Geodetic Dominating Sets and Geodetic Dominating Polynomials of Paths

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ARTICLE INFO

Article history:
Received: 26 April 2017;
Received in revised form: 29 May 2017;
Accepted: 9 June 2017;

Keywords
Geodetic dominating set,
Geodetic domination number,
Geodetic domination polynomial.

ABSTRACT

Let $G = (V,E)$ be a simple graph. A set $S \subseteq V$ is a dominating set of $G$ if every vertex in $V \setminus S$ has at least one neighbor in $S$. Let $D_i(G)$ be the family of geodetic dominating sets of $G$ of cardinality $i$. In this paper, we obtain a recursive formula for $d_i(G)$. Using the recursive formula, we construct the geodetic dominating polynomial of $G$ and obtain some properties of this polynomial.

1. Introduction

For any graph $G$, the set of vertices is denoted by $V(G)$ and the edge set by $E(G)$. The order and size of $G$ are denoted by $p$ and $q$ respectively. For a vertex $v \in V(G)$, the open neighborhood $N(v)$ is the set of all vertices adjacent to $v$, and $N[v] = N(v) \cup \{v\}$, is the closed neighborhood of $v$. The degree $d(v)$ of a vertex $v$ is defined by $d(v) = |N(v)|$.

A subset $S$ of vertices in a graph $G$ is a geodetic set if every vertex not in $S$ lies on a shortest path between two vertices from $S$. A subset $D$ of the set of vertices $G$ is called dominating set if every vertex not in $D$ has at least one neighbour in $D$.

1.1 Geodetic dominating set

A set of vertices $S$ in a graph $G$ is a geodetic dominating set if $S$ is both a geodetic set and a dominating set. The minimum cardinality of geodetic dominating set of $G$ is its geodetic domination number, and is denoted by $\gamma(G)$. A geodetic dominating set of size $\gamma(G)$ is said to be a $\gamma(G)$-set.

2. Geodetic dominating sets of path

Let $D_i(P_n)$ be the family of geodetic dominating sets of $P_n$ with cardinality $i$. We investigate the geodetic dominating sets of the path $P_n$. We need the following lemmas to prove our main results in this section.

Lemma 2.1

$$\gamma(P_n) = \left\lceil \frac{n + 2}{5} \right\rceil$$

By Lemma 2.1, and the definition of domination number, one has the following lemma:

Lemma 2.2

$$D_i(P_n) = \Phi \text{if and only if } i \geq n \text{ or } i < \left\lceil \frac{n + 2}{5} \right\rceil.$$ A simple path is a path in which all internal vertices have degree two.

Lemma 2.3

Let $P_n, n \geq 2$ be the path with $|V(P_n)| = n$

(i) If $D_i(P_{n,1}) = D_i(P_{n,3}) = \Phi$ then $D_i(P_{n,2}, i-1) = \Phi$.

(ii) If $D_i(P_{n,1}, i-1) = \Phi$ and $D_i(P_{n,3}, i-1) = \Phi$ then $D_i(P_{n,2}, i-1) = \Phi$.

(iii) If $D_i(P_{n,1}, i-1) = \Phi$ and $D_i(P_{n,3}, i-1) = \Phi$ then $D_i(P_{n,2}, i-1) = \Phi$.

Proof

(1) If $D_i(P_{n,1}, i-1) = \Phi$ and $D_i(P_{n,3}, i-1) = \Phi$ then $i - 1 < \left\lceil \frac{n + 1}{5} \right\rceil$ or $i - 1 > n - 1$ and...
Since, \( D(P_n, i - 1) = \Phi \), then we have,

(i) \( D(P_n, i - 1) = D(P_{n+1}, i - 1) = D(P_{n+2}, i - 1) = D(P_{n+3}, i - 1) = D(P_{n+4}, i - 1) = \Phi \) and \( D(P_{n+5}, i - 1) = \Phi \) if \( n = 5k + 3, i = k +1 \), for some positive integer \( k \).

