The Split Majority Domatic Number of a Graph

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ABSTRACT
Let \( G = (V, E) \) be any simple finite graph. A subset \( D \) of \( V(G) \) is said to be Split Majority Dominating set of \( G \) if \( |N[D]| \geq \left\lceil \frac{|E|}{2} \right\rceil \) and the induced subgraph \( \langle V - D \rangle \) is disconnected. A split majority dominating set \( D \) is said to be minimal if there exists a vertex \( v \) of \( V \) such that \( D - \{v\} \) is not a split majority dominating set of \( G \). The Split Majority Domatic Number denoted by \( d_{sm}(G) \) is the maximum number of disjoint minimal split majority dominating sets obtained for a graph \( G \). In this article, we have initiated the study of this concept.

Keywords
Set, Split Majority Dominating set.

Introduction
Let \( G = (V, E) \) be any simple finite graph with \( |V(G)| = p \) and \( |E(G)| = q \). With usual notations, the degree of a vertex \( v \), the maximum and the minimum degree of a graph \( G \) are denoted by \( d(v), \Delta(G) \) and \( \delta(G) \) respectively.

A set \( D \subseteq V(G) \) is said to be a dominating set [2] of \( G \) if for every vertex \( v \) in \( V-D \) there exists at least one vertex \( u \) in \( D \) such that \( u \) and \( v \) are adjacent in \( G \). A Dominating set \( D \) is said to be minimal if for some vertex \( v \) of \( G \), \( D - \{v\} \) is not a dominating set. The minimum cardinality of a minimal dominating set is called the domination number of \( G \) and it is denoted by \( \gamma(G) \).

A set \( D \subseteq V(G) \) is said to be a majority dominating set [3] of \( G \) if \( \frac{|N[D]|}{2} \geq \left\lceil \frac{q}{2} \right\rceil \). A majority Dominating set \( D \) is said to be minimal if for some vertex \( v \) of \( G \), \( D - \{v\} \) is not a majority dominating set. The minimum cardinality of a minimal majority dominating set is called the ma-jority domination number of \( G \) and it is denoted by \( M(G) \). This parameter was defined by Swaminathan and Joseline Manora.

A Dominating set \( D \subseteq V(G) \) is said to be a split dominating set[8] if the induced subgraph \( \langle V - D \rangle \) is disconnected. With usual inferences, the minimum cardinality of minimal split dominating set is denoted by \( s(G) \). This parameter was introduced by Kulli and Janakiram.

A subset \( D \) of \( V(G) \) is said to be Split Majority Dominating set[5] of \( G \) if \( |N[D]| \geq \left\lceil \frac{q}{2} \right\rceil \) and the induced subgraph \( \langle V - D \rangle \) is disconnected. As usual, the minimum cardinality of minimal split majority dominating set is called split majority domination number of a graph denoted by \( \gamma_{sm}(G) \). This parameter was defined and studied by Joseline Manora and Veeramanikandan.

A partition \( \Delta \) of its vertex set \( V(G) \) is called a domatic partition of \( G \) if each class of \( \Delta \) is a dominating set in \( G \). The maximum number of classes of a domatic partition of \( G \) is called the domatic number of \( G \) and is denoted by \( d(G) \). The domatic number was introduced by Cockayne and Hedetniemi. In a similar fashion, a majority domatic partition of a graph \( G \) was introduced and each class of it is a dominating set in \( G \). The maximum number of classes of a majority domatic partition of \( G \) is called the majority domatic number [4] and is denoted by \( d_{M}(G) \). This parameter was introduced by Swaminathan and Joseline Manora.

2 Split Majority Domatic Number of a Graph
In this section, we define Split Majority Domatic Number of a graph \( G \) and this number \( d_{sm}(G) \) is determined for some families of graphs.

Definition 2.1
Let \( D \) be the family of all disjoint minimal split majority dominating sets of \( G \). The split majority domatic number of a graph \( G \) is defined to be the maximum number of disjoint minimal split majority dominating sets of \( G \) and is denoted by \( d_{sm}(G) \).

