

(1.1)

where

\[
\frac{\partial M_j(D_j(t))}{\partial D_j(t_j)} = a_j \geq 0
\]

or

\[
M_j(D_j(t)) = c_je^{a_j D_j(t) + b_j}, \quad \text{with } a_j, c_j \geq 0
\]  

(1.2)

where,

\[
\frac{\partial M_j(D_j(t_j))}{\partial D_j(t_j)} = c_j a_j e^{a_j D_j(t) + b_j} \geq 0
\]

But

\[
\frac{\partial M_j(D_j(t_j))}{\partial t_j} = \frac{\partial M_j(D_j(t_j))}{\partial D_j(t_j)} \cdot \frac{\partial t_j}{\partial t_j}
\]

In (1.1), the assumption \( a_j \geq 0 \) guarantee that \( M_j(\cdot) \) is a non-decreasing function of \( D_j(t) \), whereas in (1.2), the assumption \( a_j, c_j \) have the same sign guarantee that \( M_j(\cdot) \) is a non-decreasing function of \( D_j(t) \). The case \( a_j = 0 \) give us a period dependent trade credit \( M_j(D_j(t))= b_j \cdot M_j(D_j(t)) = c_j e^{b_j} \) as the case considered in Balkhi (2011).

The above non-decreasing functions given in formula (1.1) and (1.2) or any other similar non-decreasing function of the demand rate can be considered so that the parameters \( a_j \geq 0 \) or \( a_j, c_j \) with the same sign can be treated as a decision variables from the supplier.

(4) The ordered items deteriorate while they are effectively in stock and the deterioration rates is an arbitrary and known function of time, and is period dependent denoted as \( \theta_j(t) \).

But there is no repair or replacement of deteriorated items.

(5) \( t_0 = 0 \) and \( t_n = T \).

(6) \( Q_j \) is the quantity ordered at the beginning of the period \( j \), and, \( I_j(t) \) is the inventory level at time \( t \). Note also that both \( Q_j \) and \( I_j(t) \) are implicitly dependent on \( M_j(D_j(t)) \).

(7) Lead time is negligible and the replenishment rate is infinite.

(8) Shortages are not allowed in any period. Such assumption can be justified by the fact that the supplier shall provide his retailer by some kind of open permissible trade credit period of demand dependent which in turn shall encourage the retailer to demand more quantities of the required items which in turn does not allow shortages.

(9) A complete order of \( Q_j \) units arrives by the beginning of the period \( j \), but there is a demand dependent permissible delay \( M_j(D_j(t)) \) for payment till the period agreed upon.

(10) The time and period dependent cost structure for period \( j \) of the underlying inventory system is as follows:

(i) \( c_j(t) \) is the unit cost in period \( j \) at time \( t \).

(ii) \( h_j(t) \) is the holding cost in period \( j \) per unit per unit time at time \( t \).

(iii) \( s_j(t) \) is the unit selling price in period \( j \) at time \( t \).

(iv) \( k_j(t) \) is the ordering cost in period \( j \) per order at time \( t \).

Note that making the above unit cost to be period and time dependent gives the flexibility of specifying each period of the \( n \) different periods for one or more of a specific item (commodity) as well as some or all periods to one item. Note also, that the
period dependent of the demand rate \( D_j(t) \) and the trade credit \( M_j(t) \) as well as all cost parameters give us the flexibility to apply this model for each period separately with its separate item(s) or with the same item for one or more of the other periods.

(11) All above time and period dependent costs are free of interest charges.

(12) All costs are affected by inflation rate and time value of money. We shall denote by \( r_1 \) the inflation rate and by \( r_2 \) the discount rate representing the time value of money so that \( r = r_2 - r_1 \) is the discount rate net of inflation.

