On Decomposition of Neo-Curvature Tensor Field in Finsler Recurrent Spaces

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ABSTRACT

Takano [1967] has studied the decomposition of curvature tensor in a recurrent space. Chandra [1972] has defined Neo-covariant derivative and its applications. In this paper, I have studied decomposition of Neo-curvature tensor field in Finsler recurrent spaces and several theorems have been established.

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Introduction


Let $F_m$ be a subspace of the Finsler space $F_n$. The metric tensors $g_{\alpha\beta}(u, \dot{u})$ of the spaces $F_m$ and $F_n$ are such that, given by Rund [1959].

$$g_{\alpha\beta}(u, \dot{u}) = g_{ij}(x, \dot{x})X^i X^j, \quad X^i = \frac{\partial x^i}{\partial u^\alpha}$$

(1.1)

The neo-covariant differentiation of any vector field $\nabla_\beta Y^\alpha (u, \dot{u})$ with respect to $u^\beta$ is given by Chandra [1972].

$$\nabla_\beta Y^\alpha = \partial_\beta Y^\alpha + \hat{\delta_\beta} Y^\alpha \partial_\beta \dot{u}^\delta + F_{\beta\gamma}^\alpha Y^\gamma$$

(1.2)

where $\nabla$ is the notation for the neo-covariant differentiation, $F_{\beta\gamma}^\alpha$ is the neo-connection of $F_m$. $\partial_\beta$ and $\dot{\partial}_\beta$ denote $\partial/\partial u^\beta$ and $\partial/\partial \dot{u}^\beta$ respectively.

Now differentiating (1.2) neo-covariantly with to $u^\gamma$ and commuting the indices $\beta$ and $\gamma$, we have

$$\nabla_\gamma (\nabla_\beta Y^\alpha) - \nabla_\beta (\nabla_\gamma Y^\alpha) = N_{\delta\beta\gamma}^\alpha Y^\delta$$

(1.3)

where $N_{\delta\beta\gamma}^\alpha (u, \dot{u})$ is the neo-curvature tensor field given by Chandra [1972].

The neo-curvature tensor field $N_{\delta\beta\gamma}^\alpha$ satisfies the following identities given by Chandra [1972]

$$N_{\delta\beta\gamma}^\alpha + N_{\delta\gamma\beta}^\alpha = 0$$

(1.4a)

$$N_{\delta\beta\gamma}^\alpha + N_{\beta\gamma\delta}^\alpha + N_{\gamma\delta\beta}^\alpha = 0$$

(1.4b)

$$\nabla_\gamma N_{\delta\beta\gamma}^\alpha + \nabla_\beta N_{\delta\gamma\beta}^\alpha + \nabla_\delta N_{\beta\delta\gamma}^\alpha = 0$$

(1.4c)

Let $T_{\beta\gamma}^\alpha$ be any tensor field. The following commutation formula will be used in sequel

$$\nabla_\gamma T_{\beta\gamma}^\alpha + \nabla_\beta T_{\gamma\theta}^\alpha = T_{\beta\gamma}^\alpha \partial_\gamma N_{\epsilon\delta\theta}^\alpha - T_{\gamma\theta}^\alpha N_{\epsilon\delta\theta}^\alpha - T_{\beta\epsilon}^\alpha N_{\epsilon\beta\theta}^\alpha - T_{\beta\epsilon}^\alpha N_{\epsilon\beta\theta}^\alpha$$

(1.5)
Definition (1.1): The subspace $F_m$ is said to be neo-recurrent if the neo-curvature tensor field $N_{\delta \beta \gamma}^\alpha$ satisfies the relation

$$
\nabla_{\bar{\theta}} N_{\delta \beta \gamma}^\alpha - \nu_{\bar{\theta}} N_{\delta \beta \gamma}^\alpha = 0,
$$

where $\nu_{\bar{\theta}}$ is non-zero neo-recurrent vector field.

The neo-curvature tensor field in this case is called neo-recurrent curvature tensor field.

In view of (1.6), (1.4c) yields

$$
v_{\bar{\theta}} N_{\delta \beta \gamma}^\alpha + \nu_{\beta} N_{\delta \phi \gamma}^\alpha + \nu_{\gamma} N_{\delta \phi \beta}^\alpha = 0.
$$

2. Decomposition of neo-curvature tensor field in finsler recurrent spaces.

Let us consider the decomposition of the neo-curvature tensor field $N_{\delta \beta \gamma}^\alpha$ is the form

$$
N_{\delta \beta \gamma}^\alpha = W_\delta X_{\beta \gamma}^\alpha,
$$

where $W_\delta$ is decomposition tensor field and $W_\delta(u, \hat{u})$ is a non-zero vector-field.

