Level Subset of Bipolar Valued Fuzzy Subsemirings of a Semiring

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ABSTRACT

In this paper, we study some of the properties of \((\alpha, \beta)\)-level subsets of bipolar valued fuzzy subsemiring and prove some results on these.

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Introduction

In 1965, Zadeh [15] introduced the notion of a fuzzy subset of a set, fuzzy sets are a kind of useful mathematical structure to represent a collection of objects whose boundary is vague. Since then it has become a vigorous area of research in different domains, there have been a number of generalizations of this fundamental concept such as intuitionistic fuzzy sets, interval valued fuzzy sets, vague sets, soft sets etc [7]. Lee [9] introduced the notion of bipolar valued fuzzy sets. Bipolar valued fuzzy sets are an extension of fuzzy sets whose membership degree range is enlarged from the interval \([0, 1]\) to \([-1, 1]\). In a bipolar valued fuzzy set, the membership degree 0 means that elements are irrelevant to the corresponding property, the membership degree \([0, 1]\) indicates that elements somewhat satisfy the property and the membership degree \([-1, 0]\) indicates that elements somewhat satisfy the implicit counter property. Bipolar valued fuzzy sets and intuitionistic fuzzy sets look similar each other. However, they are different each other [9,10]. Anitha.M.S., Muruganantha Prasad & K. Arjunan[1] defined as bipolar valued fuzzy subgroups of a group. We introduce the concept of \((\alpha, \beta)\)-level subsets of bipolar valued fuzzy subsemirings of a semiring are discussed. Using these concepts, some results are established.

1. PRELIMINARIES:

1.1 Definition:
A bipolar valued fuzzy set (BVFS) An \(X\) is defined as an object of the form \(A = \{ < x, A^-(x), A^+(x) > | x \in X \}\), where \(A^+: X \rightarrow [0, 1] \) and \(A^-: X \rightarrow [-1, 0]\). The positive membership degree \(A^+(x)\) denotes the satisfaction degree of an element \(x\) to the property corresponding to a bipolar valued fuzzy set \(A\) and the negative membership degree \(A^-(x)\) denotes the satisfaction degree of an element \(x\) to some implicit counter-property corresponding to a bipolar valued fuzzy set \(A\). If \(A^-(x) \neq 0\) and \(A^+(x) = 0\), it is the situation that \(x\) is regarded as having only positive satisfaction for \(A\) and if \(A^-(x) = 0\) and \(A^+(x) \neq 0\), it is the situation that \(x\) does not satisfy the property of \(A\), but somewhat satisfies the counter property of \(A\). It is possible for an element \(x\) to be such that \(A^+(x) \neq 0\) and \(A^-(x) \neq 0\) when the membership function of the property overlaps that of its counter property over some portion of \(X\).

1.2 Example:
\(A = \{ < a, 0.5, -0.3 >, < b, 0.1, -0.7 >, < c, 0.5, -0.4 > \}\) is a bipolar valued fuzzy subset of \(X = \{ a, b, c \}\).

1.3 Definition:
Let \(R\) be a semiring. A bipolar valued fuzzy subset \(A\) of \(R\) is said to be a bipolar valued fuzzy subsemiring of \(R\) (BVFR) if the following conditions are satisfied,

\(i\) \(A^+(x+y) \geq \min\{ A^+(x), A^+(y) \}\)

\(ii\) \(A^+(xy) \geq \min\{ A^+(x), A^+(y) \}\)

\(iii\) \(A^-(x+y) \leq \max\{ A^-(x), A^-(y) \}\)

\(iv\) \(A^-(xy) \leq \max\{ A^-(x), A^-(y) \}\) for all \(x\) and \(y\) in \(R\).

1.4 Example:
Let \(R = Z_2 = \{ 0, 1, 2 \}\) be a semiring with respect to the ordinary addition and multiplication. Then \(A = \{ < 0, 0.5, -0.6 >, < 1, 0.4, -0.5 >, < 2, 0.4, -0.5 > \}\) is a bipolar valued fuzzy subsemiring of \(R\).

1.5 Definition:
Let \(A = \{ A^+, A^- \}\) be a bipolar valued fuzzy subset of \(X\). For \(\alpha\) in \([0, 1]\) and \(\beta\) in \([-1, 0]\), the \((\alpha, \beta)\)-level subset of \(A\) is the set \(A_{(\alpha, \beta)} = \{ x \in X: A^+(x) \geq \alpha \text{ and } A^-(x) \leq \beta \}\).
1.6 Example: Consider the set $X = \{0, 1, 2, 3, 4\}$. Let $A = \{(0, 0.5, -0.1), (1, 0.4, -0.3), (2, 0.6, -0.05), (3, 0.45, -0.2), (4, 0.2, -0.5)\}$ be a bipolar valued fuzzy subset of $X$ and $\alpha = 0.4, \beta = -0.1$. Then $(0.4, -0.1)$-level subset of $A$ is $A_{(0.4, -0.1)} = \{0, 1, 3\}$.

1.7 Definition: Let $A = \langle A^+, A^- \rangle$ be a bipolar valued fuzzy subset of $X$. For $\alpha$ in $[0, 1]$, the $A^+$-level $\alpha$-cut of $A$ is the set $P(A^+, \alpha) = \{x \in X: A^+(x) \geq \alpha\}$.

1.8 Example: Consider the set $X = \{0, 1, 2, 3, 4\}$. Let $A = \{(0, 0.5, -0.1), (1, 0.4, -0.3), (2, 0.6, -0.05), (3, 0.45, -0.2), (4, 0.2, -0.5)\}$ be a bipolar valued fuzzy subset of $X$ and $\alpha = 0.4$. Then $A^+$-level $0.4$-cut of $A$ is $P(A^+, 0.4) = \{0, 1, 2, 3\}$.

1.9 Definition: Let $A = \langle A^+, A^- \rangle$ be a bipolar valued fuzzy subset of $X$. For $\beta$ in $[-1, 0]$, the $A^-$-level $\beta$-cut of $A$ is the set $N(A^-, \beta) = \{x \in X: A^-(x) \leq \beta\}$.