(13) For any the \( n \) different periods, there is an interest charged for the items being held in stock after the given trade credit period \( M_j(D_j(t)) \) concerned with that period, i.e. for those items not being sold during \( M_j(D_j(t)) \). We denote by \( i_e \) the per monetary unit per unit time charge payable at time \( t = 0 \).

(14) All generated sales revenue can be deposited into an interest bearing account at a rate \( i_e \) (at time \( t = 0 \)) per monetary unit per unit time. We generally have \( i_c \) and \( i_e \) are the same for all periods and \( i_c > i_e \).

The proposed inventory trade credit system operates as follows. A quantity of \( Q_j \) units is ordered and stored at the beginning of period \( j \). The supplier (or seller) offers the buyer (or retailer) an agreed up on trade period which is a demand and period dependent \( M_j(D_j(t)) \), during which there is no charged interest so that the account is to be settled by or after the end of this credit period. Otherwise an interest on rate \( i_e \) will be charged for any amount delayed beyond that allowed of the agreed up on period. On the other hand the retailer may deposit all generated sales revenue into an interest bearing account of rate \( i_e \) and may pay off his supplier by at most the end of the agreed up on period. Whereas the inventory system operates as follows.

A quantity of \( Q_j \) units is ordered at the beginning of period \( j \). This quantity is subject to consumption of rate \( D_j(t) \) and deterioration rate \( \theta_j(t) \) till the inventory level reaches to zero by the time where a new order for the next period, depending on the demand and the period, is ordered. The objective of the retailer is to minimize his net total relevant costs in any or all periods and for all all ordered items. The process is repeated for each period. Fig 1. Shows the variation of the underlying inventory system.

![Fig 1. The variation of inventory levels in the time horizon [0,H].](image)

**MODEL BUILDING.**

Following the above assumptions and notations The changes of \( I_j(t) \) in period \( j (j=1,2,\ldots,n) \) which is \([t_{j-1}, t_j]\) are given by the following differential equation

\[
\frac{dI_j(t)}{dt} = -D_j(t) - \theta_j(t)I_j(t) \\
\text{with the boundary condition } I_j(t) = 0. \text{ The solution of (2.1) is given by:}
\]

\[
I_j(t) = e^{-\theta_j(t) \int_t^{t_j} D_j(u) du} I_j(t) \quad ; \quad t_{j-1} \leq t \leq t_j 
\]

\[
\delta_j(t) = \int_0^t \theta_j(u) du 
\]

Next we derive the present worth of all cost components for period \( j (j=1,2,\ldots,n) \).

The present worth of the holding cost, say (PW1), of the amount being held in stock during the period \([t_{j-1}, t_j]\) is given by

\[
PW1 = \int_{t_{j-1}}^{t_j} e^{-rt} \int_{t_{j-1}}^{t_j} h_j(t)I_j(t) dt = \int_{t_{j-1}}^{t_j} e^{-rt} \int_{t_{j-1}}^{t_j} h_j(t)(\int_t^{t_j} e^{\delta(u)} D_j(u) du) dt 
\]

Integrating by parts, then PW1 is reduced to:
Where
\[ H_j(t) = \int_0^t e^{-(t-\delta_j(t))} h_j(t) dt \]

The number of items received in beginning of the period \([t_{j-1}, t_j]\) is given by
\[ Q_j = I_j(t_{j-1}) = \int_{t_{j-1}}^{t_j} e^{\delta_j(u)} D_j(u) du \]  

Since items are received at the beginning of each period (recall that lead time is negligible), the unit item's cost is equal to \(c_j(t_{j-1})Q_j\). Hence, the present worth of the items' cost in period \(j\), say \(PW2\), is equal to
\[ PW2 = c_j(t_{j-1})Q_j = c_j(t_{j-1}) e^{-\tau t_j} \int_{t_{j-1}}^{t_j} e^{\delta_j(u)} D_j(u) du \]  

Note that the items' cost include the cost of deteriorated and consumed (none deteriorated) items. The present worth of ordering cost in period \(j\), say \(PW3\), is equal to
\[ PW3 = e^{-\tau t_j} k_j(t_{j-1}) \]

To find the present worth for each of the interest charged and the interest earned. We distinguish two cases.