Theorem (2.1): Under the decomposition (2.1), the tensor fields $X_{\beta \gamma}^\alpha$ satisfies the identities

$$
X_{\beta \gamma}^\alpha + X_{\gamma \beta}^\alpha = 0, \quad (2.2a)
$$

$$
W_\delta X_{\beta \gamma}^\alpha + W_\beta X_{\delta \gamma}^\alpha + W_\gamma X_{\delta \beta}^\alpha = 0, \quad (2.2b)
$$

$$
\nu_{\bar{\theta}} X_{\beta \gamma}^\alpha + \nu_{\beta} X_{\gamma \delta}^\alpha + \nu_{\gamma} X_{\delta \beta}^\alpha = 0. \quad (2.2c)
$$

Theorem (2.2): The necessary and sufficient condition that the tensor field $X_{\beta \gamma}^\alpha$ behaves like a neo-recurrence tensor field of the first order is that the vector field $W_\delta$ be neo-covariant constant.

Proof: Taking neo-covariant differentiation of (2.1) with respect to $u^\phi$, making use of (1.6) and (2.1) in resulting equation and simplifying, we have

$$
W_\delta \left( \frac{n}{\nabla_{\bar{\theta}}} X_{\beta \gamma}^\alpha - \nu_{\bar{\theta}} X_{\beta \gamma}^\alpha \right) = - \left( \frac{n}{\nabla_{\bar{\theta}}} W_\delta \right) X_{\beta \gamma}^\alpha.
$$

Which proves the theorem.

Theorem (2.3): If the vector field $W_\delta$ be neo-covariant constant, then under the decomposition (2.1), the tensor $n_{\bar{\theta}}$ is neo-recurrent, where the square bracket denotes the skew symmetric part.

Proof: Since $n_{\bar{\theta}}$ then from (2.3), we have

$$
\nabla_{\bar{\theta}} W_\delta = 0, \quad (2.4)
$$

Differentiating (2.4) neo-covariantly with respect to $u^\phi$ and using (2.4), we have

$$
\nabla_{\bar{\theta}} X_{\beta \gamma}^\alpha = \left( \nabla_{\bar{\theta}} \nu_{\phi} + \nu_{\beta} \nu_{\phi} \right) X_{\beta \gamma}^\alpha. \quad (2.5)
$$

Commuting the indices $\theta$ and $\phi$ in (2.5) and using (1.5), we have

$$
( \nabla_{\bar{\theta}} \nu_{\phi} - \nabla_{\beta} \nu_{\phi} ) X_{\beta \gamma}^\alpha = \nu_{\phi} N_{\beta \gamma}^\alpha - \nu_{\beta} N_{\phi \gamma}^\alpha - \nu_{\beta} N_{\phi \gamma}^\alpha - \nu_{\beta} N_{\phi \gamma}^\alpha. \quad (2.6)
$$

Again differentiating (2.6) neo-covariantly with respect to $u^\epsilon$ and making use of (1.6), (2.4) and (2.6), we have the result

Theorem (2.4): If the vector field $W_\delta$ be the neo-covariant constant, then under the decomposition (2.1), the recurrence vector field $v_{\bar{\theta}}$ satisfies the relation

$$
v_{\bar{\theta}} \left( \frac{n}{\nabla_{\phi}} \nu_{\phi} - \frac{n}{\nabla_{\theta}} \nu_{\phi} \right) + \nu_{\phi} \left( \frac{n}{\nabla_{\bar{\theta}}} \nu_{\phi} - \frac{n}{\nabla_{\phi}} \nu_{\phi} \right) = 0. \quad (2.7)
$$
Proof: Differentiating (2.5) neo-covariantly with respect to $\partial^\varphi$ and using (2.4), we have

\[
\begin{align*}
\nabla_{\varphi \theta} X_\beta^\alpha - \nabla_\theta X_\beta^\alpha &= n \left[ \nabla_{\varphi} \left( \nabla_{\theta} X_\beta^\alpha - \nabla_{\varphi} X_\beta^\alpha \right) \right] \\
\varphi \theta &\nabla \theta \varphi \\
\nabla_{\varphi \theta} - \nabla_\theta X_\beta^\alpha &= \left[ \nabla_{\varphi} \left( \nabla_{\theta} X_\beta^\alpha - \nabla_{\varphi} X_\beta^\alpha \right) \right] \\
\varphi \theta &\nabla \theta \varphi \\
\nabla_{\varphi} \left( \nabla_{\theta} X_\beta^\alpha - \nabla_{\varphi} X_\beta^\alpha \right)
\end{align*}
\]

(2.8)