1.10 Example: Consider the set $X = \{0, 1, 2, 3, 4\}$. Let $A = \{(0, 0.5, -0.1), (1, 0.4, -0.3), (2, 0.6, -0.05), (3, 0.45, -0.2), (4, 0.2, -0.5)\}$ be a bipolar valued fuzzy subset of $X$ and $\beta = -0.1$. Then $A^-$-level $-0.1$-cut of $A$ is $N(A^-, -0.1) = \{0, 1, 3, 4\}$.

1.11 Definition: Let $A = \langle A^+, A^- \rangle$ be a bipolar valued fuzzy subset of $X$ and $\alpha$ in $[0, 1]$, sup $\{A^+(x)\}$, $\beta$ in $[-1, \inf \{A^-(x)\}, 0]$. Then $T = \langle T^+, T^- \rangle$ is called a bipolar valued fuzzy translation of $A$ if $T^+(x) = T^+A^+_\alpha(x) = A^+(x) + \alpha$, $T^- = T^-A^-_{\beta}(x) = A^-(x) + \beta$, for all $x$ in $X$.

1.12 Example: Consider the set $X = \{0, 1, 2, 3, 4\}$. Let $A = \{(0, 0.5, -0.1), (1, 0.4, -0.3), (2, 0.6, -0.05), (3, 0.45, -0.2), (4, 0.2, -0.5)\}$ be a bipolar valued fuzzy subset of $X$ and $\alpha = 0.1, \beta = -0.1$. Then the bipolar valued fuzzy translation of $A$ is $T = T^+A^+_{0.1}(x) = (0, 0.6, -0.2), (1, 0.5, -0.4), (2, 0.7, -0.15), (3, 0.55, -0.3), (4, 0.3, -0.6)$.

1.13 Definition: Let $X$ and $X'$ be any two sets. Let $f: X \rightarrow X'$ be any function and let $A$ be a bipolar valued fuzzy subset in $X$, $V$ be a bipolar valued fuzzy subset in $f(X) = X'$, defined by $V^+(y) = \sup_{x \in f^+(y)} A^+(x)$ and $V^-(y) = \inf_{x \in f^-(y)} A^-(x)$, for all $x$ in $X$ and $y$ in $X'$. $A$ is called a preimage of $V$ under $f$ and is defined as $A^+(x) = V^+(f(x))$, $A^-(x) = V^-(f(x))$ for all $x$ in $X$ and is denoted by $f^{-1}(V)$.

2. Properties:

2.1 Theorem: Let $R$ and $R'$ be any two semirings. The homomorphic image of a bipolar valued fuzzy subsemiring of $R$ is a bipolar valued fuzzy subsemiring of $R'$.

2.2 Theorem: Let $R$ and $R'$ be any two semirings. The homomorphic preimage of a bipolar valued fuzzy subsemiring of $R'$ is a bipolar valued fuzzy subsemiring of $R$.

2.3 Theorem: Let $R$ and $R'$ be any two semirings. The antihomomorphic image of a bipolar valued fuzzy subsemiring of $R$ is a bipolar valued fuzzy subsemiring of $R'$.

2.4 Theorem: Let $R$ and $R'$ be any two semirings. The antihomomorphic preimage of a bipolar valued fuzzy subsemiring of $R'$ is a bipolar valued fuzzy subsemiring of $R$.

2.5 Theorem: Let $A = \langle A^+, A^- \rangle$ be a bipolar valued fuzzy subsemiring of a semiring $R$. Then for $\alpha$ in $[0, 1]$ and $\beta$ in $[-1, 0]$ such that $\alpha \leq A^+(e)$ and $\beta \geq A^-(e)$. $A_{(\alpha, \beta)}$ is a $(\alpha, \beta)$-level subsemiring of $R$.

Proof: For all $x$ and $y$ in $A_{(\alpha, \beta)}$, we have, $A^+(x) \geq \alpha$ and $A^-(x) \leq \beta$ and $A^+(y) \geq \alpha$ and $A^-(y) \leq \beta$.

Suppose there exists $x$ in $R$ such that $\alpha > A^+(x) > \delta$ and $\beta < A^-(x) < \phi$. Therefore $x+y$ in $A_{(\alpha, \beta)}$. Hence $A_{(\alpha, \beta)}$ is a $(\alpha, \beta)$-level subsemiring of $R$.
Then \( A_{(\alpha, \beta)} \subseteq A_{(\delta, \delta)} \) implies \( x \) belongs to \( A_{(\delta, \delta)} \), but not in \( A_{(\alpha, \beta)} \).

This is contradictory to \( A_{(\alpha, \beta)} = A_{(\delta, \delta)} \).

Therefore there is no \( x \) in \( R \) such that \( \alpha > A'(x) > \delta \) and \( \beta < A'(x) < \phi \).

Conversely, if there is no \( x \) in \( R \) such that \( \alpha > A'(x) > \delta \) and \( \beta < A'(x) < \phi \).

Then \( A_{(\alpha, \beta)} = A_{(\delta, \delta)} \) (By the definition of \((\alpha, \beta)\)-level subset).

2.7 Theorem: Let \( A = (A', A^\phi) \) be a bipolar valued fuzzy subsemiring of a semiring \( R \). If any two \((\alpha, \beta)\)-level subsemirings of \( A \) belongs to \( R \), then their intersection is also \((\alpha, \beta)\)-level subsemiring of \( A \) in \( R \).

**Proof:** Let \( \alpha, \delta \in [0, 1], \beta, \phi \in [-1, 0], \alpha \leq A'(e), \delta \leq A'(e), \beta \geq \Lambda'(e), \phi \geq A'(e) \).

**Case (i):** If \( \alpha > A'(x) > \delta \) and \( \beta < A'(x) < \phi \), then \( A_{(\alpha, \beta)} \subseteq A_{(\delta, \delta)} \).

Therefore \( A_{(\alpha, \beta)} \cap A_{(\delta, \delta)} = A_{(\alpha, \beta)} \), but \( A_{(\alpha, \beta)} \) is a \((\alpha, \beta)\)-level subsemiring of \( A \).

**Case (ii):** If \( \alpha < A'(x) < \delta \) and \( \beta > A'(x) > \phi \), then \( A_{(\delta, \delta)} \subseteq A_{(\alpha, \beta)} \).