**Case (1). Cycles with \(M_j(D_j(t_j)) \leq t_j\).**

In this case we need to find the present worth of the cost which will result from interest charged for items in inventory not being sold after \(M_j(D_j(t_j))\). But the number of items not being sold in a small time interval \(dt\) after \(t_{j-1}\) is equal to
\[ J(t_{j-1}) = \int_{t_{j-1}}^{t_j} e^{-\tau t_j} c_j(t) I_j(t) dt \]

Integrating by parts, we find
\[ PW4 = \int_{M_j(D_j(t_j))}^{t_j} e^{-\tau t_j} c_j(t) I_j(t) dt = \int_{M_j(D_j(t_j))}^{t_j} e^{-\tau t_j} c_j(t) e^{-\tau t_j} (\int_{t_{j-1}}^{t_j} e^{\delta_j(u)} D_j(u) du) dt \]

Where
\[ G_j(t) = \int_0^{t_j} e^{-\tau t_j} c_j(t) dt \]

Similarly, the present worth of the interest earned, say \(PW5\), in the permissible period \([t_{j-1}, M_j(D_j(t_j))]\), which has positive stock, and in the rest of the period \([M_j(D_j(t_j)), t_j]\) from the remaining cash, is given by:
\[ PW5 = \int_{t_{j-1}}^{M_j(D_j(t_j))} e^{-\tau t_j} s_j(t) D_j(t) dt + \int_{M_j(D_j(t_j))}^{t_j} e^{-\tau t_j} s_j(t) D_j(t) dt = \int_{t_{j-1}}^{t_j} e^{-\tau t_j} s_j(t) D_j(t) dt \]

Hence, the net total variable cost in period \(j\) in Case (1) as a function of \(t_{j-1}, t_j\), and \(M_j(D_j(t_j))\) say \(W_{ij}(t_{j-1}, t_j, M_j(D_j(t_j)))\), is given by
\[ W_{ij}(t_{j-1}, t_j, M_j(D_j(t_j))) = \int_{t_{j-1}}^{t_j} e^{\delta_j(t)} [H_j(t) - H_j(t_{j-1})] D_j(t) dt + \int_{t_{j-1}}^{t_j} e^{-\tau t_j} c_j(t_{j-1}) e^{-\tau t_j} (-\delta_j(t_{j-1})) \int_{t_{j-1}}^{t_j} e^{\delta_j(u)} D_j(u) du + \int_{t_{j-1}}^{t_j} e^{-\tau t_j} k_j(t_{j-1}) dt + \int_{M_j(D_j(t_j))}^{t_j} e^{-\tau t_j} s_j(t) D_j(t) dt \]

**Case (2). Cycles with \(M_j(D_j(t_j)) \geq t_j\).**

In this case we do not have an interest charge for inventory not being sold after \(M_j(D_j(t_j))\) since, then, we have \(M_j(D_j(t_j)) \geq t_j\). But the interest earned per period \(j\) consists, here, from two parts. The first part is the interest earned during the positive inventory in period \([t_{j-1}, t_j]\) which is given by
\[ i_e \int_{t_{j-1}}^{t_j} e^{-rt} s_j(t) D_j(t) dt \]

The second part is the interest earned from the remaining cash during the time period \([t_j, M_j(D_j(t_j))]\) after the depletion of inventory. The present worth of this last part is equal to

\[ i_e [M_j(D_j(t_j)) - t_j] \int_{t_j}^{M_j(D_j(t_j))} e^{-rt} s_j(t) D_j(t) dt \]

