Connected Total Dominating Sets and Connected Total Domination Polynomials of Extended Grid Graphs

A. Vijayan1 and T. Anitha Baby2

1Department of Mathematics, Nesamony Memorial Christian College, Marthandam, Tamil Nadu, India.
2Department of Mathematics, Women's Christian College, Nagercoil, Tamil Nadu, India.

ABSTRACT

Let G be a simple connected graph of order n. Let $D_{ct}(G, i)$ be the family of connected total dominating sets of G with cardinality i. The polynomial $d_{ct}(G, x) = \sum_{i = 1}^{n} d_{ct}(G, i) x^i$ is called the connected total domination polynomial of G. In this paper, we study some properties of connected total domination polynomials of the Extended grid graph $G_n$. We obtain a recursive formula for $d_{ct}(G_n, i)$. Using this recursive formula, we construct the connected total domination polynomial $d_{ct}(G_n, x) = \sum_{i = n-2}^{2n} d_{ct}(G_n, i) x^i$, of $G_n$, where $d_{ct}(G_n, i)$ is the number of connected total dominating sets of $G_n$ with cardinality i and some properties of this polynomial have been studied.

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1. Introduction

Let $G = (V, E)$ be a simple graph of order n. For any vertex $v \in V$, the open neighbourhood of v is the set $N(v) = \{u \in V \mid uv \in E\}$ and the closed neighbourhood of v is the set $N[v] = N(v) \cup \{v\}$. For a set $S \subseteq V$, the open neighbourhood of S is $N(S) = \bigcup_{v \in S} N(v)$ and the closed neighbourhood of S is $N[S] = N(S) \cup S$. The maximum degree of the graph G is denoted by $\Delta(G)$ and the minimum degree is denoted by $\gamma(G)$.

A set S of vertices in a graph G is said to be a total dominating set if every vertex $v \in V$ is adjacent to an element of S.

A total dominating set S of G is called a connected total dominating set if the induced subgraph (S) is connected.

The minimum cardinality taken over all connected total dominating sets S of G is called the connected total domination number of G and is denoted by $\gamma_{ct}(G)$.

A connected total dominating set with cardinality $\gamma_{ct}(G)$ is called $\gamma_{ct}$-set. We denote the set {1, 2, ..., 2n - 1} by [2n], throughout this paper.

2. Connected Total Dominating Sets of Extended Grid Graphs

Consider two paths $[u_1u_2...u_n]$ and $[v_1v_2...v_n]$. Join each pair of vertices $u_i, v_i \in [u_1u_2...u_n] \times [v_1v_2...v_n]$, $i = 1, 2, ..., n$. The resulting graph is an Extended grid graph.

Let $G_n$ be an Extended grid graph with 2n vertices. Label the vertices of $G_n$ as given in Figure 1.

![Figure 1. Extended Grid Graph $G_n$.](image)

Then, $V(G_n) = \{1, 2, 3, ..., 2n-2, 2n-1, 2n\}$ and $E(G_n) = \{(1,3),(3,5), (5,7), ..., (2n-3, 2n-1) \}$.

For the construction of the connected total dominating sets of the Extended grid graphs $G_n$, we need to investigate the connected total dominating sets of $G_n - \{2n\}$. In this section, we investigate the connected total dominating sets of $G_n$. Let $D_{ct}(G_n)$ be the family of connected total dominating sets of $G_n$ with cardinality i. We shall find the recursive formula for $d_{ct}(G_n, i)$.

**Lemma 2.1** [7]

$\gamma_{ct}(P_n) = n - 2$.

**Lemma 2.2**

For every $n \in N$ and $n \geq 4$,

(i) $\gamma_{ct}(G_n) = n - 2$.

(ii) $\gamma_{ct}(G_n - \{2n\}) = n - 2$.

(iii) $D_{ct}(G_n, i) = \phi$ if and only if $i < n - 2$ or $i > 2n$.

(iv) $D_{ct}(G_n - \{2n\}, i) = \phi$ if and only if $i < n - 2$ or $i > 2n - 1$.