(ii) \( D(P_{n+2}, i - 1) = D(P_{n+3}, i - 1) = D(P_{n+4}, i - 1) = \Phi \) and \( D(P_{n+5}, i - 1) = \Phi \) if \( i = n \).

(iii) \( D(P_{n+3}, i - 1) = \Phi \) and \( D(P_{n+4}, i - 1) = \Phi \) and \( D(P_{n+5}, i - 1) = \Phi \) if \( i = n - 3 \).

(iv) \( D(P_{n+5}, i - 1) = \Phi \) if \( i = n - 4 \).

Therefore, \( i - 1 < \left\lceil \frac{n + 1}{5} \right\rceil \) or \( i - 1 > n - 3 \).

Therefore, \( i - 1 < \left\lceil \frac{n + 1}{5} \right\rceil \) or \( i - 1 > n - 1 \).

Hence, the theorem.

**Lemma 2.4**

Let \( P_n \) be the path with \( |V(P_n)| = n \). Suppose that \( D_5(P_n, i) = \Phi \), then we have,

(i) \( D_5(P_{n+i+1}, i) = D_5(P_n, i) = D_5(P_{n+1}, i) = D_5(P_{n+2}, i) = D_5(P_{n+3}, i) = D_5(P_{n+4}, i) = D_5(P_{n+5}, i) = \Phi \) if \( n = 5k + 3, i = k +1 \), for some positive integer \( k \).

(ii) \( D_5(P_{n+2}, i) = D_5(P_{n+3}, i) = D_5(P_{n+4}, i) = D_5(P_{n+5}, i) = \Phi \) if \( i = n \).

(iii) \( D_5(P_{n+3}, i) = D_5(P_{n+4}, i) = D_5(P_{n+5}, i) = \Phi \) if \( i = n - 3 \).

(iv) \( D_5(P_{n+4}, i) = D_5(P_{n+5}, i) = \Phi \) if \( i = n - 4 \).

Therefore, \( D_5(P_n, i) = \Phi \) if \( i - 1 < \left\lceil \frac{n + 1}{5} \right\rceil \) or \( i - 1 > n - 3 \).

**Proof**

(i) Since, \( D_5(P_{n+i+1}, i) = D_5(P_n, i) = D_5(P_{n+1}, i) = D_5(P_{n+2}, i) = D_5(P_{n+3}, i) = D_5(P_{n+4}, i) = D_5(P_{n+5}, i) = \Phi \), by Lemma 2.2,

\[ i - 1 > n - 3 \text{ or } i - 1 < \left\lceil \frac{n + 1}{5} \right\rceil. \]
\[ i - 1 > n - 3 \text{ or } i - 1 < \left\lfloor \frac{n - 1}{5} \right\rfloor \] and
\[ i - 1 > n - 4 \text{ or } i - 1 < \left\lfloor \frac{n - 2}{5} \right\rfloor \]

Therefore, \[ i - 1 < \left\lfloor \frac{n - 2}{5} \right\rfloor \text{ or } i - 1 > n - 1 \]

If \[ i - 1 > n - 1 \] then \[ i > n \]

Therefore, \[ D_E(P_n, i) = \Phi \] which is a contradiction.

Therefore, \[ i - 1 < \left\lfloor \frac{n - 2}{5} \right\rfloor \]

Therefore, \[ i < \left\lfloor \frac{n - 2}{5} \right\rfloor + 1 \] and since \[ D_E(P_n, i) \neq \Phi \], we have \[ \frac{n + 2}{5} \leq i < \left\lfloor \frac{n - 2}{5} \right\rfloor + 1 \] which implies that \[ n = 5k + 3 \] and \[ i = k + 1 \] for some \( k \in \mathbb{N} \).

Conversely assume \( n = 5k + 3 \) and \( i = k + 1 \) for some \( k \in \mathbb{N} \).