Thus the net total variable cost in period \(j\) in Case (2) as a function of \(t_{j-1}, t_j\) and \(M_j(D_j(t_j))\), say \(W_{2j}(t_{j-1}, t_j, M_j(D_j(t_j)))\), is given by

\[
W_{2j}\left(t_{j-1}, t_j, M_j(D_j(t_j))\right) = \int_{t_{j-1}}^{t_j} e^{\delta(t)} [H_j(t) - H_j(t_{j-1})] D_j(t) dt \\
+ c(t_{j-1})e^{-rt_{j-1}} \int_{t_{j-1}}^{t_j} e^{\delta(t)} D_j(u) du + e^{-rt_{j-1}} K_j(t_{j-1}) \\
- i_e \int_{t_{j-1}}^{t_j} e^{-ru} s_j(t) D_j(t) dt \quad i_e \left[M_j\left(D_j(t)\right) - t_j\right] \int_{t_j}^{M_j(D_j(t_j))} e^{-ru} s_j(t) D_j(t) dt
\]

(2.15)

Now we define

\[
\delta_j = \begin{cases} 
1 & \text{if } M_j(D_j(t_j)) \leq t_j \\
0 & \text{otherwise}
\end{cases}
\]

(2.16)

Then by (2.16), we can unify (2.12) and (2.13) by the following formula

\[
W_{2j}\left(t_{j-1}, t_j, M_j(D_j(t_j))\right) = \int_{t_{j-1}}^{t_j} e^{\delta(t)} [H_j(u) - H_j(t_{j-1})] D_j(t) du \\
+ c(t_{j-1})e^{-rt_{j-1}} \int_{t_{j-1}}^{t_j} e^{\delta(t)} D_j(u) du + e^{-rt_{j-1}} K_j(t_{j-1}) \\
+ \alpha_j c \int_{M_j(D_j(t_j))}^{t_j} [G_j(u) - G_j(M_j(D_j(t)))] e^{\delta(u)} D_j(u) du \\
- i_e \int_{t_{j-1}}^{t_j} e^{-ru} s_j(u) D_j(u) du - i_e (1 - \alpha_j) [M_j\left(D_j(t)\right) - t_j] \int_{t_j}^{M_j(D_j(t_j))} e^{-ru} s_j(u) D_j(u) du
\]

(2.17)

Note that

\[
W_j(t_{j-1}, t_j, M_j(D_j(t_j))) = \begin{cases} 
W_{1j}(t_{j-1}, t_j, M_j(D_j(t_j))) & \text{if } \alpha_j = 1 \\
W_{2j}(t_{j-1}, t_j, M_j(D_j(t_j))) & \text{if } \alpha_j = 0
\end{cases}
\]

(2.18)

Hence, the net total relevant cost in the whole time horizon \([0, H]\), where \(t_0 = 0\) and \(t_n = H\), is given by

\[ W = \sum_{j=1}^{j=n} W_j(t_{j-1}, t_j, M_j(D_j(t_j))) \]
Thus our problem, which we shall refer to as \((P)\), is
Minimize \(W\) given by (2.19) subject to:
\[
0 = t_0 \leq t_1 \leq t_2 \leq \ldots \ldots \leq t_n = H
\] (2.20)
Note that constraints (2.20) must be satisfied for any feasible solution of the underlying inventory system since otherwise the problem would have no meaning.

**PROBLEMS’ SOLUTION**

To solve problem \((P)\) we shall first ignore constraints (2.20) and consider \(n\) to be fixed. We refer to the resulting unconstrained problem as \((P_1)\).

Then, the necessary conditions for having a minimizing solution for \((P_1)\) are:
\[
\sum_{j=1}^{n-1} \int_{t_j}^{t_{j+1}} e^{-r_t s_j(u)} D_j(u) du - \int_{t_n}^{t_f} e^{-r_t s_j(u)} D_j(u) du - i_e (1 - \alpha_j) [M_j (D_j(t_j))] - t_0 \int_{t_0}^{t_f} e^{-r_t s_j(u)} D_j(u) du = 0
\] (2.21)

Note that \(\partial W_j (t_j, M_j (D_j(t_j))) / \partial t_j = 0\) for all \(j \Rightarrow \partial W = 0\). Hence, (2.21) give the critical points for both \(W_j\) and \(W\).