Remark 2.2
In this article, we consider only the family of disjoint minimal split majority dominating sets of \( G \) rather than the partition of vertices of \( G \). The reason is that there are some vertices that are not the elements of any minimal split majority dominating set \( D \) of \( G \) since the definition of split majority dominating set is violated when these vertices are included in any set \( D \).
2.1 $d_{sm}(G)$ for some families of graphs

1. For $G = K_p$, a totally disconnected graph.
\[
d_{sm}(K_p) = \begin{cases} 
1 & \text{if } p \text{ is odd} \\
2 & \text{if } p \text{ is even} 
\end{cases}
\]

2. If $G$ is a star $K_{1,p-1}$, $p \geq 3$. Then $d_{sm}(G) = 1$.

3. Suppose $G$ being double star $D_{r,s}$, $p = r + s + 2$. Then $d_{sm}(G) = 2$.

4. For $G = \zeta_p$, $p \geq 3$. Then $d_{sm}(G) = 3$.

5. If $G$ is a complete bipartite graph $K_{m,n}$, $m \leq n$, $d_{sm}(G) = 2$.

6. Let $G$ be a Petersen graph. Then $d_{sm}(G) = 2$.

7. Suppose $G$ is a fan $F_p$, $p \geq 4$. Then $d_{sm}(G) = 1$.

3 Main Results on $d_{sm}(G)$

**Theorem 3.1**

If $G$ has a full degree vertex, $d_{sm}(G) = 1$.

**Proof**

Suppose $G$ has a full degree vertex $v$. If $v$ is a cut vertex then $D = \{v\}$ is a split majority dominating set of $G$. Assume that there exists another split majority dominating set $S$ of $G$. Then $S$ must contain $v$. If not, $(V - S)$ is connected, a contradiction. Therefore $d_{sm}(G) = 1$. If $v$ is not a cut vertex, and $v$ is in every split majority dominating set of $G$. Applying the same argument as above, we get a contradiction. Therefore $d_{sm}(G) = 1$.

**Theorem 3.2**

If every vertex of a graph is such that $d(v) > \left\lceil \frac{p}{2} \right\rceil$ then $d_{sm}(G) = 1$.

**Proof**

Suppose $\delta(G) > \left\lceil \frac{p}{2} \right\rceil$. Then every vertex is a majority dominating vertex. Let $D$ be a minimum split majority dominating set of $G$. Then $\gamma_{sm}(G) \geq \delta(G)$. That is $|D| > \left\lceil \frac{p}{2} \right\rceil$. This implies that $D$ contains at least one vertex more than $\left\lceil \frac{p}{2} \right\rceil$ vertices. Then $|V - D| < \left\lceil \frac{p}{2} \right\rceil$, implying that $V - D$ is not a majority dominating set. Therefore there exists only one split majority dominating set of $G$ and hence $d_{sm}(G) = 1$.

**Theorem 3.3**

For any graph $G$, $1 \leq d_{sm}(G) \leq \left\lceil \frac{p}{2} \right\rceil + 1$.

**Proof**

If $G$ has a full degree vertex then the lower bound is attained. When $\delta(G) = \left\lceil \frac{p}{2} \right\rceil$ then $d_{sm}(G) = 1$. Consider a minimally connected graph $G$, namely a tree $T$. If $T$ has exactly two end vertices then it is a path $P_p$. When $p \leq 6$, then every intermediate vertex is a split majority dominating set of $G$. Therefore $d_{sm}(G) = 4 \leq \left\lceil \frac{p}{2} \right\rceil + 1$. Suppose $p > 7$. Then $\gamma_{sm}(G) \geq 2$ but $d_{sm}(G) < \left\lceil \frac{p}{2} \right\rceil$ then only intermediate vertices constitute split majority dominating sets of $G$ and $d_{sm}(G) < \left\lceil \frac{p}{2} \right\rceil$. Thus $1 \leq d_{sm}(G) \leq \left\lceil \frac{p}{2} \right\rceil + 1$.

**Proposition 3.4**

If $G$ is any graph with $\text{diam}(G) = 2$ then $d_{sm}(G) = 1$.

**Proof**

Suppose $G$ is a graph with $\text{diam}(G) = 2$. Let $v$ be the center of the graph $G$. If $D$ is the minimal split majority dominating set of a graph $G$ and containing the vertex $v$ then no other minimal split majority dominating set is obtained without $v$. Therefore there exists only one split majority dominating set of $G$. Thus $d_{sm}(G) = 1$.