Commuting the indices $\theta$ and $\varphi$ in (2.8), we have

\[
\begin{align*}
\nabla_{\varphi} X_\beta^\alpha &= \nabla_{\varphi} - \nabla_{\theta} X_\beta^\alpha \\
\varphi \theta &\nabla \theta \varphi \\
\nabla_{\varphi} \left( \nabla_{\theta} X_\beta^\alpha - \nabla_{\varphi} X_\beta^\alpha \right)
\end{align*}
\]

(2.9)

Which may be written as

\[
\begin{align*}
\nabla_{\varphi} \left( \nabla_{\theta} X_\beta^\alpha - \nabla_{\varphi} X_\beta^\alpha \right) &= \left[ \nabla_{\varphi} \left( \nabla_{\theta} X_\beta^\alpha - \nabla_{\varphi} X_\beta^\alpha \right) \right] \\
\varphi \theta &\nabla \theta \varphi \\
\nabla_{\varphi} \left( \nabla_{\theta} X_\beta^\alpha - \nabla_{\varphi} X_\beta^\alpha \right)
\end{align*}
\]

(2.10)

Applying (1.5) and (2.4) in (2.10) and simplifying, we have

\[
\begin{align*}
v_\delta \left( X_\beta^\delta N_{\delta \varphi} - X_\delta^\gamma N_{\delta \varphi} - X_\beta^\delta N_{\varphi \delta} - X_\delta^\gamma N_{\varphi \delta} \right)
\varphi \theta &\nabla \theta \varphi \\

v_\delta \left( X_\beta^\delta N_{\delta \varphi} - X_\delta^\gamma N_{\delta \varphi} - X_\beta^\delta N_{\varphi \delta} - X_\delta^\gamma N_{\varphi \delta} \right)
\end{align*}
\]

(2.11)

Cyclic permutation of the indices $\varphi$, $\theta$ and $\varphi$ in (2.11) yield two more relations. On adding these three relations and making use of (1.4a) and (1.7), we have (2.7)

Theorem (2.5): Under the decomposition (2.1) the decomposition tensor field $X_\beta^\alpha$ satisfies the relation

\[
\begin{align*}
\nabla_{\varphi} X_\beta^\alpha - \nabla_\theta X_\beta^\alpha &= \left[ \nabla_{\varphi} \left( \nabla_{\theta} X_\beta^\alpha - \nabla_{\varphi} X_\beta^\alpha \right) \right] \\
\varphi \theta &\nabla \theta \varphi \\
\nabla_{\varphi} \left( \nabla_{\theta} X_\beta^\alpha - \nabla_{\varphi} X_\beta^\alpha \right)
\end{align*}
\]

(2.12)

Provided that the vector field $W_\delta$ be neo-covariant constant.

Proof: In view of commutation formula (1.5), equation (2.9) yields

\[
\begin{align*}
\nabla_{\varphi} X_\beta^\alpha - \nabla_\theta X_\beta^\alpha &= \left[ \nabla_{\varphi} \left( \nabla_{\theta} X_\beta^\alpha - \nabla_{\varphi} X_\beta^\alpha \right) \right] \\
\varphi \theta &\nabla \theta \varphi \\
\nabla_{\varphi} \left( \nabla_{\theta} X_\beta^\alpha - \nabla_{\varphi} X_\beta^\alpha \right)
\end{align*}
\]

(2.13)

Interchanging the indices $\varphi$, $\theta$ and $\varphi$ cyclically in (2.13) and adding all the three equations and using (1.4b) and (2.7), we have (2.12).

Now considering the decomposition of the tensor field $X_\beta^\alpha$ in the form

\[
X_\beta^\alpha = U_\alpha Y_\beta^\gamma
\]

(2.14)

Where $U_\alpha (u, \dot{u})$ is any non-zero vector field and $Y_\beta^\gamma (u, \dot{u})$ is non-zero tensor field.

Under the decomposition (2.14), the theorem (2.1) yields the following results:
We may establish the following theorem.

**Theorem (2.6):** If the subspace \( F_m \) undergoes the decompositions (2.1) and (2.14) and the vector-field \( W_\delta \) is neo-covariant constant then the tensor \( Y_{\beta\gamma} \) behaves like neo-recurrent tensor field of the first order provided that \( U^\alpha \) is neo-covariant constant.

**Proof:** Differentiating (2.14) neo-covariantly with respect to \( u^\alpha \), we have

\[
\nabla_{\xi} X_{\beta\gamma} = (\nabla_{\xi} U^\alpha) Y_{\beta\gamma} + U^\alpha (\nabla_{\xi} Y_{\beta\gamma})
\]

Since \( W_\delta \) is neo-covariant constant, hence using (2.4) and (2.14) in (2.16), we have

\[
(\nabla_{\xi} Y_{\beta\gamma} - v_\delta Y_{\beta\gamma}) U^\alpha = - (\nabla_{\xi} U^\alpha) Y_{\beta\gamma}
\]

from which we get the theorem.

**References**