To find \(\partial W_j (t_j, M_j (D_j(t_j))) / \partial t_j\) we note that \(t_j\) appears in the following Terms

\[
\int_{t_{j-1}}^{t_j} e^{\delta t(u)} [H_j(u) - H_j(t_{j-1})] D_j(u) du + c(t_{j-1}) e^{-r_{t_{j-1}} t_{j-1}} \int_{t_{j-1}}^{t_j} e^{\delta t(u)} D_j(u) du - i_e \int_{t_{j-1}}^{t_j} e^{-r u s_j(u)} D_j(u) du - \int_{t_j}^{t_{j+1}} e^{\delta t_{j+1}(t_{j+1})} [H_{j+1}(u) - H_j(t_j)] D_{j+1}(u) du + c(t_j) e^{-r_{t_j} t_j} \int_{t_j}^{t_{j+1}} e^{\delta t_{j+1}(u)} D_j(u) du + \alpha_j i_e \int_{t_j}^{t_f} e^{\delta t_{j+1}(t_j)} [G_j(u) - G_j(M_j (D_j(t_j)))] e^{\delta t(u)} D_j(u) du + \alpha_j i_e \int_{t_j}^{t_f} e^{-r t_k t_j} e^{-r u s_j(u)} D_j(u) du - i_e (1 - \alpha_j) [M_j (D_j(t_j))] - t_j \int_{t_j}^{t_f} e^{-r t_k t_j} e^{-r u s_j(u)} D_j(u) du = 0
\] (2.22)

Note that (2.22) are \((n-1)\) equations with \((n-1)\) decision variable, namely \(t_1, t_2, \ldots, t_{n-1}\) (recall that \(0 = t_0, t_n = H\)). The solution of these equations (if it exists) gives the critical points \(W_j (t_j, \ldots, t_n, M_j (D_j(t_j)))\), hence of \(W\).

Next we deliver sufficient conditions for which any existing solution of \((P1)\) is a minimizing solution to \((P1)\). For this purpose let \(T = (0 = t_0, t_1, t_2, \ldots, t_n = H)\) be a solution of equations (2.21) and let \(HM(T)\) be the value of the Hessian Matrix at \(T\), then, by Bazara et al. (1993), Stewart (1973, page 143 Chapter 3) and Theorem 3 of Balkhi and Benkhrouf (2004), \(HM(T)\) is positive definite if
\[
W_{ij} > \left| W_{(j+1)i} \right| + \left| W_{(j-1)i} \right| \quad \text{for} \quad j = 2, \ldots, n-2; \quad W_{jn} > \left| W_{(j+1)n} \right| \quad \text{for} \quad j = 1.
\]

and

\[
W_{ij} > \left| W_{(j-1)i} \right| \quad \text{for} \quad j = n-1
\]

Where

\[
W_{ij} = \frac{\partial^2 W}{\partial t_i \partial t_j} \quad W_{jk} = \frac{\partial^2 W}{\partial t_k \partial t_j}
\]

The above arguments lead to the following theorem.

**Theorem 1.** For fixed \( n \), any existing solution of \((P1)\) is a minimizing solution to \((P1)\) if this solution satisfies conditions \((2.23)\).

**UNIQUENESS AND GLOBAL OPTIMALITY OF THE SOLUTION.**

In this section we shall show that any existing and minimizing solution of \((P1)\) is unique (hence global optimal) of both \((P1)\) and \((P)\). We shall show this in three steps:

First we show that \((P1)\) depend only on one of the variables \( t_0, t_2, \ldots, t_{n-1}. \) Second, we show that under the hypothesis of Theorem 1., then any existing and minimizing solution of \((P1)\) is unique.