**Proof**

(i) Clearly $\{3, 5, 7, ..., 2n - 3\}$ is a minimum connected total dominating set for $G_n$. If n is even or odd it contains n - 2 elements. Hence, $\gamma_{ct}(G_n) = n - 2$.

(ii) Clearly $\{3, 5, 7, ..., 2n - 3\}$ is a minimum connected total dominating set for $G_n - \{2n\}$. If n is even or odd it contains n - 2 elements. Hence, $\gamma_{ct}(G_n - \{2n\}) = n - 2$.

(iii) follows from (i) and the definition of connected total dominating set.
Lemma 2.3

(i) If $D_\text{ct} (G_{n-2} \{2n-4\}, i - 1) = \phi$, then $D_\text{ct} (G_{n-1}, i - 1) = \phi$, and $D_\text{ct} (G_{n-1} \{2n-2\}, i - 1) = \phi$, then $D_\text{ct} (G_{n-2}, i - 1) = \phi$.

(ii) If $D_\text{ct} (G_{n-2} \{2n-4\}, i - 1) \neq \phi$, then $D_\text{ct} (G_{n-1}, i - 1) \neq \phi$, and $D_\text{ct} (G_{n-1} \{2n-2\}, i - 1) \neq \phi$, then $D_\text{ct} (G_{n-2}, i - 1) \neq \phi$.

(iii) If $D_\text{ct} (G_{n-1} \{2n-2\}, i - 1) \neq \phi$, then $D_\text{ct} (G_{n-1}, i - 1) \neq \phi$, and $D_\text{ct} (G_{n-1} \{2n\}, i - 1) \neq \phi$, then $D_\text{ct} (G_{n-1}, i) \neq \phi$.

(iv) If $D_\text{ct} (G_{n-1} \{2n-2\}, i - 1) \neq \phi$, then $D_\text{ct} (G_{n-1}, i - 1) = \phi$, and $D_\text{ct} (G_{n-1} \{2n\}, i - 1) = \phi$, then $D_\text{ct} (G_{n-1}, i) = \phi$.

(v) If $D_\text{ct} (G_{n-1} \{2n\}, i - 1) = \phi$, then $D_\text{ct} (G_{n-1}, i - 1) = \phi$, and $D_\text{ct} (G_{n-1} \{2n\}, i - 1) = \phi$, then $D_\text{ct} (G_{n-1}, i) = \phi$.

Proof

(i) Since, $D_\text{ct} (G_{n-2} \{2n-4\}, i - 1) = \phi$, then $D_\text{ct} (G_{n-1}, i - 1) = \phi$, and $D_\text{ct} (G_{n-1} \{2n-2\}, i - 1) = \phi$, by Lemma 2.2(iii) & (iv), we have, $i - 1 < n - 4$ or $i - 1 > 2n - 5$, $i - 1 < n - 3$ or $i - 1 > 2n - 2$, and $i - 1 < n - 3$ or $i - 1 > 2n - 3$.

Therefore, $i - 1 < n - 4$ or $i - 1 > 2n - 3$. Therefore, $i - 1 < n - 4$ or $i - 1 > 2n - 4$ holds.

Hence, $D_\text{ct} (G_{n-2}, i - 1) = \phi$.

(ii) Since, $D_\text{ct} (G_{n-2} \{2n-4\}, i - 1) \neq \phi$, then $D_\text{ct} (G_{n-1}, i - 1) \neq \phi$, and $D_\text{ct} (G_{n-1} \{2n-2\}, i - 1) \neq \phi$, by Lemma 2.2(iii) & (iv), we have, $n - 4 \leq i - 1 \leq 2n - 5$, $n - 3 \leq i - 1 \leq 2n - 2$, and $n - 3 \leq i - 1 \leq 2n - 3$.

Suppose, $D_\text{ct} (G_{n-2}, i - 1) = \phi$, then $i - 1 < n - 4$ or $i - 1 > 2n - 4$.

Suppose, $i - 1 < n - 4$ or $i - 1 > 2n - 4$, then $D_\text{ct} (G_{n-1}, i - 1) = \phi$, a contradiction.