**Proposition 3.5**

For a tree $T$ with $\text{diam}(T) = 3$, $d_{sm}(G) = 2$.

**Proof**

Suppose $T$ is a tree with $\text{diam}(T) = 3$. Since every tree has at least two end vertices. If $\text{diam}(T) = 3$ then $T$ is a double star $D_{r,s}$ or $P_4$ or pendants adjacent to intermediate vertices. If $T$ is $D_{r,s}$ or $P_4$, $d_{sm}(G) = 2$. If pendants are adjacent to intermediate vertices, $d_{sm}(G) = 2$.

**Theorem 3.6**

Let $G = P_p$ be a path on $p$ vertices, $p > 4$, $d_{sm}(P_p) = \frac{p}{\gamma_{sm}(G)}$ if and only if $p = 8, 10, 15, 20, 25$.
Proof
Let $P_p = \{u_1, u_2, \ldots, u_p\}$ be a path on $p$ vertices and $\gamma_{\text{sm}}(P_p) = \left\lfloor \frac{p}{6} \right\rfloor$.

Suppose $p = 8, 10, 15, 20, 25$. Then $\gamma_{\text{sm}}(G) = 2, 2, 3, 4, 5$. It is clear that $\gamma_{\text{sm}}(G)$ divides $p$. When $p = 8$, $d_{\text{sm}}(P_8) = 4$. When $p = 6k + 2$, $\gamma_{\text{sm}}(G) = 4$ if $k = 1$. Hence $d_{\text{sm}}(P_8) = 4 = \frac{p}{\gamma_{\text{sm}}(G)}$. Therefore the split majority domatic partition of $P_k$ is $\{\{u_1, u_2\}, \{u_3, u_4\}, \{u_5, u_6\}, \{u_7, u_8\}\}$. 

Let $p = 10, 15, 20, 25$. Then $d_{\text{sm}}(P_p) = 5$.

When $p = 10$, (i.e.) $p = 6k + 4$, then $\frac{p}{\gamma_{\text{sm}}(G)} = 5$ if $k = 1$.

When $p = 15$, (i.e.) $p = 6k + 3$, then $\frac{p}{\gamma_{\text{sm}}(G)} = 5$ if $k = 2$.

When $p = 20$, (i.e.) $p = 6k + 2$, then $\frac{p}{\gamma_{\text{sm}}(G)} = 5$ if $k = 3$.

When $p = 25$, (i.e.) $p = 6k + 1$, then $\frac{p}{\gamma_{\text{sm}}(G)} = 5$ if $k = 4$.

Therefore the split majority domatic partition of $P_p$ are

$D_1 = \{u_1, u_2, \ldots, u_{(\gamma_{\text{sm}}(G) - 1)}\} \bigcup \{u_{(\gamma_{\text{sm}}(G) - 1)} + 1\}$

$D_2 = \{u_2, u_7, \ldots, u_{(\gamma_{\text{sm}}(G) - 1)}\} \bigcup \{u_{(\gamma_{\text{sm}}(G) - 1)} + 2\}$

$D_3 = \{u_3, u_8, \ldots, u_{(\gamma_{\text{sm}}(G) - 1)}\} \bigcup \{u_{(\gamma_{\text{sm}}(G) - 1)} + 3\}$

$D_4 = \{u_4, u_9, \ldots, u_{(\gamma_{\text{sm}}(G) - 1)}\} \bigcup \{u_{(\gamma_{\text{sm}}(G) - 1)} + 4\}$

$D_5 = \{u_5, u_{10}, \ldots, u_{(\gamma_{\text{sm}}(G) - 1)}\} \bigcup \{u_{(\gamma_{\text{sm}}(G) - 1)} + 5\}$

In all cases, $\frac{p}{\gamma_{\text{sm}}(G)} = 5 = d_{\text{sm}}(P_p)$ if $p = 8, 10, 15, 20, 25$.