By Lemma 2.2,
\[ \gamma_E(P_n) = \left\lfloor \frac{n + 2}{5} \right\rfloor \]

Therefore, \[ D_E(P_{n-1}, i - 1) = D_E(P_{5k+3-1}, k) = \Phi, \]

since \( k < \left\lfloor \frac{5k + 3 + 2}{5} \right\rfloor = \left\lfloor \frac{5k + 5}{5} \right\rfloor \)

Similarly, \[ D_E(P_{n-2}, i - 1) = \Phi \]; \[ D_E(P_{n-3}, i - 1) = \Phi \] and \[ D_E(P_{n-4}, i - 1) = \Phi \].

\( D_E(P_{n-5}, i - 1) = D_E(P_{5k+3-5}, k + 1 - 1) = D_E(P_{5k-2}, k), \)

since \( k \geq \left\lfloor \frac{5k - 2 + 2}{5} \right\rfloor = \left\lfloor \frac{5k}{5} \right\rfloor \)

Therefore, \[ D_E(P_{n-5}, i - 1) \neq \Phi . \]

Hence, \[ D_E(P_{n-1}, i - 1) = \Phi ; D_E(P_{n-2}, i - 1) = \Phi ; \]
\[ D_E(P_{n-3}, i - 1) = \Phi ; D_E(P_{n-4}, i - 1) = \Phi \] and \[ D_E(P_{n-5}, i - 1) \neq \Phi . \]

(ii) Since \[ D_E(P_{n-2}, i - 1) = \Phi ; D_E(P_{n-3}, i - 1) = \Phi ; \]
\[ D_E(P_{n-4}, i - 1) = \Phi \] and \[ D_E(P_{n-5}, i - 1) = \Phi \]

By Lemma 2.2,
\[ i - 1 > n - 2 \text{ or } i - 1 < \left\lfloor \frac{n}{5} \right\rfloor \]

If \[ i - 1 < \left\lfloor \frac{n}{5} \right\rfloor \] then \[ i - 1 < \left\lfloor \frac{n + 1}{5} \right\rfloor \]

Therefore, by lemma 2.2, \[ D_E(P_{n-1}, i - 1) = \Phi , \] which is a contradiction.

So we have \[ i - 1 > n - 2 \]

i.e., \[ i > n - 1 \]

Therefore, \[ i \geq n \]

Since, \[ D_E(P_{n-1}, i - 1) \neq \Phi , \] then \[ \left\lfloor \frac{n + 1}{5} \right\rfloor \leq i - 1 \leq n - 1 \]

Therefore, \[ i \leq n \]

Hence, \[ i = n \]

Conversely, if \( i = n \), then \[ D_E(P_{n-2}, i - 1) = D_E(P_{n-2}, n - 1) = \Phi , \]
\[ D_E(P_{n-3}, i - 1) = D_E(P_{n-3}, n - 1) = \Phi , \]
\[ D_E(P_{n-4}, i - 1) = D_E(P_{n-4}, n - 1) = \Phi , \]
\[ D_E(P_{n-5}, i - 1) = D_E(P_{n-5}, n - 1) = \Phi \] and \[ D_E(P_{n-6}, i - 1) = D_E(P_{n-6}, n - 1) \neq \Phi . \]

Since, \[ D_E(P_{n-1}, n - 1) = 1 \].

(iii) Since, \[ D_E(P_{n-5}, i - 1) = \Phi , \] by Lemma 2.2,
\[ i - 1 > n - 5 \text{ or } i - 1 < \left\lfloor \frac{n - 3}{5} \right\rfloor . \]
Since, $D(g(P_{n,2}, i - 1) \neq \Phi$, \(\left[\frac{n}{5}\right] \leq i \leq n - 2\)

i.e., $i - 1 < \left[\frac{n - 3}{5}\right]$ is not possible.