Third, we show that under the result of Theorem 1., then the total net relevant cost \( W \) is convex with respect to \( n \).

Now, from equations (2.22) and for \( j = 1 \) we have

\[
e^{\delta(t_1)}[H_1(t_1) - H_1(0)]D_1(t_1) + c(0)e^{\delta(t_1)}e^{\delta(t_1)}D_1(t_1) - i_{e^{\delta(t_1)}s_1(t_1)} + H_1'(t_1)e^{\delta(t_1)}[H_2(t_1) - H_1(t_1)]D_2(t_1) + [c(t_1)e^{\delta(t_1)} - c(t_1)(r + \delta(t_1))]e^{-\delta(t_1)} - \delta_2(t_1)] \int_{t_1}^{t_2} e^{\delta(u)} D_1(u)du - c(t_1)e^{-\delta(t_1)}D_1(t_1) - a_{j_1}D_1(t_1)G_1(M_1(D_1(t_1))) \times \int_{M_1(D_1(t_1))}^{t_1} G_1(u) - G_1(M_1(D_1(t_1)))e^{\delta(u)}D_1(u)du + a_{j_1}G_1(M_1(D_1(t_1)))e^{\delta(t_1)}D_1(t_1) + i_{e^{\delta(t_1)}s_1(t_1)}D_2(t_1) + i_{e^{\delta(t_1)}s_1(t_1)}[M_1(D_1(t_1)) - t_1] \times e^{-\delta(t_1)}s_1(t_1)D_1(t_1) - e^{-\delta(M_2(t_1))}s_1(t_1)D_1(t_1)
\]

\[
e^{-\delta(t_1)}s_1(t_1)D_1(t_1) + [D_1(t_1)M_1(D_1(t_1))] + [D_1(t_1)M_1(D_1(t_1))] - 1 \int_{t_1}^{M_1(D_1(t_1))} e^{-\delta(t_1)}s_1(u)D_1(u)du - r e^{-\delta(t_1)}K_1'(t_1) = 0
\]

Now, from relations (3.2), and for \( j = 2 \), we can see that \( t_1 \) is a function of \( t_2 \) and \( t_2 \) hence it is a function of \( t_1 \) and so forth we can see that each of the variables \( t_0, t_2, \ldots, t_{n-1}, \) is a function of \( t_1 \), say

\[
t_1 = f_1(t_1), \quad j = 1, \ldots, n-1 \quad \text{with} \quad t_1 = f_j(t_1) = t_1
\]

Next, we state some important results.

**Lemma.** Under the hypothesis of Theorem 1., we have

\[
t_1 > t_j > 0 \quad j = 1, 2, \ldots, n-1
\]

where

\[
t_j = f_j(t_1)
\]

**Proof.** The proof of this theorem is similar to the proof of Lemma 1. of Balkhi (2011), so it will not be repeated again.

Now relations (3.3) and (3.2) lead to the following corollary.

**Corollary 1.** Under the hypothesis of Theorem 1., then

(i) All variables \( t_0, t_2, \ldots, t_{n-1} \) are increasing functions of \( t_1 \) and of each other’s.

(ii) For fixed \( n \), any existing and minimizing solution to problem \((P1)\) is an existing and minimizing solution to problem \((P)\).

**Proof.** The proof of part (i) is clear from relations (3.2) and (3.3). Again, from (3.3) and by the theory of Real Analysis we have

\[
t_{j+1} t_j \geq 0
\]

Thus constraints (2.20) hold for any existing and minimizing solution of \((P1)\) if \( t_1 \geq 0 \). But, as an implication of Kohn-Tucker necessary conditions, the last inequality \( t_1 \geq 0 \) needs not to be considered. Hence, constraints (2.20) are satisfied for any feasible solution of (2.22). Hence, such a solution is an existing and minimizing solution to \((P)\).

Some other main results follow.

