Minimal Total Dominating Color Transversal Set of Generalised Petersen Graph \( P(n, 1) \)

D.K.Thakkar\(^1\) and A.B.Kothiya\(^2\)

\(^1\)Department of Mathematics, Saurashtra University, Rajkot.
\(^2\)G.K.Bharad Institute of Engineering, Kasturbadham, Rajkot.

**Abstract**

Total Dominating Color Transversal Set of a graph is a Total Dominating Set which is also Transversal of some \( \chi \) - Partition of vertices of \( G \). Here \( \chi \) is the Chromatic number of the graph \( G \). A Total Dominating Color Transversal Set of a graph \( G \) is called Minimal Total Dominating Color Transversal Set of the graph if no proper subset of it is a Total Dominating Color Transversal Set of \( G \). In this paper, we determine a necessary and sufficient condition under which a Total Dominating Color Transversal Set becomes Minimal. We also obtain Minimal Total Dominating Color Transversal set of Generalised Petersen graph \( P(n, 1) \).

**Keywords**

Total Dominating Color Transversal Set, Minimal Total Dominating Color Transversal Set, \( \chi \)- Partition of a graph.

1. Introduction

We begin with simple, finite, connected and undirected graph without isolated vertices. We know that proper coloring of vertices of graph \( G \) partitions the vertex set \( V \) of \( G \) into equivalence classes (also called the color classes of \( G \)). Using minimum number of colors to properly color all the vertices of \( G \) yields \( \chi \)- equivalence classes. Transversal of a \( \chi \)- Partition of \( G \) is a collection of vertices of \( G \) that meets all the color classes of the \( \chi \)- Partition. That is, if \( T \) is a subset of \( V \) (the vertex set of \( G \)) and \( \{V_1, V_2, \ldots, V_{\chi}\} \) is a \( \chi \)- Partition of \( G \) then \( T \) is called a Transversal of this \( \chi \)- Partition if \( T \cap V_i \neq \emptyset \) \( \forall \ i \in \{1, 2, \ldots, \chi\} \). Total Dominating Color Transversal Set of graph \( G \) is a Total Dominating Set with the extra property that it is also Transversal of some such \( \chi \)- Partition of \( G \).

We first mention definitions.

2. Definitions

**Definition 2.1[4]: (Total Dominating Set)** Let \( G = (V, E) \) be a graph. Then a subset \( S \) of \( V \) (the vertex set of \( G \)) is said to be a Total Dominating Set of \( G \) if for each \( v \in V \), \( v \) is adjacent to some vertex in \( S \).

**Definition 2.2[4]: (Minimum Total Dominating Set/Total Domination number)** Let \( G = (V, E) \) be a graph. Then a Total dominating set \( S \) is said to be a Minimum Total Dominating set of \( G \) if \( |S| = \min \{|D| : D \text{ is a Total Dominating set of } G\} \). Here \( S \) is called \( Y_t \)-set and its cardinality, denoted by \( Y_t(G) \) or just by \( Y_t \), is called the Total Domination number of \( G \).

**Definition 2.3[1]: (\( \chi \)-partition of a graph)** Proper coloring of vertices of a graph \( G \), by using minimum number of colors, yields minimum number of independent subsets of vertex set of \( G \) called equivalence classes (also called color classes of \( G \)). Such a partition of a vertex set of \( G \) is called a \( \chi \)- Partition of the graph \( G \).

**Definition 2.4[1]: (Transversal of a \( \chi \)-Partition of a graph)** Let \( G = (V, E) \) be a graph with \( \chi \)- Partition \( \{V_1, V_2, \ldots, V_{\chi}\} \). Then a set \( S \subseteq V \) is called a Transversal of this \( \chi \)- Partition if \( S \cap V_i \neq \emptyset \ \forall \ i \in \{1, 2, \ldots, \chi\} \). Total Dominating Color Transversal Set of the graph \( G \) is a Total Dominating Set of \( G \). Hence any Total Dominating Set of \( G \) will be \( \chi \)- Partition of \( G \).

**Definition 2.5[1]: (Total Dominating Color Transversal Set)** Let \( G = (V, E) \) be a graph. Then a Total Dominating Set \( S \) of \( G \) is called a Total Dominating Color Transversal Set of \( G \) if no proper subset of \( S \) is a Total Dominating Color Transversal Set of \( G \).