Therefore, $i - 1 > n - 5$
Therefore, $i - 1 \geq n - 4$
But $i - 1 \leq n - 4$
Therefore, $i = n - 3$

Conversely, suppose $i = n - 3$, then
$D(g(P_{n,1}, i - 1) = D(g(P_{n,1}, n - 4) \neq \Phi, D(g(P_{n,2}, i - 1) = D(g(P_{n,2}, n - 4) \neq \Phi, D(g(P_{n,3}, i - 1) = D(g(P_{n,3}, n - 4) \neq \Phi, D(g(P_{n,4}, i - 1) = D(g(P_{n,4}, n - 4) \neq \Phi, but $D(g(P_{n,5}, i - 1) = D(g(P_{n,5}, n - 4) = \Phi$.

By Lemma 2.2,
$D(g(P_{n,1}, i - 1) \neq \Phi, D(g(P_{n,2}, i - 1) \neq \Phi, D(g(P_{n,3}, i - 1) \neq \Phi, D(g(P_{n,4}, i - 1) \neq \Phi and $D(g(P_{n,5}, i - 1) = \Phi$.

(v) Since $D(g(P_{n,1}, i - 1) = \Phi$, by Lemma 2.2,

\[
i - 1 > n - 1 \text{ or } i - 1 < \left[\frac{n + 1}{5}\right]
\]

If $i - 1 > n - 1$ then $i - 1 > n - 2$, by Lemma 2.2,

$D(g(P_{n,2}, i - 1) = D(g(P_{n,3}, i - 1) = D(g(P_{n,4}, i - 1) = D(g(P_{n,5}, i - 1) = \Phi$ which is a contradiction.

Therefore, $i - 1 < \left[\frac{n + 1}{5}\right]

\[
i < \left[\frac{n + 1}{5}\right] + 1.
\]

But $i - 1 \geq \left[\frac{n}{5}\right]$, because $D(g(P_{n,2}, i - 1) \neq \Phi$

Therefore, $i - 1 \geq \left[\frac{n}{5}\right] + 1$

Hence, $\left[\frac{n}{5}\right] + 1 \leq i < \left[\frac{n + 1}{5}\right] + 1$

This holds only if $n = 5k$ and $i = 2k$ for some $k \in \mathbb{N}$.

Conversely, assume $n = 5k$ and $i = 2k$ for some $k \in \mathbb{N}$, then by Lemma 2.2,

$D(g(P_{n,1}, i - 1) = \Phi; D(g(P_{n,2}, i - 1) \neq \Phi; D(g(P_{n,3}, i - 1) \neq \Phi; D(g(P_{n,4}, i - 1) \neq \Phi; D(g(P_{n,5}, i - 1) \neq \Phi$.

Therefore, \(i - 1 \leq n - 3\) and hence \(\left[\frac{n + 1}{5}\right] + 1 \leq i \leq n - 4\)

Therefore, \(i - 1 \leq n - 3\) and \(\left[\frac{n + 1}{5}\right] \leq i \leq n - 2\),

\[
\left[\frac{n - 1}{5}\right] \leq i - 1 \leq n - 3; \left[\frac{n - 2}{5}\right] \leq i - 1 \leq n - 4 \text{ and } \left[\frac{n - 3}{5}\right] \leq i - 1 \leq n - 5.
\]

From these, we obtain that $D(g(P_{n,1}, i - 1) \neq \Phi; D(g(P_{n,2}, i - 1) \neq \Phi; D(g(P_{n,3}, i - 1) \neq \Phi; D(g(P_{n,4}, i - 1) \neq \Phi and $D(g(P_{n,5}, i - 1) \neq \Phi$. 


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Lemma 2.5

For every \( n \geq 4 \) and \( i > \left\lceil \frac{n + 2}{5} \right\rceil \):

(i) If \( D_\emptyset(P_{n-2}, i - 1) = D_{\emptyset}(P_{n-3}, i - 1) = D_\emptyset(P_{n-4}, i - 1) = D_\emptyset(P_{n-5}, i - 1) = \emptyset \) and \( D_\emptyset(P_{n-1}, i - 1) \neq \emptyset \) then \( D_\emptyset(P_n, i) = D_\emptyset(P_n, n) = \{1, 2, 3, \ldots, n\} \).