Suppose, $i - 1 > 2n - 4$, then $i - 1 > 2n - 5$ holds, which implies $D_\text{ct} (G_{n-2} \{2n - 4\}, i - 1) = \phi$, a contradiction.

Therefore, $D_\text{ct} (G_{n-2}, i - 1) \neq \phi$.

(iii) Since, $D_\text{ct} (G_{n-2} \{2n-2\}, i - 1) \neq \phi$, then $D_\text{ct} (G_{n-1}, i - 1) \neq \phi$, and $D_\text{ct} (G_{n-1} \{2n\}, i - 1) \neq \phi$, by Lemma 2.2(iii) & (iv), we have, $n - 3 \leq i - 1 \leq 2n - 3$, $n - 3 \leq i - 1 \leq 2n - 2$, and $n - 3 \leq i - 1 \leq 2n - 1$.

Suppose, $D_\text{ct} (G_{n-1}, i) = \phi$, then, by Lemma 2.2(iii), we have $i < n - 2$ or $i > 2n$.

Suppose, $i < n - 2$, then $i - 1 < n - 3$, which implies $D_\text{ct} (G_{n-1} \{2n-2\}, i - 1) = \phi$, a contradiction.

Suppose, $i > 2n$, then $i - 1 > 2n - 1$, which implies $D_\text{ct} (G_{n-1} \{2n\}, i - 1) = \phi$, a contradiction.

Therefore, $D_\text{ct} (G_{n-1}, i) \neq \phi$.

(iv) Since, $D_\text{ct} (G_{n-1} \{2n-2\}, i - 1) \neq \phi$, then $D_\text{ct} (G_{n-1}, i - 1) \neq \phi$, by Lemma 2.2(iii) & (iv), we have, $n - 3 \leq i - 1 \leq 2n - 3$ and $n - 3 \leq i - 1 \leq 2n - 2$.

Suppose, $D_\text{ct} (G_{n-1}, i) = \phi$, then, by Lemma 2.2(iii), we have $i < n - 2$ or $i > 2n$.

Suppose, $i < n - 2$ then $i - 1 < n - 3$ which implies $D_\text{ct} (G_{n-1} \{2n-2\}, i - 1) = \phi$, a contradiction.

Therefore, $i < n - 2$ then $i - 1 > 2n - 1$.

Therefore, $i < n - 2$ or $i > 2n$.

Suppose, $i < n - 2$ then $i - 1 < n - 3$ which implies $D_\text{ct} (G_{n-1} \{2n-2\}, i - 1) = \phi$, a contradiction.

Therefore, $i < 2n$.

Suppose, $i > 2n$, then $i - 1 > 2n - 1$.

Therefore, $i < n - 2$ or $i > 2n$.

(v) Since, $D_\text{ct} (G_{n-1} \{2n - 2\}, i - 1) = \phi$, then $D_\text{ct} (G_{n-1}, i - 1) = \phi$ and $D_\text{ct} (G_{n-1} \{2n\}, i - 1) = \phi$, by Lemma 2.2(iv), we have, $i - 1 < n - 3$ or $i - 1 > 2n - 2$.

Suppose, $i - 1 < n - 3$ then $i - 1 < n - 2$, which implies, $D_\text{ct} (G_{n-1}, i) = \phi$, a contradiction.

Therefore, $i - 1 > 2n - 2$.

Therefore, $i - 1 < n - 3$ or $i - 1 > 2n - 2$.

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Therefore, $i - 1 < n - 3$ or $i - 1 > 2n - 2$.

Therefore, $i - 1 < n - 3$ or $i - 1 > 2n - 2$.
(⇐) follows from Lemma 2.2 (iii) & (iv).