Therefore \( A_{(\alpha, \beta)} \cap A_{(\delta, \delta)} = A_{(\delta, \delta)} \), but \( A_{(\delta, \delta)} \) is a \((\alpha, \beta)\)-level subsemiring of \( A \).

**Case (iii):** If \( \alpha < A'(x) < \delta \) and \( \beta < A'(x) < \phi \), then \( A_{(\delta, \delta)} \subseteq A_{(\alpha, \beta)} \).

Therefore \( A_{(\alpha, \beta)} \cap A_{(\delta, \delta)} = A_{(\delta, \delta)} \), but \( A_{(\delta, \delta)} \) is a \((\alpha, \beta)\)-level subsemiring of \( A \).

**Case (iv):** If \( \alpha > A'(x) > \delta \) and \( \beta > A'(x) > \phi \), then \( A_{(\alpha, \beta)} \subseteq A_{(\delta, \delta)} \).

Therefore \( A_{(\alpha, \beta)} \cap A_{(\delta, \delta)} = A_{(\alpha, \beta)} \), but \( A_{(\delta, \delta)} \) is a \((\alpha, \beta)\)-level subsemiring of \( A \).

**Case (v):** If \( \alpha = \alpha \) and \( \beta = \beta \), then \( A_{(\alpha, \beta)} = A_{(\delta, \delta)} \).

In other cases are true,

So, in all the cases, intersection of any two \((\alpha, \beta)\)-level subsemirings is a \((\alpha, \beta)\)-level subsemiring of \( A \).

2.8 Theorem: Let \( A = (A', A^\phi) \) be a bipolar valued fuzzy subsemiring of a semiring \( R \). The intersection of a collection of \((\alpha, \beta)\)-level subsemirings of \( A \) is also a \((\alpha, \beta)\)-level subsemiring of \( A \).

2.9 Theorem: Let \( A = (A', A^\phi) \) be a bipolar valued fuzzy subsemiring of a semiring \( R \). If any two \((\alpha, \beta)\)-level subsemirings of \( A \) belongs to \( R \), then their union is also \((\alpha, \beta)\)-level subsemiring of \( A \) in \( R \).

**Proof:** Let \( \alpha, \delta \in [0, 1], \beta, \phi \in [-1, 0], \alpha \leq A'(e), \delta \leq A'(e), \beta \geq \Lambda'(e), \phi \geq A'(e) \).

**Case (i):** If \( \alpha > A'(x) > \delta \) and \( \beta < A'(x) < \phi \), then \( A_{(\alpha, \beta)} \subseteq A_{(\delta, \delta)} \).

Therefore \( A_{(\alpha, \beta)} \cup A_{(\delta, \delta)} = A_{(\delta, \delta)} \), but \( A_{(\delta, \delta)} \) is a \((\alpha, \beta)\)-level subsemiring of \( A \).

**Case (ii):** If \( \alpha < A'(x) < \delta \) and \( \beta > A'(x) > \phi \), then \( A_{(\delta, \delta)} \subseteq A_{(\alpha, \beta)} \).

Therefore \( A_{(\alpha, \beta)} \cup A_{(\delta, \delta)} = A_{(\alpha, \beta)} \), but \( A_{(\alpha, \beta)} \) is a \((\alpha, \beta)\)-level subsemiring of \( A \).

**Case (iii):** If \( \alpha < A'(x) < \delta \) and \( \beta < A'(x) < \phi \), then \( A_{(\alpha, \beta)} \subseteq A_{(\delta, \delta)} \).

Therefore \( A_{(\alpha, \beta)} \cup A_{(\delta, \delta)} = A_{(\alpha, \beta)} \), but \( A_{(\alpha, \beta)} \) is a \((\alpha, \beta)\)-level subsemiring of \( A \).

**Case (iv):** If \( \alpha > A'(x) > \delta \) and \( \beta > A'(x) > \phi \), then \( A_{(\delta, \delta)} \subseteq A_{(\alpha, \beta)} \).

Therefore \( A_{(\alpha, \beta)} \cup A_{(\delta, \delta)} = A_{(\alpha, \beta)} \), but \( A_{(\delta, \delta)} \) is a \((\alpha, \beta)\)-level subsemiring of \( A \).

**Case (v):** If \( \alpha = \alpha \) and \( \beta = \beta \), then \( A_{(\alpha, \beta)} = A_{(\delta, \delta)} \).

In other cases are true,

so, in all the cases, union of any two \((\alpha, \beta)\)-level subsemirings is a \((\alpha, \beta)\)-level subsemiring of \( A \).

2.10 Theorem: Let \( A = (A', A^\phi) \) be a bipolar valued fuzzy subsemiring of a semiring \( R \). The union of a collection of \((\alpha, \beta)\)-level subsemirings of \( A \) is also a \((\alpha, \beta)\)-level subsemiring of \( A \).

2.11 Theorem: The homomorphic image of a \((\alpha, \beta)\)-level subsemiring of a bipolar valued fuzzy subsemiring of a semiring \( R \) is a \((\alpha, \beta)\)-level subsemiring of a bipolar valued fuzzy subsemiring of a semiring \( R \).

**Proof:** Let \( V = f(A) \). Here \( A = (A', A^\phi) \) is a bipolar valued fuzzy subsemiring of \( R \). By Theorem 2.1, \( V = (V', V^\phi) \) is a bipolar valued fuzzy subsemiring of \( R \). Let \( x \) and \( y \) in \( R \). Then \( f(x) \) and \( f(y) \) in \( R \).

Let \( A_{(\alpha, \beta)} \) be a \((\alpha, \beta)\)-level subsemiring of \( A \).

That is, \( A'(x) \geq \alpha \) and \( A'(y) \leq \beta \); \( A'(y) \geq \alpha \) and \( A'(y) \leq \beta \);

\( A'(x+y) \geq \alpha, A'(x+y) \leq \beta, A'(xy) \geq \alpha, A'(xy) \leq \beta \).

We have to prove that \( f(A_{(\alpha, \beta)}) \) is a \((\alpha, \beta)\)-level subsemiring of \( V \).