(ii) If \( D_\emptyset(P_{n-2}, i - 1) = \emptyset \neq D_\emptyset(P_{n-1}, i - 1) \) then \( D_\emptyset(P_n, i) = \{\lceil n \rceil - \{x\}, x \in [n] \} \).

(iii) If \( D_\emptyset(P_{n-1}, i - 1) = \emptyset \neq D_\emptyset(P_{n-2}, i - 1) \neq \emptyset \) then \( D_\emptyset(P_n, i) \neq \emptyset \).

\[ \text{Proof}\]

(i) We have \( D_\emptyset(P_{n-2}, i - 1) = D_\emptyset(P_{n-3}, i - 1) = D_\emptyset(P_{n-4}, i - 1) = D_\emptyset(P_{n-5}, i - 1) = \emptyset \) by Lemma 2.4(ii) we have \( i = n \), therefore, \( D_\emptyset(P_n, i) = D_\emptyset(P_n, n) = \{\{1, 2, 3, \ldots, n\}\} \).

(ii) If \( D_\emptyset(P_{n-3}, i - 1) = \emptyset \neq D_\emptyset(P_{n-1}, i - 1) \neq D_\emptyset(P_{n-2}, i - 1) \neq \emptyset \) by Lemma 2.4.

(iii) Suppose \( D_\emptyset(P_{n-1}, i - 1) \neq \emptyset \), \( D_\emptyset(P_{n-2}, i - 1) \neq \emptyset \), \( D_\emptyset(P_{n-3}, i - 1) \neq \emptyset \). Let \( x_1 \in D_\emptyset(P_n, i - 1) \), then \( n - 2 \) or \( n - 3 \) is \( \emptyset \).

If \( n - 2 \) or \( n - 3 \), then \( x_1 \in D_\emptyset(P_n, i) \).

Let \( x_2 \in D_\emptyset(P_{n-2}, i - 1) \), then \( n - 3 \) or \( n - 4 \) is in \( x_2 \).

If \( n - 3 \) or \( n - 4 \), then \( x_2 \in D_\emptyset(P_{n-3}, i - 1) \), then \( n - 4 \) or \( n - 5 \) is in \( x_2 \).

If \( n - 4 \) or \( n - 5 \), then \( x_3 \in D_\emptyset(P_{n-4}, i - 1) \), then \( n - 5 \) or \( n - 6 \) is in \( x_3 \).

If \( n - 5 \), then \( x_4 \in D_\emptyset(P_{n-5}, i - 1) \), then \( n - 6 \) is in \( x_4 \).

Thus, we have \( \{x_1, x_2, x_3, x_4\} \subseteq D_\emptyset(P_n, i) \).

So, \( D_\emptyset(P_n, i) \subseteq \{\{x_1, x_2, x_3, x_4\}\} \subseteq \{\{x_1, x_2, x_3\}\} \subseteq \{\{x_1, x_2\}\} \subseteq \{\{x_1\}\} \subseteq \emptyset \).

\[ \text{III. Geodetic Domination Polynomial of Path } P_n \]

Let \( D_\emptyset(P_n, x) = \sum_{i=\lceil n/2 \rceil}^{n} d_\emptyset(P_n, i) x^i \) be the geodetic domination polynomial of a Path \( P_n \). In this section, we derive an expression for \( D_\emptyset(P_n, x) \).

Theorem 3.1

a) If \( D_\emptyset(P_n, i) \) is the family of geodetic dominating sets with cardinality \( i \) of \( P_n \), then
The following properties hold for the coefficient of $D_g$.