**Definition 2.6[1]: (Minimal Total Dominating Color Transversal Set)** Let \( G = (V, E) \) be a graph and \( S \subseteq V \) be a Total Dominating Transversal Set of \( G \). Then \( S \) is called Minimal Total Dominating Color Transversal Set of \( G \) if it is Transversal of at least one \( \chi \)- Partition of \( G \).

**Definition 2.7[5]: (Generalised Petersen Graph)** Let \( n, k \) be the positive integers such that \( n \geq 3 \) and \( 1 \leq k \leq \lfloor \frac{n}{2} \rfloor \). The generalised Petersen graph \( P(n, k) \) is the graph whose vertex set is \( \{a_i, b_i / 1 \leq i \leq n\} \) and whose edge set is \( \{a_i, b_{i+1}, \{a_i, a_{i+k}\}, \{b_i, b_{i+k}\} / 1 \leq i \leq n \} \) where \( a_{i+k} = a_i \) and \( b_{i+k} = b_i \) for every \( c \geq 1 \).

3. Main results

First we state the following theorem taken from [1].

**Theorem 3.1 [1]:** If \( G \) is a graph with \( \chi(G) = 2 \) then \( Y_{tud}(G) = Y_t(G) \).

**Remark 3.2:** Let \( G \) be a graph with \( \chi(G) = 2 \). Then any Total Dominating Set of \( G \) will be a Transversal of every \( \chi \)-Partition of \( G \). Hence any Total Dominating Set of \( G \) will be Total Dominating Color Transversal Set of \( G \).
We now state necessary and sufficient condition under which Total Dominating Color Transversal Set of a graph is Minimal.

**Theorem 3.3:** A Total Dominating Color Transversal Set $D$ of a graph $G = (V, E)$ is Minimal iff for every $u \in D$ at least one of the following conditions hold:

1. There exists $v \in V \setminus \{u\}$ such that $N(v) \cap D = \{u\}$.
2. For every $\chi$ Partition $\{V_1, V_2, V_3, \ldots, V_\chi\}$ there exists $V_i$ such that $V_i \cap D = \{u\}$ or $\phi$.

**Proof:** Let $D$ be a Total Dominating Color Transversal Set of a graph $G$. First assume that $D$ is a Minimal Total Dominating Color Transversal Set of $G$. Then for every $u \in D$, $D \setminus \{u\}$ is not a Total Dominating Color Transversal Set of $G$. Then $D \setminus \{u\}$ is not a Total Dominating Set of $G$ or $D \setminus \{u\}$ is not a transversal for every $\chi$ Partition of $G$.

Case (i): $D \setminus \{u\}$ is not a Total Dominating of $G$.

In such case there exists $v \in V$ such that it is not adjacent to any vertex of $D \setminus \{u\}$. Note that $v \neq u$ as if $v = u$ then $u$ is an isolate of $D$ which is not possible as $D$ is a Total Dominating Set of $G$. Hence we are left with two possibilities viz., (a) $v \in V \setminus D$ or (b) $v \in D \setminus \{u\}$.

(a) $v \in V \setminus D$. $v$ is not adjacent to any of $D \setminus \{u\}$. $v \in V \setminus D$ and $D$ is a Total Dominating set together implies that $v$ is adjacent to $u$ only. Hence $N(v) \cap D = \{u\}$.

(b) $v \in D \setminus \{u\}$. $v$ is not adjacent to any of $D \setminus \{u\}$ implies that $v$ is an isolate of $D \setminus \{u\}$. As $D$ is a Total Dominating set, $v$ is adjacent to $u$ only in $D$. Hence $N(v) \cap D = \{u\}$.

Case (ii): $D \setminus \{u\}$ is not a transversal of every $\chi$ Partition $\{V_1, V_2, V_3, \ldots, V_\chi\}$ of $G$.

In such case $D \setminus \{u\} \cap V_i = \phi$ for some $i \in \{1, 2, \ldots, n\}$. Hence $D \cap V_i = \phi$ or $\{u\}$ for some $i \in \{1, 2, \ldots, n\}$.

Conversely it is easy to prove the above results assuming (II) is true.

Consider $\chi$ Partition $\{V_1, V_2, V_3, \ldots, V_\chi\}$ of $G$ such that $D \setminus \{u\}$ and $D$ are transversals of it. Then $D \setminus \{u\} \cap V_i \neq \phi$ and $D \cap V_i \neq \phi$ for every $i \in \{1, 2, \ldots, n\}$. This means that $D \cap V_i \neq \phi \cap \{u\}$. Therefore condition (II) also fails to hold. Hence both the conditions fails to hold, which is a contradiction.