(iii) Since, $D_{ct}(G_n - \{2n\}, i - 1) \neq \phi$, $D_{ct}(G_{n-1}, i - 1) \neq \phi$ and $D_{ct}(G_{n-1}, \{2n-2\}, i-1) \neq \phi$, by Lemma 2.2 (iii) & (iv), we have,

\[
\begin{align*}
&n-2 \leq i-1 \leq 2n-1, \\
&n-3 \leq i-1 \leq 2n-2 \\
&n-3 \leq i-1 \leq 2n-3.
\end{align*}
\]

Therefore, $n-2 \leq i-1 \leq 2n-3$. (5)

Also, since, $D_{ct}(G_{n-2}, i - 1) = \phi$, by Lemma 2.2 (iii), we have,

\[i-1 < n-4, \text{ or } i-1 > 2n-4.\]

Suppose, $i-1 < n-4$, then $i-1 < n-3$. Therefore, $i < n-2$, which implies $D_{ct}(G_n, i) = \phi$, a contradiction.

Therefore, $i-1 > 2n-4$. (6)

From (5) and (6), we have, $i-1 = 2n-3$.

Therefore, $i = 2n-2$.

(⇐) follows from Lemma 2.2 (iii) & (iv).

(iv) Since, $D_{ct}(G_{n-1}, i-1) \neq \phi$ and $D_{ct}(G_{n-1}, \{2n-2\}, i-1) \neq \phi$, by Lemma 2.2 (iii) & (iv), we have,

\[
\begin{align*}
&n-3 \leq i-1 \leq 2n-2 \\
&n-3 \leq i-1 \leq 2n-3.
\end{align*}
\]

Therefore, $n-3 \leq i-1 \leq 2n-3$. (7)

Also, since, $D_{ct}(G_n - \{2n\}, i - 1) = \phi$, by Lemma 2.2 (iv), we have,

\[i-1 < n-2 \text{ or } i-1 > 2n-1.\]

Suppose, $i-1 > 2n-1$, then $i > 2n$, which implies $D_{ct}(G_n, i) = \phi$, a contradiction.

Therefore, $i-1 < n-2$. (8)

From (7) and (8), we have, $n-2 \leq i < n-1$. (9)

When $n = k + 2$, we get an inequality of the form $s \leq i < s$, which is not possible. When $n = k + 2$, we have $s \leq i < s + 1$. Therefore (9) holds. In this case $i = k$.

Conversely, assume $n = k + 2$ and $i = k$.

Therefore, $n = 2k$ and $i-1 = k-1$.

$k-1 < k = n-2$.

Therefore, $D_{ct}(G_{n-1}, i-1) = \phi$.

Also,

\[D_{ct}(G_{n-1} - \{2n-2\}, i-1) = D_{ct}(G_{k+1} - \{2(k+1)\}, k-1) \neq \phi.\]

and $D_{ct}(G_{n-1}, i-1) = D_{ct}(G_{k+1}, k-1) \neq \phi$. \hfill (Theorem 2.5)

For every $n \geq 4$,

(i) If $D_{ct}(G_{n-\{2n\}}, i-1) \neq \phi$, $D_{ct}(G_{n-1}, i-1) \neq \phi$, and $D_{ct}(G_{n-1} - \{2n-2\}, i-1) \neq \phi$, then

\[D_{ct}(G_n, i) = \{X \cup \{2n\} / X \in D_{ct}(G_n - \{2n\}, i-1)\} \cup \{X \cup \{2n-3\} / X \in D_{ct}(G_n - \{2n-3\}, i-1)\}.\]

(ii) If $D_{ct}(G_{n-\{2n\}}, i-1) \neq \phi$, $D_{ct}(G_{n-1}, i-1) \neq \phi$, and $D_{ct}(G_{n-1} - \{2n-2\}, i-1) \neq \phi$, then

\[D_{ct}(G_n, i) = \{X \cup \{2n-2\} / X \in D_{ct}(G_n - \{2n-2\}, i-1)\} \cup \{X \cup \{2n-3\} / X \in D_{ct}(G_n - \{2n-3\}, i-1)\}.\]