Now \( V'( f(x)) \geq A'(x) \geq \alpha \) which implies that \( V'( f(x)) \geq \alpha \); and \( V'( f(y)) \geq A'(y) \geq \alpha \) which implies that \( V'( f(y)) \geq \alpha \).
Then $V'(f(x)+f(y)) = V'(f(x+y)) \geq A'(x+y) \geq \alpha$, which implies that $V'(f(x)+f(y)) \geq \alpha$.

And $V'(f(x)f(y)) = V'(f(xy)) \geq A'(xy) \geq \alpha$, which implies that $V'(f(x)f(y)) \geq \alpha$.

And $V'(f(x)) \leq A'(x) \leq \beta$ which implies that $V'(f(x)) \leq \beta$;

and $V'(f(y)) \leq A'(y) \leq \beta$ which implies that $V'(f(y)) \leq \beta$.

Then $V'(f(x)+f(y)) = V'(f(x+y)) \geq A'(x+y) \leq \beta$, which implies that $V'(f(x)+f(y)) \leq \beta$.

And $V'(f(x)f(y)) = V'(f(xy)) \leq A'(xy) \leq \beta$, which implies that $V'(f(x)f(y)) \leq \beta$.

Hence $f(A_{(\alpha, \beta)})$ is a $(\alpha, \beta)$-level subsemiring of a bipolar valued fuzzy subsemiring $V$ of $R'$.

12.2 Theorem: The homomorphic pre-image of a $(\alpha, \beta)$-level subsemiring of a bipolar valued fuzzy subsemiring of a semiring $R'$ is a $(\alpha, \beta)$-level subsemiring of a bipolar valued fuzzy subsemiring of a semiring $R$.

Proof: Let $V = f(A)$. Here $V = \langle V', V^- \rangle$ is a bipolar valued fuzzy subsemiring of $R'$. By Theorem 2.2, $A = \langle A', A^- \rangle$ is a bipolar valued fuzzy subsemiring of $R$. Let $f(x)$ and $f(y)$ in $R$.

Then $x$ and $y$ in $R$. Let $f(A_{(\alpha, \beta)})$ be a $(\alpha, \beta)$-level subsemiring of $V$. That is $V'(f(x)) \geq \alpha$ and $V'(f(y)) \geq \alpha$ and $V'(f(x)+f(y)) \geq \alpha$, $V'(f(x)f(y)) \geq \alpha$, $V'(f(x)(f(y))) \leq \beta$.

We have to prove that $A_{(\alpha, \beta)}$ is a $(\alpha, \beta)$-level subsemiring of $A$. Now $A'(x) = V'(f(x)) \geq \alpha$ implies that $A'(x) \geq \alpha$;

$A'(y) = V'(f(y)) \geq \alpha$ implies that $A'(y) \geq \alpha$.

Then $A'(x+y) = V'(f(x+y)) = V'(f(x)+f(y)) \geq \alpha$, which implies that $A'(x+y) \geq \alpha$.

And $A'(xy) = V'(f(xy)) = V'(f(x)f(y)) \geq \alpha$, which implies that $A'(xy) \geq \alpha$.

Hence $f(A_{(\alpha, \beta)})$ is a $(\alpha, \beta)$-level subsemiring of bipolar valued fuzzy subsemiring $A$ of $R$.

12.3 Theorem: The anti-homomorphic image of a $(\alpha, \beta)$-level subsemiring of a bipolar valued fuzzy subsemiring of a semiring $R$ is a $(\alpha, \beta)$-level subsemiring of a bipolar valued fuzzy subsemiring of a semiring $R'$.

Proof: Let $V = f(A)$. Here $A = \langle A', A^- \rangle$ is a bipolar valued fuzzy subsemiring of $R$. By Theorem 2.3, $V = \langle V', V^- \rangle$ is a bipolar valued fuzzy subsemiring of $R$. Let $x$ and $y$ in $R$.

Then $x$ and $y$ in $R$. Let $f(A_{(\alpha, \beta)})$ be a $(\alpha, \beta)$-level subsemiring of $A$.

That is $A'(x) \geq \alpha$ and $A'(x) \leq \beta$; $A'(y) \geq \alpha$ and $A'(y) \leq \beta$.

And $A'(x+y) \geq \alpha$ and $A'(x+y) \leq \beta$, $A'(yx) \geq \alpha$ and $A'(yx) \leq \beta$.

We have to prove that $f(A_{(\alpha, \beta)})$ is a $(\alpha, \beta)$-level subsemiring of $V$.

Now $V'(f(x)) \geq A'(x) \geq \alpha$ which implies that $V'(f(x)) \geq \alpha$;

and $V'(f(y)) \geq A'(y) \geq \alpha$ which implies that $V'(f(y)) \geq \alpha$.

Also $V'(f(x)+f(y)) = V'(f(x)+f(y)) \geq A'(x+y) \geq \alpha$, which implies that $V'(f(x)+f(y)) \geq \alpha$.

And $V'(f(x)f(y)) = V'(f(xy)) \geq A'(xy) \geq \alpha$, which implies that $V'(f(x)f(y)) \geq \alpha$.

And $V'(f(x)) \leq A'(x) \leq \beta$ which implies that $V'(f(x)) \leq \beta$;

and $V'(f(y)) \leq A'(y) \leq \beta$ which implies that $V'(f(y)) \leq \beta$.

Also $V'(f(x)+f(y)) = V'(f(x)+f(y)) \leq A'(x+y) \leq \beta$, which implies that $V'(f(x)+f(y)) \leq \beta$.

And $V'(f(x)f(y)) = V'(f(xy)) \leq A'(xy) \leq \beta$, which implies that $V'(f(x)f(y)) \leq \beta$.

Hence $f(A_{(\alpha, \beta)})$ is a $(\alpha, \beta)$-level subsemiring of bipolar valued fuzzy subsemiring $V$ of $R$.

12.4 Theorem: The anti-homomorphic pre-image of a $(\alpha, \beta)$-level subsemiring of a bipolar valued fuzzy subsemiring of a semiring $R'$ is a $(\alpha, \beta)$-level subsemiring of a bipolar valued fuzzy subsemiring of a semiring $R$.