Hence $D$ is a Minimal Total Dominating Color Transversal Set of $G$.

Hence the theorem.

Now we obtain the Minimal Total Dominating Color Transversal Set of very Interesting and Special Graph called Generalised Petersen Graph $P(n, 1)$.

**Generalised Petersen Graph $P(n, 1)$**

![Figure 1](Image)

![Proposition 3.4 - $\chi(P(n, 1)) = 2$ if $n$ is even $= 3$ if $n$ is odd.](Image)

**Proof:** Consider the above drawn Generalised Petersen Graph $P(n, 1)$. In our proof color pair $(i, j)$ assigned to vertex pair $(u, v)$ will mean colors $i$ and $j$ are assigned to $u$ and $v$ respectively.

Case 1: $n$ is even

Assign color pair $(1, 2)$ to each pairs $(u_1, v_1), (u_3, v_3), \ldots, (u_{n-1}, v_{n-1})$ and color pair $(2, 1)$ to $(u_2, v_2), (u_4, v_4), \ldots, (u_n, v_n)$.

Therefore in this case $\chi(P(n, 1)) = 2$.

Case 2: $n$ is odd

In this case $P(n, 1)$ contains odd cycle. So at least three colors are required to color it. We prove that exactly three colors are required to properly color $P(n, 1)$.

Divide then vertices pairs $(u_i, v_i)$ into groups of three pairs are $\{(u_1, v_1), (u_2, v_2), (u_3, v_3),\ldots, (u_{n-1}, v_{n-1})\}$.

In this case $\chi(P(n, 1)) = 3$.
When $n$ is odd

$$D = \{u_1, u_2, u_3\} \text{ with } |D| = 3, \text{ (mod } 3)$$

$$\{u_1, u_2, v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9, v_{10}, v_{11} \} \text{ with } |D| = \frac{2n + 1}{3}, n \equiv 1 \text{ (mod } 3)$$

$$\{u_1, u_2, v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9, v_{10}, v_{11} \} \text{ with } |D| = \frac{2n + 2}{3}, n \equiv 2 \text{ (mod } 3)$$

**Proof:** Consider the Generalized Petersen Graph drawn in Figure 1.

Divide the vertices into groups of three pairs as $(u_1, u_2, u_3)$, $(u_4, u_5, u_6)$, ... where the last group may have one, two or three pairs.

Case 1: $n$ is even

We know by Proposition 3.4 that $P(n, 1)$ is bipartite. So obviously any Total Dominating Set of $P(n, 1)$ is a Total Dominating Color Transversal Set of $P(n, 1)$.

Sub Case 1(a): $n \equiv 0 \text{ (mod } 3)$

By above Result 3.5, the number of groups of three pairs will be even. From each odd group of three pairs select first two $u_i$'s and from each even group of three pairs select first two $v_i$'s. The resultant set $D = \{u_1, u_2, u_3\}$ is a Minimal Total Dominating Color Transversal Set of $P(n, 1)$ with $|D| = \frac{2n}{3}$.

Sub Case 1(b): $n \equiv 1 \text{ (mod } 3)$

In this case $n \geq 4$.

Note that as $n - 1 \equiv 0 \text{ (mod } 3)$ and $n - 1$ is odd, by Result 3.5, the number of groups of three pairs will be odd and last group will have just one pair $(u_n)$. From each odd group of three pairs select first two $u_i$'s and from each even group of three pairs select first two $v_i$'s.

Case 2: $n$ is odd

Sub Case 2(a): $n \equiv 0 \text{ (mod } 3)$

In this case $n \geq 3$.

For $n = 3$

$$D = \{u_1, u_2, u_3\} \text{ is a Minimal Total Dominating Color Transversal Set of } P(n, 1) \text{ with proper } 3 - \text{ coloring is shown in Figure 2.}$$

For $n > 3$ (That is in this case $n \geq 9$)

By Result 3.5, number of groups of three pairs will be odd. Select the vertices as in Sub Case 1(a), we get a Minimal Total Dominating Color Transversal Set of $P(n, 1)$ with $|D| = \frac{2n}{3}$.

Sub Case 2(b): $n \equiv 1 \text{ (mod } 3)$

In this case $n \geq 7$.

Here as $n - 1 \equiv 0 \text{ (mod } 3)$ and as $n - 1$ is even by Result 3.5, the number of groups of three pairs will be even. From each odd group of three pairs select first two $u_i$'s and from each even group of three pairs select first two $v_i$'s. The last group, which is an odd group, will have $(u_n)$. From this group select $v_n$.