(iii) If $D_{ct}(G_{n-\{2n\}}, i-1) \neq \phi$, $D_{ct}(G_{n-1}, i-1) \neq \phi$, and $D_{ct}(G_{n-1} - \{2n-2\}, i-1) \neq \phi$,

then,

\[D_{ct}(G_n, i) = \{X_1 \cup \{2n\} / X_1 \in D_{ct}(G_n - \{2n\}, i-1)\} \cup \{X_2 \cup \{2n-1\} / X_2 \in D_{ct}(G_n - \{2n-1\}, i-1)\} \cup \{X_3 \cup \{2n-2\} / X_3 \in D_{ct}(G_n - \{2n-2\}, i-1)\} \cup \{X_4 \cup \{2n-3\} / X_4 \in D_{ct}(G_n - \{2n-3\}, i-1)\}.\]

Proof

(i) Since, $D_{ct}(G_n - \{2n\}, i-1) \neq \phi$, $D_{ct}(G_{n-1}, i-1) \neq \phi$ and $D_{ct}(G_n - \{2n-2\}, i-1) \neq \phi$, by Theorem 2.4 (i), $i = 2n$.

Therefore, $D_{ct}(G_n, i) = D_{ct}(G_n, 2n) = \{2n\}$ and $D_{ct}(G_{n-1} - \{2n-2\}, i-1) = \phi$.

Obviously, $Y_1 \cup Y_2 \subseteq D_{ct}(G_n, i)$. (1)

Now, let $Y \in D_{ct}(G_n, i)$.

If $2n \not\in Y$, then at least one of the vertices labeled $2n-4$ or $2n-5$ is in $Y$. In either cases, $Y = \{X_1 \cup \{2n\}\}$ for some $X_1 \in D_{ct}(G_n, i-1)$.

Therefore, $Y \subseteq Y_1$.

If $2n-3 \in Y$, then at least one of the vertices labeled $2n-4$ or $2n-5$ is in $Y$. In either cases, $Y = \{X_2 \cup \{2n-3\}\}$ for some $X_2 \in D_{ct}(G_{n-1} - \{2n-2\}, i-1)$.

Therefore, $Y \subseteq Y_2$.

Therefore, $D_{ct}(G_n, i) \subseteq Y_1 \cup Y_2$.

From (1) and (2), we have,

$D_{ct}(G_n, i) = \{X_1 \cup \{2n\} / X_1 \in D_{ct}(G_n - \{2n\}, i-1)\} \cup \{X_2 \cup \{2n-1\} / X_2 \in D_{ct}(G_{n-1} - \{2n-1\}, i-1)\} \cup \{X_3 \cup \{2n-2\} / X_3 \in D_{ct}(G_{n-1} - \{2n-2\}, i-1)\} \cup \{X_4 \cup \{2n-3\} / X_4 \in D_{ct}(G_{n-1} - \{2n-3\}, i-1)\}$. (3)

Now, let $Y \in D_{ct}(G_n, i)$.

If $2n \in Y$, then at least one of the vertices labeled $2n-1$ or $2n-2$ or $2n-3$ is in $Y$. In each cases, $Y = \{X_1 \cup \{2n\}\}$ for some $X_1 \in D_{ct}(G_n - \{2n\}, i-1)$.

Therefore, $Y \subseteq Y_1$.

Now suppose that, $2n-1 \not\in Y$, $2n \not\in Y$, then at least one of the vertices labeled $2n-2$ or $2n-3$ is in $Y$.

In each cases, $Y = \{X_2 \cup \{2n\}\}$ for some $X_2 \in D_{ct}(G_{n-1} - \{2n\}, i-1)$.

Now suppose that, $2n-2 \not\in Y$ and $2n, 2n-1 \not\in Y$, then at least one of the vertices labeled $2n-3$ or $2n-4$ is in $Y$.

If $2n-3 \not\in Y$, then $Y = \{X_3 \cup \{2n-3\}\}$ for some $X_3 \in D_{ct}(G_{n-1} - \{2n-3\}, i-1)$.

If $2n-4 \not\in Y$, then $Y = \{X_4 \cup \{2n-4\}\}$ for some $X_4 \in D_{ct}(G_{n-1} - \{2n-4\}, i-1)$. 