**Theorem 2.**

(i) $d_g(P_n, n) = 1$ for every $n \geq 2$.
(ii) $d_g(P_n, n-1) = n-2$ for every $n \geq 3$.
(iii) $d_g(P_n, n-2) = \frac{1}{2} [n^2 - 5n + 6]$ for every $n \geq 4$.
(iv) $d_g(P_n, n-3) = \frac{1}{6} (n^3 - 9n^2 + 26n - 36)$ for every $n \geq 5$.

We obtain $d_g(P_n, i)$ for $1 \leq n \leq 12$ as shown in the table

<table>
<thead>
<tr>
<th>$n$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
</tr>
</thead>
<tbody>
<tr>
<td>$d_g(P_n, i)$</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>1</td>
<td>5</td>
<td>3</td>
<td>3</td>
<td>1</td>
<td>6</td>
<td>2</td>
</tr>
</tbody>
</table>

In the following theorem we obtain some properties of $d_g(P_n, i)$.

**Theorem 3.**

The following properties hold for the co-efficient of $D_g(P_n, x)$.

(i) $d_g(P_n, n) = |D_g(P_n, i)|$.
(ii) $d_g(P_n, n-1) = |D_g(P_{n-1}, i)|$.
(iii) $d_g(P_n, n-2) = \sum_{1 \leq i < n} d_g(P_n, i)$.
(iv) $d_g(P_n, n-3) = \sum_{1 \leq i < n} d_g(P_{n-3}, i)$.
(v) \( d_6(P_n, n-4) = \frac{1}{24} (n^4 - 14n^3 + 71 n^2 - 202n + 360) \).

**Proof**

(i) \( d_6(P_n, n) = \{[n]\} \), we have the result.

(ii) \( d_6(P_n, n-1) = n-2 \), for every \( n \geq 3 \).

Since \( D_6(P_n, n-1) = \{[n] - \{x\}, x \in [x]\} \), we have \( d_6(P_n, n-1) = n-2 \).

(iii) By induction on \( n \), the result is true for \( n = 4 \)

L.H.S = \( d_6(4, 2) = 1 \) (from table).

R.H.S = \( \frac{1}{2} (4^2 - 5(4) + 6) = 1 \)

Therefore, the result is true for \( n = 4 \).

Now, suppose that the result is true for all numbers less than \( n \), and we prove it for \( n \).

By theorem 3.1, we have,

\[ d_6(P_n, n-2) = d_6(P_{n-1}, n-3) + d_6(P_{n-2}, n-3) + d_6(P_{n-3}, n-3) + d_6(P_{n-4}, n-3) + d_6(P_{n-5}, n-3) \]

\[ = \frac{1}{2} \left[ (n-1)^2 - 5(n-1) + 6 \right] + (n-2) - 2 + 1 \]

\[ = \frac{1}{2} \left[ n^2 - 2n + 1 - 5n + 5 + 6 + 2n - 6 \right] \]

\[ = \frac{1}{2} \left[ n^2 - 5n + 6 \right] \]

Hence, the result is true for all \( n \).

(v) By induction on \( n \), the result is true for \( n = 6 \).

L.H.S = \( d_6(6, 3) = 2 \) (from table)

R.H.S = \( \frac{1}{6} (6^3 - 9 \times 6^2 + 26 \times 6 - 36) \)

\[ = 2 \]

Therefore, the result is true for all natural numbers less than \( n \).

By Theorem 3.1, we have,

\[ d_6(P_n, n-3) = d_6(P_{n-1}, n-4) + d_6(P_{n-2}, n-4) + d_6(P_{n-3}, n-4) + d_6(P_{n-4}, n-4) + d_6(P_{n-5}, n-4) \]

\[ = \frac{1}{6} \left[ (n-3)^3 - 9(n-3)^2 + 26(n-3) - 36 \right] + \frac{1}{2} \left[ (n-2)^2 - 5(n-2) + 6 \right] + (n-3)^2 - 2 + 1 \]

\[ = \frac{1}{6} \left[ n^3 - 3n^2 + 3n - 1 - 9(n^2 - 2n + 1) + 26n - 26 - 36 \right] + \frac{1}{2} \left[ n^2 - 4n + 4 - 5n + 10 + 6 \right] + n-4 \]