Proof: Let $V = f(A)$. Here $V = \langle V', V^- \rangle$ is a bipolar valued fuzzy subsemiring of $R'$. By Theorem 2.4, $A = \langle A', A^- \rangle$ is a bipolar valued fuzzy subsemiring of $R$. Let $f(x)$ and $f(y)$ in $R$.

Then $x$ and $y$ in $R$.

Let $f(A_{(\alpha, \beta)})$ be a $(\alpha, \beta)$-level subsemiring of $V$.

That is $V'(f(x)) \geq A'(x) \geq \alpha$ and $V'(f(x)) \leq \beta$, $V'(f(y)) \geq \alpha$ and $V'(f(y)) \leq \beta$;

$V'(f(x)+f(y)) \geq \alpha$, $V'(f(x)+f(y)) \leq \beta$, $V'(f(y)+f(xy)) \geq \alpha$, $V'(f(y)+f(xy)) \leq \beta$.

We have to prove that $A_{(\alpha, \beta)}$ is a $(\alpha, \beta)$-level subsemiring of $A$.

Now $A'(x) = V'(f(x)) \geq \alpha$, which implies that $A'(x) \geq \alpha$;

and $A'(y) = V'(f(y)) \geq \alpha$, which implies that $A'(y) \geq \alpha$.

Then $A'(x+y) = V'(f(x+y)) = V'(f(x)+f(y)) \geq \alpha$. 
which implies that $A'(x+y) \geq \alpha$.
And $A'(xy) = V'(f(xy)) = V'(f(y)f(x)) \geq \alpha$,
which implies that $A'(xy) \geq \alpha$.

And $A'(x) = V'(f(x)) \leq \beta$ which implies that $A'(x) \leq \beta$
and $A'(y) = V'(f(y)) \leq \beta$ which implies that $A'(y) \leq \beta$.

Also $A'(x+y) = V'(f(x+y)) = V'(f(y)+f(x)) \leq \beta$,
which implies that $A'(x+y) \leq \beta$.

And $A'(xy) = V'(f(xy)) = V'(f(y)f(x)) \leq \beta$,
which implies that $A'(xy) \leq \beta$.

Hence $A_{(\alpha, \beta)}$ is a $(\alpha, \beta)$-level subsemiring of bipolar valued fuzzy subsemiring $A$ of $R$.

2.15 Theorem: Let $A$ be a bipolar valued fuzzy subsemiring of a semiring $R$. Then for $\alpha$ in $[0, 1]$, $A^\alpha$-level $\alpha$-cut $P(A^\alpha, \alpha)$ is a $A^\alpha$-level $\alpha$-cut subsemiring of $R$.

Proof: For all $x$ and $y$ in $P(A^\alpha, \alpha)$ we have $A'(x) \geq \alpha$ and $A'(y) \geq \alpha$.

Now $A'(x+y) \geq \min \{ A'(x), A'(y) \} \geq \min \{ \alpha, \alpha \} = \alpha$,
which implies that $A'(x+y) \geq \alpha$.

And $A'(xy) \geq \min \{ A'(x), A'(y) \} \geq \min \{ \alpha, \alpha \} = \alpha$,
which implies that $A'(xy) \geq \alpha$.

Therefore $x+y, xy$ in $P(A^\alpha, \alpha)$.

Hence $P(A^\alpha, \alpha)$ is a $A^\alpha$-level $\alpha$-cut subsemiring of $R$.

2.16 Theorem: Let $A$ be a bipolar valued fuzzy subsemiring of a semiring $R$. Then for $\beta$ in $[-1, 0]$, $A^{-\beta}$-level $\beta$-cut $N(A^{-\beta}, \beta)$ is a $A^{-\beta}$-level $\beta$-cut subsemiring of $R$.

Proof: For all $x$ and $y$ in $N(A^{-\beta}, \beta)$, we have $A'(x) \leq \beta$ and $A'(y) \leq \beta$.

Now $A'(x+y) \leq \max \{ A'(x), A'(y) \} \leq \max \{ \beta, \beta \} = \beta$,
which implies that $A'(x+y) \leq \beta$.

And $A'(xy) \leq \max \{ A'(x), A'(y) \} \leq \max \{ \beta, \beta \} = \beta$,
which implies that $A'(xy) \leq \beta$.

Therefore $x+y, xy$ in $N(A^{-\beta}, \beta)$.

Hence $N(A^{-\beta}, \beta)$ is a $A^{-\beta}$-level $\beta$-cut subsemiring of $R$.

3. Properties of Bipolar Valued Fuzzy Translations:

3.1 Theorem: If $M$ and $N$ are two bipolar valued fuzzy translations of bipolar valued fuzzy subsemiring $A$ of a semiring $R$, then their intersection $M \cap N$ is also a bipolar valued fuzzy translation of $A$.

Proof: Let $x$ and $y$ belong to $R$.

Let $M = T_{(x, y)}^A = \{ (x, A^+(x) + \alpha, A^+(x) + \beta) / x \in R \}$ and

$N = T_{(x, y)}^A = \{ (x, A^+(x) + \gamma, A^+(x) + \delta) / x \in R \}$ be two bipolar valued fuzzy translations of bipolar valued fuzzy subsemiring $A = (A^+, A^{-})$ of $R$.

Let $C = M \cap N$ and $C = \{ (x, C^+(x), C^{-}(x)) / x \in R \}$, where

$C^+(x) = \min \{ A^+(x) + \alpha, A^+(x) + \gamma \}$ and $C^{-}(x) = \max \{ A^+(x) + \beta, A^+(x) + \delta \}$.

Case (i): $\alpha \geq \gamma$ and $\beta \leq \delta$.

Now $C^+(x) = \min \{ M^+(x), N^+(x) \}$

$= \min \{ A^+(x)+\alpha, A^+(x)+\gamma \} = A^+(x)+\alpha = M^+(x)$ for all $x$ in R.

And $C^{-}(x) = \max \{ M^-(x), N^-(x) \}$

$= \max \{ A^-(x)+\beta, A^-(x)+\delta \} = A^-(x)+\delta = N^-(x)$ for all $x$ in R.

Therefore $C = T_{(x, y)}^A = \{ (x, A^+(x) + \alpha, A^+(x) + \beta) / x \in R \}$ is a bipolar valued fuzzy translation of bipolar valued fuzzy subsemiring $A$ of $R$.