Therefore, $Y \subseteq Y_2$.
Therefore, $Y \notin Y_2$ or $Y \notin Y_3$.
Now suppose that, $2n-3 \notin Y$ and $2n, 2n-1, 2n-2 \notin Y$, then $2n-4$ is in $Y$.
In this case, $Y = \{X_3 \cup \{2n-3\}\}$ for some $X_3 \in D_{ct}(G_{n-1} - \{2n-2\}, i-1)$. Therefore, $Y \notin Y_3$.
Hence, $D_{ct}(G_n,i) \subseteq Y_1 \cup Y_2 \cup Y_3$.

From (3) and (4) we have,
$$D_{ct}(G_n,i) = \left\{ \begin{array}{l}
\{X_1 \cup \{2n\} / X_1 \in D_{ct}(G_n - \{2n\},i-1) \cup
\{X_2 \cup \{2n-1\} / X_2 \in D_{ct}(G_{n-1},i-1) \cup
\{X_3 \cup \{2n-2\} / X_3 \in D_{ct}(G_{n-1} - \{2n-2\},i-1) \cup
\{X_4 \cup \{2n-3\} / X_4 \in D_{ct}(G_{n-1} - \{2n-2\},i-1) \cup
\end{array} \right\}$$

**Theorem 2.6**
If $D_{ct}(G_n,i)$ is the family of connected total dominating sets of $G_n$ with cardinality $i$, where $i \geq n - 2$, then $d_{ct}(G_n,i) = d_{ct}(G_{n-1} - \{2n\},i-1) + d_{ct}(G_{n-1},i-1) + d_{ct}(G_{n-1} - \{2n-2\},i-1)$.

**Proof**
We consider all the three cases given in Theorem 2.5. By Theorem 2.5 (i), we have,
$$D_{ct}(G_n,i) = \{X \cup \{2n\} / X \in D_{ct}(G_n - \{2n\}, i-1) \}.$$ Since, $D_{ct}(G_n,i-1) = \phi$ and $d_{ct}(G_n,i-1) = 0$, we have $d_{ct}(G_n,i) = 0$ and $d_{ct}(G_n - \{2n\}, i-1) = 0$.
Therefore, $d_{ct}(G_n,i) = d_{ct}(G_{n-1} - \{2n\}, i-1)$.

By Theorem 2.5 (ii), we have,
$$D_{ct}(G_n,i) = \left\{ \begin{array}{l}
\{X_1 \cup \{2n-2\} / X_1 \in D_{ct}(G_n - \{2n\},i-1) \cup
\{X_2 \cup \{2n-3\} / X_2 \in D_{ct}(G_n - \{2n\},i-1) \cup
\end{array} \right\}$$

Since, $D_{ct}(G_n,i) = \phi$, we have $d_{ct}(G_n,i) = 0$ and $d_{ct}(G_n - \{2n\}, i-1) = 0$.
Therefore, $d_{ct}(G_n,i) = d_{ct}(G_{n-1} - \{2n\}, i-1)$.

By Theorem 2.5 (iii), we have,
$$D_{ct}(G_n,i) = \{X_1 \cup \{2n\} / X_1 \in D_{ct}(G_n - \{2n\},i-1) \cup$$

Therefore, $d_{ct}(G_n,i) = d_{ct}(G_{n-1} - \{2n\}, i-1) + d_{ct}(G_{n-1},i-1) + d_{ct}(G_{n-1} - \{2n-2\},i-1)$.

**3. Connected Total Domination Polynomials of Extented Grid Graphs.**

**Definition 3.1**
Let $D_{ct}(G_n,i)$ be the family of connected total dominating sets of $G_n$ with cardinality $i$ and let $d_{ct}(G_n,i) = \mid D_{ct}(G_n,i) \mid$.
Then the connected total domination polynomial $D_{ct}(G_n,x)$ of $G_n$ is defined as,
$$D_{ct}(G_n,x) = \sum_{i=0}^{\gamma_{ct}(G_n)} d_{ct}(G_n,i) x^i.$$  

**Theorem 3.2**
For every $n \geq 5$,
Table 1. $d_d(G_n, i)$ and $d_d(G_n−\{2n\}, i)$ for $2 \leq n \leq 9$.

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<th>i/n</th>
<th>2</th>
<th>3</th>
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<td>$G_2$</td>
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