**Theorem 2.** Under the hypothesis of Theorem 1., and Lemma 1., any existing and minimizing solution of \((P)\) is the unique solution of \((P)\).

**Proof.** Let \( T = (0 = t_0, t_1, t_2, \ldots, t_n, t_n = H) \) be an existing and minimizing solution of \((P)\). By relations (3.2), the amount

\[
\sum_{j=1}^{n} (t_j - t_{j-1}) - H = 0
\]

as an equation of \( t_j \), either has a unique solution or it does not have any solution. To see this, let us denote by \( Z(t_1) \) the left hand side of (3.5). Then \( Z'(t_1) = \sum_{j=1}^{n} (t_j - t_{j-1}) > 0 \) by (3.3), which means that \( Z(t_1) \) is an increasing function of \( t_1 \). Now, if \( Z(0) \leq 0 \), then (3.5) has a unique solution. If however \( Z(0) > 0 \) then (3.5) does not have any solution (see Fig. 2). This leads to the desired result.

**Theorem 3.** Consider the following two different existing schedules with \( n \) and \((n+1)\) entries, say \( T = (0 = t_0, t_1, t_2, \ldots, t_n, t_n = H) \), \( S = (T, n) \) and \( \bar{T} = (0 = t_0, t_1, t_2, \ldots, t_{n+1} = H) \), \( \bar{S} = (\bar{T}, n+1) \). If the hypothesis of Theorem 1., and
Lemma 1. hold, then the entries of \( \hat{S} \) lies between the entries of \( S \), that is;

\[
0 = \hat{t}_0 \leq \hat{t}_1 \leq t_1 \leq \hat{t}_2 \leq t_2 \leq \ldots \leq \hat{t}_n = \hat{t}_{n+1} = H
\]  

(3.6)

Proof. The proof of this theorem mimics the proof of Theorem 3 of Balkhi (2001) so it will not be repeated again.

**Theorem 4.** Suppose that the hypothesis of Theorem 1, and Lemma 1, hold and \( MS \) is the set of all minimizing solutions of the form \( S=(T,n) \) for the underlying inventory system, then under some seemingly possible conditions, there exists a unique vector \( S^*=(T^*,n^*) \) from \( MS \) for which the net total relevant cost of this system is minimum.

Proof. From (2.17) and (2.19) we can rewrite \( W \) as the sum of set up costs, say \( W_0 \), and the rest of the cost components, say \( WR \). That is

\[
W=W_0+WR \quad \text{where} \quad W_0 = \sum_{j=1}^{n^*} e^{-r_{j-1}} k(t_{j-1})
\]

and

\[
WR=W-W_0.
\]

If we ignored the terms \( -re^{-r_{j-1}} k(t_{j-1}) + e^{-r_j} k(t_j) \), then from (3.1) we have

\[
\frac{\partial W}{\partial t_j} = e^{-r_j} \left( [H_1(t_j) - H_1(0)] D_1(t_j) + c(t_j) \right) e^{-\delta(t_j)} D_1(t_j) + H_1(t_j) e^{-\delta(t_j)} D_2(t_j) + \left[ c'(t_j) e^{-rt} - \delta_2(t_j) \right] \int_{t_{j-1}}^{t_j} D_1(u) du + c(t_j) \int_{t_{j-1}}^{t_j} \left[ G_1(t_j) - G_1(M_1(D_1(t_j))) \right] e^{-r_{j-1} t} D_1(t_j) + \int_{t_{j-1}}^{t_j} e^{-r_{j-1} s_2} S_2(t_1(t_j)) D_2(t_j) + \int_{t_{j-1}}^{t_j} e^{-r_{j-1} s_2} D_1(u) du - c(t_j) \int_{t_{j-1}}^{t_j} D_1(t_1) - \alpha_j e^{-r_{j-1} s_1(t_1)} D_1(t_1) - c(t_j) \int_{t_{j-1}}^{t_j} D_1(t_1) - \alpha_j e^{-r_{j-1} s_1(t_1)} D_1(t_1) - c(t_j) \int_{t_{j-1}}^{t_j} D_1(t_1) - \alpha_j e^{-r_{j-1} s_1(t_1)} D_1(t_1) - c(t_j) \int_{t_{j-1}}^{t_j} D_1(t_1)
\]