\[ = \frac{1}{6} \left[ n^3 - 3n^2 + 3n - 1 - 9n^2 - 18n - 9 + 26n - 26 - 36 \right] + \frac{1}{2} \left[ n^2 - 9n + 20 \right] + n-4 \]

\[ = \frac{1}{6} \left[ n^3 - 12n^2 + 47n - 72 - 3n^2 - 72n + 60 + 6n - 24 \right] \]

\[ = \frac{1}{6} \left[ n^3 - 9n^2 + 26n - 36 \right] \]

Hence, the result is true for all \( n \).

(vi) By induction on \( n \). Let \( n = 7 \).

L.H.S = \( d_6(P_7, 3) = 1 \) (from table)

R.H.S = \( \frac{1}{24} (74 - 14 \times 7^3 + 71 \times 7^2 - 202 \times 7 + 360) \)

\[ = 1 \]

Therefore, the result is true for \( n = 1 \).

Now, suppose that the result is true for all natural numbers less than \( n \).

\[ d_6(P_{n-4}, n-4) = d_6(P_{n-1}, n-5) + d_6(P_{n-2}, n-5) + d_6(P_{n-3}, n-5) + d_6(P_{n-4}, n-5) + d_6(P_{n-5}, n-5) \]

\[ = \frac{1}{24} \left[ (n-1)^3 - 14(n-1)^2 + 71(n-1) - 202(n-1) + 360 \right] + \]

\[
\frac{1}{6} \left[ (n - 2)^2 - 9(n - 2)^2 + 26(n - 2) - 36 \right] + \\
\frac{1}{2} \left( (n - 2)^3 - 5(n - 3) + 6 \right) + (n - 4) - 2 + 1 \\
= \frac{n^4 - 18n^3 + 119n^2 - 390n + 648}{24} + \frac{n^3 - 15n^2 + 74n - 132}{6} + \frac{n^2 - 11n + 30}{6} + n - 5 \\
= \frac{1}{2} \left[ n^2 - 18n^3 + 119n^2 - 390n + 648 + 4n^3 - 60n^2 + 296n - 528 + 12n^2 - 132n + 460 + 24n - 120 \right] \\
= \frac{n^4 - 14n^3 + 7n^2 - 202n + 360}{24}
\]

Hence the result is true for all n.

**Theorem 3.3**

\[
\sum_{i=n}^{3n} d_g(P_i, n) = 5 \sum_{i=4}^{3n-5} d_g(P_i, n-1) \quad \text{for every} \quad n \geq 4.
\]

**Proof**

Proof by induction on n.

First suppose that n = 4. Then

\[
\sum_{i=4}^{12} d_g(P_i, 4) = 45 = 5 \sum_{i=4}^{7} d_g(4, 3).
\]

\[
\sum_{i=k}^{3k} d_g(P_i, k) = \sum_{i=k}^{3k} d_g(P_{i-1}, k-1) + \sum_{i=k}^{3k} d_g(P_{i-2}, k-1) + \sum_{i=k}^{3k} d_g(P_{i-3}, k-1) + \sum_{i=k}^{3k} d_g(P_{i-4}, k-1) + \sum_{i=k}^{3k} d_g(P_{i-5}, k-1).
\]

\[
= 5 \sum_{i=k-1}^{3k-1} d_g(P_{i-1}, k-2) + 5 \sum_{i=k-1}^{3k-2} d_g(P_{i-2}, k-2) + 5 \sum_{i=k-1}^{3k-3} d_g(P_{i-3}, k-2) + 5 \sum_{i=k-1}^{3k-4} d_g(P_{i-4}, k-2) + 5 \sum_{i=k-1}^{3k-5} d_g(P_{i-5}, k-2).
\]

\[
= 5 \sum_{i=k-1}^{3(k-5)} d_g(P_i, k-1).
\]

We have the result.

**References**