Conversely let $d_{\text{sm}}(P_p) = \frac{p}{\gamma_{\text{sm}}(G)}$. Suppose $p \equiv 0 \pmod{6}$. Then $d_{\text{sm}}(P_p) = 5$. But $d_{\text{sm}}(P_p) = \frac{p}{\gamma_{\text{sm}}(G)}$ implies that $d_{\text{sm}}(P_p) = 6$ which is a contradiction. Hence $p \not\equiv 0 \pmod{6}$.

Suppose $p \equiv 1, 2, 3, 4, 5 \pmod{6}$. Let $p = 6k + 1, 1 \leq l \leq 5$. Then $\gamma_{\text{sm}}(G) = \left\lfloor \frac{p}{6} \right\rfloor = k + 1$ and $\frac{p}{\gamma_{\text{sm}}(G)} = \frac{6k + 1}{k + 1} = m$ (say), $m \neq 0$. It implies that $k = \frac{m - 1}{6 - m}$.

Take $l = 1$. Then $m = 2, 3, 4, 5$. $k = \frac{m - 1}{6 - m}$. Then

$$k = \begin{cases} 
\frac{1}{4} & \text{if } m = 2 \\
\frac{2}{3} & \text{if } m = 3 \\
\frac{1}{2} & \text{if } m = 4 \\
\frac{3}{2} & \text{if } m = 5.
\end{cases}$$

Hence $k = 4$ is an integer if $l = 1$. Therefore for $k = 4$ and $l = 1$ implies $p = 6k + 1 = 25$. In a similar way, take $l = 2$. Then $m = 3, 4, 5$. $k = \frac{m - 1}{6 - m} = 1$ is an integer if $m = 4$ and $k = 3$ if $m = 5$. Therefore for $k = 1$ and $l = 2$ implies $p = 6k + 1 = 8$ and for $k = 3$ and $l = 2$ implies $p = 6k + 1 = 20$.

Take $l = 3$. Then $m = 4, 5$. $k = \frac{m - 1}{6 - m} = 2$ is an integer if $m = 5$. For $k = 2$ and $l = 3$ implies $p = 6k + 1 = 15$.

Take $l = 4$. Then $m = 5$. $k = \frac{m - 1}{6 - m} = 1$ is an integer if $m = 5$. For $k = 1$ and $l = 4$ implies $p = 6k + 1 = 10$.

Take $l = 5$. Then $m = 5$. Then there is no integer value for $k$. Hence, $p = 8, 10, 15, 20, 25$ if $d_{\text{sm}}(P_p) = \frac{p}{\gamma_{\text{sm}}(G)}$.

Theorem 3.7
Let $G = C_p$ be a cycle on $p$ vertices, $p > 4$. Then $d_{\text{sm}}(C_p) = \frac{p}{\gamma_{\text{sm}}(G)}$ if and only if $p = 8, 10, 15, 20, 25$, or $p \equiv 0 \pmod{6}$.
Proof

Let $C_p = \{u_1, u_2, \ldots, u_p\}$ be a cycle on $p$ vertices. Then $\gamma_{sm}(C_p) = \left\lceil \frac{p}{6} \right\rceil$. Suppose $p = 8, 10, 15, 20, 25$, then $\gamma_{sm}(G) = 2, 3, 4, 5$ and suppose $p \equiv 0 \pmod{6}$, then $\gamma_{sm}(C_p) = \frac{6k}{6} = k$. It is clear that $\gamma_{sm}(G)$ divides $p$. When $p = 8$, $d_{sm}(P_8) = 4$. When $p = 6k + 2$, then $\frac{p}{\gamma_{sm}(G)} = 4$.

If $k = 1$. Hence $d_{sm}(C_8) = 4 = \frac{p}{\gamma_{sm}(G)}$. Therefore a split majority domatic partition of $C_8$ is

$\{\{u_1, u_2\}, \{u_3, u_6\}, \{u_4, u_5\}, \{u_7, u_8\}\}.$

Let $p = 10, 15, 20, 25$. Then $d_{sm}(C_p) = 5$.

When $p = 10$ (i.e.) $p = 6k + 4$, then $\frac{p}{\gamma_{sm}(G)} = 5$ if $k = 1$.

When $p = 15$, (i.e.) $p = 6k + 3$, then $\frac{p}{\gamma_{sm}(G)} = 5$ if $k = 2$.