(3.7)

Suppose that in (3.7) the following condition hold:

\[
\text{sum of all positive terms in (3.7) \geq \sum \text{sum of negative terms (3.8)}
\]

Recalling that

\[
\delta_j(t) = \int_{0}^{t} \delta_j(u) du, H_j(t) = \int_{0}^{t} e^{-r_j s_j} h_j(u, t_j) dt, \text{and } G_j(t) = \int_{0}^{t} e^{-r_j s_j} c_j(u, t_j) dt, \text{then } \frac{\partial W}{\partial t_j} \geq 0
\]

(3.8)

Holds. In this case, WR is an increasing function of \( t_j \). Now, consider the two replenishment schedules \( S=(T,n) \) and \( \hat{S}=(\hat{T},n+1) \), then we have that \( \hat{t}_1 \leq t_1 \) by (3.6), hence \( W_0(\hat{S}) \leq W_0(S) \). This means that WR is an increasing function of \( t_j \) but a non-increasing function of \( n \). On the other hand we have \( W_0 = \sum_{j=1}^{n^*} e^{-r_{j-1} k(t_{j-1})} > k(0) > 0 \). As a set up cost it is clear that \( W_0 \) is an increasing function of of the period number \( n \). Now, combining the above results we can reach to the following conclusion. While WR decreases with \( n \), \( W_0 \) increases with \( n \) so that \( W_0 \) will eventually dominate WR after certain value of \( n \) say \( n^* \) where \( W=W_0+WR \) starts to increase with \( n \). Hence, there is a unique vector say \( S^*=(T^*,n^*) \) that solves the underlying inventory problem. This completes the proof of the theorem.

![Figure 2](image-url)

CONCLUSION

In this paper we have presented a general economic order quantity (EOQ) inventory model for deteriorating item for a given time horizon under a trade credit policy which may motivate the buyer to demand more quantities. The given time horizon consists of \( n \) different periods each of which has its own trade credit as a non decreasing function of its demand rate, has its own parameters and has its own cost structure. Both demand and deterioration rates as well as all cost parameters are time and period dependent, and general continuous functions of time. As examples of such trade credit periods introduced linear and/or exponent functions. These assumptions lead to the possibility of making the demand not only to be a stock dependent, but also to make a possibility of ordering the required item for one or more of the \( n \) periods. The two possibilities, that the trade credit of any period
may be less or greater than the period length, are conventionally incorporated for each period so that the proposed model can suite more practical and possible cases, including the cases of prepayments, immediate payments, and late payments. The effects of both inflation and time value of money are also incorporated in all cost components. The period dependent trade credit of the demand rate, as well as all cost parameters via the index j of the model give us the flexibility of applying our model for each period separately with a separate item (commodity) or with the same item for one or more of the other periods. Each of the demand, deterioration rates as well as all cost parameters are known and arbitrary functions of time and are dependent on the related period. The two possibilities, that the credit period may be less or greater than the period length, are conventionally incorporated for each period so that the proposed model can suite more practical and possible cases, including the cases of prepayments, immediate payments, and late payments. Both inflation and time value of money are, also, incorporated in all cost components. Our main concern were on the theoretical results. The proposed model with the above mentioned general features is developed, and no approximations were used neither in the total net cost nor in any other relations. A closed form of the net total cost is derived and the resulting model is solved. Then, a rigorous mathematical methods are used to show that, under some seemingly Possible conditions, there exists a unique vector of the relevant decision variables that solve the underlying inventory system.

REFERENCES.