Case (ii): $\alpha \leq \gamma$ and $\beta \geq \delta$.

Now $C^+(x) = \min \{ M^+(x), N^+(x) \}$

$= \min \{ A^+(x)+\alpha, A^+(x)+\gamma \} = A^+(x)+\gamma = N^+(x)$ for all $x$ in R.

And $C^{-}(x) = \max \{ M^-(x), N^-(x) \}$

$= \max \{ A^-(x)+\beta, A^-(x)+\delta \} = A^-(x)+\delta = M^-(x)$ for all $x$ in R.

Therefore $C = T_{(x, y)}^A = \{ (x, A^+(x)+\gamma, A^+(x)+\delta) / x \in R \}$ is a bipolar valued fuzzy translation of bipolar valued fuzzy subsemiring $A$ of $R$.

Case (iii): $\alpha \leq \gamma$ and $\beta \geq \delta$.

Clearly $C = T_{(x, y)}^A = \{ (x, A^+(x)+\alpha, A^+(x)+\beta) / x \in R \}$ is a bipolar valued fuzzy translation of bipolar valued fuzzy subsemiring $A$ of $R$.

Case (iv): $\alpha \geq \gamma$ and $\beta \leq \delta$.

Clearly $C = T_{(x, y)}^A = \{ (x, A^+(x)+\gamma, A^+(x)+\delta) / x \in R \}$ is a bipolar valued fuzzy translation of bipolar valued fuzzy subsemiring $A$ of $R$.
In other cases are true, so in all the cases, the intersection of any two bipolar valued fuzzy translations of bipolar valued fuzzy subsemiring A of R is a bipolar valued fuzzy translation of A.

3.2 Theorem: The intersection of a family of bipolar valued fuzzy translations of bipolar valued fuzzy subsemiring A of a semiring R is a bipolar valued fuzzy translation of A.

Proof: It is trivial.

3.3 Theorem: Union of any two bipolar valued fuzzy translations of bipolar valued fuzzy subsemiring A of a semiring R is a bipolar valued fuzzy translation of A.

Proof: Let x and y belong to R.

Let $M = T^{A}_{(\alpha, \beta)} = \{ (x, A^*(x) + \alpha, A^*(x) + \beta) / x \in R \}$ and

$N = T^{A}_{(\gamma, \delta)} = \{ (x, A^*(x) + \gamma, A^*(x) + \delta) / x \in R \}$ be two bipolar valued fuzzy translations of bipolar valued fuzzy subs miring $A = (A^*, A^*)$ of R.

Let $C = M \cup N$ and $C = \{ (x, C^*(x), C^*(x)) / x \in R \}$, where

$C^*(x) = \max \{ A^*(x) + \alpha, A^*(x) + \gamma \}$ and $C^*(x) = \min \{ A^*(x) + \beta, A^*(x) + \delta \}$.

Case (i): $\alpha \leq \gamma$ and $\beta \leq \delta$.

Now $C^*(x) = \max \{ M^*(x), N^*(x) \}

\begin{align*}
= \max \{ A^*(x) + \alpha, A^*(x) + \gamma \} = A^*(x) + \gamma = N^*(x) \quad \text{for all } x \text{ and } y \text{ in } R.
\end{align*}

And $C^*(x) = \min \{ M^*(x), N^*(x) \}

\begin{align*}
= \min \{ A^*(x) + \beta, A^*(x) + \delta \} = A^*(x) + \beta = M^*(x) \quad \text{for all } x \text{ in } R.
\end{align*}

Therefore $C = T^{A}_{(\alpha, \beta)} = \{ (x, A^*(x) + \gamma, A^*(x) + \delta) / x \in R \}$ is a bipolar valued fuzzy translation of bipolar valued fuzzy subsemiring A of R.

Case (ii): $\alpha \geq \gamma$ and $\beta \geq \delta$.

Now $C^*(x) = \max \{ M^*(x), N^*(x) \}

\begin{align*}
= \max \{ A^*(x) + \alpha, A^*(x) + \gamma \} = A^*(x) + \alpha = M^*(x) \quad \text{for all } x \text{ and } y \text{ in } R.
\end{align*}

And $C^*(x) = \min \{ M^*(x), N^*(x) \}

\begin{align*}
= \min \{ A^*(x) + \beta, A^*(x) + \delta \} = A^*(x) + \delta = N^*(x) \quad \text{for all } x \text{ in } R.
\end{align*}

Therefore $C = T^{A}_{(\alpha, \beta)} = \{ (x, A^*(x) + \alpha, A^*(x) + \beta) / x \in R \}$ is a bipolar valued fuzzy translation of bipolar valued fuzzy subs miring A of R.

Case (iii): $\alpha \leq \gamma$ and $\beta \geq \delta$.

Clearly $C = T^{A}_{(\gamma, \beta)} = \{ (x, A^*(x) + \gamma, A^*(x) + \delta) / x \in R \}$ is a bipolar valued fuzzy translation of bipolar valued fuzzy subsemiring A of R.

Case (iv): $\alpha \geq \gamma$ and $\beta \leq \delta$.

Clearly $C = T^{A}_{(\alpha, \beta)} = \{ (x, A^*(x) + \alpha, A^*(x) + \beta) / x \in R \}$ is a bipolar valued fuzzy translation of bipolar valued fuzzy subsemiring A of R.

In other cases are true, so in all the cases, union of any two bipolar valued fuzzy translations of bipolar valued fuzzy subsemiring A of R is a bipolar valued fuzzy translation of A.

3.4 Theorem: The union of a family of bipolar valued fuzzy translations of bipolar valued fuzzy subsemiring A of a semiring R is a bipolar valued fuzzy translation of A.

Proof: It is trivial.

3.5 Theorem: A bipolar valued fuzzy translation of a bipolar valued fuzzy subsemiring A of a semiring R is a bipolar valued fuzzy subsemiring of R.

Proof: Assume that $T = \{ T^+, T^- \}$ is a bipolar valued fuzzy translation of a bipolar valued fuzzy subsemiring $A = \{ A^+, A^- \}$ of a semiring R.

Let x and y in R.