The resultant set $D = \{u_1, u_2, u_3, u_4, u_5, u_6, u_7, u_8, v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9, v_{10}, v_{11}, v_{12}, v_{13}, \}$ is a Minimal Total Dominating Color Transversal Set of $P(n, 1)$ with $|D| = 2(n - 1) + 1 = \frac{2n + 1}{3}$.

Sub Case 2(c): $n \equiv 2 \text{ (mod } 3)$

In this case $n \geq 5$.

Here as $n - 2 \equiv 0 \text{ (mod } 3)$ and $n - 2$ is odd by Result 3.5, the number of groups of three pairs will be odd. Select first two $u_i$'s and from each even group of three pairs select first two $v_i$'s. From the last group, which is an even group, will have $(u_n)$. Select $v_{n - 1}$ and $v_n$ from it. The resultant set $D = \{u_1, u_2, u_3, \}$ is a Minimal Total Dominating Color Transversal Set of $P(n, 1)$ with $|D| = 2(n - 2) + 2 = \frac{2n + 2}{3}$ with proper $3 - \text{ coloring of } P(n, 1)$ is same as follows:

$$D = \{u_1, u_2, u_3\}$$

Case 2: $n$ is odd

Sub Case 2(a): $n \equiv 0 \text{ (mod } 3)$

In this case $n \geq 3$.

For $n = 3$

$$D = \{u_1, u_2, u_3\} \text{ is a Minimal Total Dominating Color Transversal Set of } P(n, 1) \text{ with proper } 3 - \text{ coloring is shown in Figure 2.}$$

For $n > 3$ (That is in this case $n \geq 9$)

By Result 3.5, number of groups of three pairs will be odd. Select the vertices as in Sub Case 1(a), we get a Minimal Total Dominating Color Transversal Set of $P(n, 1)$ with $|D| = \frac{2n}{3}$.

Sub Case 2(b): $n \equiv 1 \text{ (mod } 3)$

In this case $n \geq 7$.

Here as $n - 1 \equiv 0 \text{ (mod } 3)$ and as $n - 1$ is even by Result 3.5, the number of groups of three pairs will be even. From each odd group of three pairs select first two $u_i$'s and from each even group of three pairs select first two $v_i$'s. The last group, which is an odd group, will have $(u_n)$. From this group select $v_n$.

The resultant set $D = \{u_1, u_2, u_3, u_4, u_5, u_6, u_7, u_8, v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9, v_{10}, v_{11} \} \text{ is a Minimal Total Dominating Color Transversal Set of } P(n, 1) \text{ with } |D| = 2(n - 1) + 1 = \frac{2n + 1}{3}$.

Sub Case 2(c): $n \equiv 2 \text{ (mod } 3)$

In this case $n \geq 5$.

Here as $n - 2 \equiv 0 \text{ (mod } 3)$ and $n - 2$ is odd by Result 3.5, the number of groups of three pairs will be odd. Select first two $u_i$'s and from each even group of three pairs select first two $v_i$'s. From the last group, which is an even group, will have $(u_n)$. Select $v_{n - 1}$ and $v_n$ from it. The resultant set $D = \{u_1, u_2, u_3\} \text{ is a Minimal Total Dominating Color Transversal Set of } P(n, 1) \text{ with } |D| = 2(n - 2) + 2 = \frac{2n + 2}{3}$.

In this case $n \geq 7$.

Here as $n - 1 \equiv 0 \text{ (mod } 3)$ and as $n - 1$ is even by Result 3.5, the number of groups of three pairs will be even. From each odd group of three pairs select first two $u_i$'s and from each even group of three pairs select first two $v_i$'s. The last group, which is an odd group, will have $(u_n)$. From this group select $v_n$.

The resultant set $D = \{u_1, u_2, u_3, u_4, u_5, u_6, u_7, u_8, v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9, v_{10}, v_{11} \} \text{ is a Minimal Total Dominating Color Transversal Set of } P(n, 1) \text{ with } |D| = 2(n - 1) + 1 = \frac{2n + 1}{3}$.

Sub Case 2(c): $n \equiv 2 \text{ (mod } 3)$

In this case $n \geq 5$.
4. Concluding Remarks

Minimal Total Dominating Color Transversal Set of Generalised Petersen graph $P(n, 1)$ is in fact Minimal Total Dominating Set of the graph. This set for Generalised Petersen Graph $P(n, k)$ ($k > 2$) is still to be obtained and it becomes an open problem.

5. References