We have $T^+(x+y) = A^+(x+y) + \alpha \geq \min \{ A^+(x), A^+(y) \} + \alpha$.

\begin{align*}
= \min \{ A^+(x) + \alpha, A^+(y) + \alpha \} = \min \{ T^+(x), T^+(y) \}. \end{align*}

Therefore $T^+(x+y) \geq \min \{ T^+(x), T^+(y) \}$ for all x and y in R.

And $T^-(x+y) = A^-(x+y) + \beta \geq \min \{ A^-(x), A^-(y) \} + \beta$

\begin{align*}
= \min \{ A^-(x) + \beta, A^-(y) + \beta \} = \min \{ T^-(x), T^-(y) \}. \end{align*}

Therefore $T^-(x+y) \leq \min \{ T^-(x), T^-(y) \}$ for all x and y in R.

Also $T^+(x+y) = A^+(x+y) + \beta \leq \max \{ A^+(x), A^+(y) \} + \beta$

\begin{align*}
= \max \{ A^+(x) + \beta, A^+(y) + \beta \} = \max \{ T^+(x), T^+(y) \}. \end{align*}

Therefore $T^-(x+y) \leq \max \{ T^-(x), T^-(y) \}$ for all x and y in R.

And $T^-(x+y) = A^-(x+y) + \beta \leq \max \{ A^-(x), A^-(y) \} + \beta$

\begin{align*}
= \max \{ A^-(x) + \beta, A^-(y) + \beta \} = \max \{ T^-(x), T^-(y) \}. \end{align*}

Therefore $T^-(x+y) \leq \max \{ T^-(x), T^-(y) \}$ for all x and y in R.

Hence T is a bipolar valued fuzzy subsemiring of R.

3.6 Theorem: Let (R, +, .) and (R, , .) be any two semirings and f be a homomorphism. Then the homomorphic image of a bipolar valued fuzzy translation of a bipolar valued fuzzy subsemiring A of R is also a bipolar valued fuzzy subsemiring of R.
Proof: Let $V = (V^+, V^-) = f(T^A_{(\alpha, \beta)})$, where $T^A_{(\alpha, \beta)}$ is a bipolar valued fuzzy translation of a bipolar valued fuzzy submiring $A = (A^+, A^-)$ of $R$. We have to prove that $V$ is a bipolar valued fuzzy subsemiring of $R^1$.

For all $f(x)$ and $f(y)$ in $R^1$, we have $V^+[f(x)+f(y)] = V^+[f(x+y)] \geq T^+_{\alpha}(x+y) = A^+(x+y) + \alpha \geq \min\{A^+(x), A^+(y)\} + \alpha$

which implies $V^+[f(x)+f(y)] \geq \min\{T^+_{\alpha}(x), T^+_{\alpha}(y)\}$

And $V^+[f(x)f(y)] = V^+[f(xy)] \geq T^+_{\alpha}(xy) = A^+(xy) + \alpha \geq \min\{A^+(x), A^+(y)\} + \alpha$

which implies $V^+[f(x)f(y)] \geq \min\{T^+_{\alpha}(x), T^+_{\alpha}(y)\}$

Also $V^+[f(x)+f(y)] = V^+[f(x+y)] \leq T^-_{\beta}(x+y) = A^-(x+y) + \beta = A^-(x) + A^-(y) + \beta = A^- \leq \max\{A^-(x), A^-(y)\} + \beta$

which implies that $V^+[f(x)+f(y)] \leq \max\{T^-_{\beta}(x), T^-_{\beta}(y)\}$

And $V^+[f(x)f(y)] = V^+[f(xy)] \leq T^-_{\beta}(xy) = A^-(xy) + \beta = A^- \leq \max\{A^-(x), A^-(y)\} + \beta$

which implies that $V^+[f(x)f(y)] \leq \max\{T^-_{\beta}(x), T^-_{\beta}(y)\}$

Therefore $V$ is a bipolar valued fuzzy subsemiring of $R^1$.

3.7 Theorem: Let $(R, +, \cdot)$ and $(R^1, +, \cdot)$ be any two semirings and $f$ be a homomorphism. Then the homomorphic pre-image of bipolar valued fuzzy translation of a bipolar valued fuzzy subsemiring $V$ of $R^1$ is a bipolar valued fuzzy subsemiring of $R$.

Proof: Let $T = T^V_{(\alpha, \beta)} = f(A)$, where $T^V_{(\alpha, \beta)}$ is a bipolar valued fuzzy translation of bipolar valued fuzzy submiring $V = (V^+, V^-)$ of $R^1$. We have to prove that $A = (A^+, A^-)$ is a bipolar valued fuzzy subsemiring of $R$.

Let $x$ and $y$ in $R$.

Then $A^+(x+y) = T^+_{\alpha}(f(x+y)) = T^+_{\alpha}(f(x)+f(y)) = V^+[f(x)+f(y)] + \alpha \geq \min\{V^+(f(x)), V^+(f(y))\} + \alpha = \min\{T^+_{\alpha}(f(x)), T^+_{\alpha}(f(y))\} = \min\{A^+(x), A^+(y)\}$

which implies $A^+(x+y) \geq \min\{A^+(x), A^+(y)\}$ for all $x$ and $y$ in $R$.

And $A^+(xy) = T^+_{\alpha}(f(xy)) = T^+_{\alpha}(f(x)f(y)) = V^+[f(x)f(y)] + \alpha \geq \min\{V^+(f(x)), V^+(f(y))\} + \alpha = \min\{T^+_{\alpha}(f(x)), T^+_{\alpha}(f(y))\} = \min\{A^+(x), A^+(y)\}$

which implies $A^+(xy) \geq \min\{A^+(x), A^+(y)\}$ for all $x$ and $y$ in $R$.

Also $A^-(x+y) = T^-_{\beta}(f(x+y)) = T^-_{\beta}(f(x)+f(y)) = V^-[f(x)+f(y)] + \beta \leq \max\{V^-(f(x)), V^-(f(y))\} + \beta = \max\{T^-_{\beta}(f(x)), T^-_{\beta}(f(y))\} = \max\{A^-(x), A^-(y)\}$

which implies $A^-(x+y) \leq \max\{A^-(x), A^-(y)\}$ for all $x$ and $y$ in $R$.

And $A^-(xy) = T^-_{\beta}(f(xy)) = T^-_{\beta}(f(x)f(y)) = V^-[f(x)f(y)] + \beta \leq \max\{V^-(f(x)), V^-(f(y))\} + \beta = \max\{T^-_{\beta}(f(x)), T^-_{\beta}(f(y))\} = \max\{A^-(x), A^-(y)\}$

which implies $A^-(xy) \leq \max\{A^-(x), A^-(y)\}$ for all $x$ and $y$ in $R$.

Therefore $A$ is a bipolar valued fuzzy subsemiring of $R$.

3.8 Theorem: Let $(R, +, \cdot)$ and $(R^1, +, \cdot)$ be any two semirings and $f$ be an anti-homomorphism. Then the anti-homomorphic image of a bipolar valued fuzzy translation of a bipolar valued fuzzy subsemiring $A$ of $R$ is also a bipolar valued fuzzy subsemiring of $R^1$. 
Proof: Let \( V = \langle V^+, V^- \rangle = f(T^A_{(a, b)}), \) where \( T^A_{(a, b)} \) is a bipolar valued fuzzy translation of a bipolar valued fuzzy subsemiring \( A = (A^+, A^-) \) of \( R. \)

We have to prove that \( V \) is a bipolar valued fuzzy subsemiring of \( R \).

For all \( f(x) \) and \( f(y) \) in \( R \),

we have \( V^+[f(x)+f(y)] = V^+[f(y)+f(x)] \geq T^{(a)}_{(y)+x}(x+y) \)

\[= A^+(y+x) + \alpha \geq \min\{ A^+(y), A^+(x) \} + \alpha \]

\[= \min\{ A^+(x) + \alpha, A^+(y) + \alpha \} = \min\{ T^{(a)}_{+a} (x), T^{(a)}_{+a} (y) \} \]

which implies that \( V^+[f(x)+f(y)] \geq \min\{ V^+[f(x)), V^+[f(y)] \} \) for all \( f(x) \) and \( f(y) \) in \( R \).

And \( V^+[f(x)f(y)] = V^+[f(y)x] \geq T^{(a)}_{+a} (xy) \)

\[= A^+(y) + \alpha \geq \min\{ A^+(y), A^+(x) \} + \alpha \]

\[= \max\{ A^+(x) + \alpha, A^+(y) + \beta \} = \max\{ T^{(a)}_{+b} (x), T^{(a)}_{+b} (y) \} \]

which implies that \( V^+[f(x)f(y)] \geq \min\{ V^+[f(x)], V^+[f(y)] \} \) for all \( f(x) \) and \( f(y) \) in \( R \).

Also \( V^+[f(x)+f(y)] = V^+[f(y)+f(x)] \leq T^{(b)}_{-b} (y+x) \)

\[= A^-(y+x) + \beta \leq \max\{ A^-(y), A^-(x) \} + \beta \]

\[= \max\{ A^-(x) + \beta, A^-(y) + \beta \} = \max\{ T^{(b)}_{-b} (x), T^{(b)}_{-b} (y) \} \]

which implies that \( V^+[f(x)f(y)] \leq \min\{ V^+[f(x)], V^+[f(y)] \} \) for all \( f(x) \) and \( f(y) \) in \( R \). Therefore \( V \) is a bipolar valued fuzzy subsemiring of \( R \).

3.9 Theorem: Let \( (R, +, \cdot) \) and \( (R^1, +, \cdot) \) be any two semirings and \( f \) be an anti-homomorphism. Then the anti-homomorphic pre-image of bipolar valued fuzzy translation of a bipolar valued fuzzy subsemiring \( V \) of \( R \) is a bipolar valued fuzzy subsemiring of \( R. \)

Proof: Let \( T = T^V_{(a, b)} = f(A), \) where \( T^V_{(a, b)} \) is a bipolar valued fuzzy translation of bipolar valued fuzzy sub miring \( V = \langle V^+, V^- \rangle \) of \( R \). We have to prove that \( A = (A^+, A^-) \) is a bipolar valued fuzzy subsemiring of \( R. \)

Let \( x \) and \( y \) in \( R. \)

Then \( A^+(x+y) = T^V_{+a} (f(x)+f(y)) = T^V_{+a} (f(y)+f(x)) \)

\[= V^+[f(x)+f(y)] + \alpha \geq \min\{ V^+[f(x)], V^+[f(y)] \} + \alpha \]

\[= \min\{ V^+[f(x)], V^+[f(y)] \} + \alpha \]

which implies that \( A^+(x+y) \geq \min\{ A^+(x), A^+(y) \} \) for all \( x \) and \( y \) in \( R. \)

And \( A^-(x) = T^V_{-a} (f(y)) = T^V_{-a} (f(y)f(x)) \)

\[= V^-[f(x)f(y)] + \alpha \geq \min\{ V^-[f(x)], V^-[f(y)] \} + \alpha \]

\[= \min\{ V^-[f(x)], V^-[f(y)] \} + \alpha \]

which implies that \( A^- (x+y) \geq \min\{ A^- (x), A^- (y) \} \) for all \( x \) and \( y \) in \( R. \)

Also \( A^+(x+y) = T^V_{+b} (f(x+y)) = T^V_{+b} (f(y)+f(x)) \)

\[= V^+[f(y)+f(x)] + \beta \leq \min\{ V^+[f(y)], V^+[f(x)] \} + \beta \]

\[= \min\{ V^+[f(y)], V^+[f(x)] \} + \beta \]

which implies that \( A^+(x+y) \leq \min\{ A^+(x), A^+(y) \} \) for all \( x \) and \( y \) in \( R. \)

And \( A^- (x) = T^V_{-b} (f(x)) = T^V_{-b} (f(y)) \)

\[= V^-[f(x)f(y)] + \beta \leq \min\{ V^-[f(x)], V^-[f(y)] \} + \beta \]

\[= \min\{ V^-[f(x)], V^-[f(y)] \} + \beta \]

which implies that \( A^- (x+y) \leq \min\{ A^- (x), A^- (y) \} \) for all \( x \) and \( y \) in \( R. \)

Therefore \( A \) is a bipolar valued fuzzy subsemiring of \( R \).
References