When $p = 20$, (i.e.) $p = 6k + 2$, then $\frac{p}{\gamma_{sm}(G)} = 5$ if $k = 3$.

When $p = 25$, (i.e.) $p = 6k + 1$, then $\frac{p}{\gamma_{sm}(G)} = 5$ if $k = 4$. Therefore the split majority domatic partitions of $V(C_p)$ are

$D_1 = \left\{u_1, u_6, \ldots, u_{\frac{p}{\gamma_{sm}(G)} - 1}\right\}$, $D_2 = \left\{u_2, u_7, \ldots, u_{\frac{p}{\gamma_{sm}(G)} - 1}\right\}$

$D_3 = \left\{u_3, u_8, \ldots, u_{\frac{p}{\gamma_{sm}(G)} - 1}\right\}$, $D_4 = \left\{u_4, u_9, \ldots, u_{\frac{p}{\gamma_{sm}(G)} - 1}\right\}$

$D_5 = \left\{u_5, u_{10}, \ldots, u_{\frac{p}{\gamma_{sm}(G)} - 1}\right\}$.

In all cases, $\frac{p}{\gamma_{sm}(G)} = 5 = d_{sm}(C_p)$ if $p = 8, 10, 15, 20, 25$. Let $p = 6k$. Then $\frac{p}{\gamma_{sm}(G)} = 6$, since $\gamma_{sm}(G) = \frac{p}{6} = k$. Therefore $\{D_1, D_2, D_3, D_4, D_5, D_6\}$ are the split majority domatic partitions of $V(G)$ and hence $\gamma_{sm}(G) = 6 = d_{sm}(C_p)$ if $p \equiv 0 \pmod{6}$.

Conversely, let $d_{sm}(C_p) = 5$, then $p = \gamma_{sm}(G)$. Therefore $p = d_{sm}(C_p) \left\lceil \frac{p}{6} \right\rceil$ (i.e.) $\frac{p}{6}$ divides $p$. If $p \equiv 0 \pmod{6}$, then $p = 6k$ and $\left\lceil \frac{p}{6} \right\rceil = k$. Thus $\frac{p}{6}$ divides $p$. Suppose $p = 6k + 1$, $i \leq l \leq 5$. Applying the same argument in the converse part of the theorem, we obtain the values as $p = 8, 10, 15, 20, 25$.

Next, We discuss the split majority domatic number for complement of a graph $G$ and Nordhaus-Gauddum type results.

Proposition 3.8

If $G$ has a full degree vertex and all other vertices are of degree less than $\left\lceil \frac{p}{2} \right\rceil$, then $d_{sm}(\overline{G}) = p - 1$.

Proof

Suppose $G$ has a full degree vertex and all other vertices are of degree less than $\left\lceil \frac{p}{2} \right\rceil$. Then $G$ has an isolate and all other vertices are of degree greater than or equal to $\left\lceil \frac{p - 2}{2} \right\rceil$.

Then every vertex except the isolate constitutes a majority dominating set of $\overline{G}$. Since $\overline{G}$ has an isolate $v$, every majority dominating set of $\overline{G}$ is split majority dominating set of $\overline{G}$. Thus $d_{sm} = p - 1$.

Theorem 3.9

For any graph $G$, $d_{sm}(G) + d_{sm}(\overline{G}) \leq p + 2$ and $d_{sm}(G), d_{sm}(\overline{G}) \leq 2p$.

Proof

If $G$ has a full degree vertex $v$, then $d_{sm}(G) = 1$ and $\overline{G}$ has an isolate $v$. Suppose $\delta(G) \geq \left\lceil \frac{p}{2} \right\rceil - 1$. Then there exists atleast one vertex $v$ in $\overline{G}$ such that $d(v) < \left\lceil \frac{p}{2} \right\rceil - 1$. In this case, there exists a minimal split majority dominating set of $\overline{G}$ with cardinality greater than or equal to two. Therefore $d_{sm}(\overline{G}) \leq p - 1$ and $d_{sm}(G) + d_{sm}(\overline{G}) \leq p$. Suppose $G$ is a complete bipartite graph with $m = n$. Then $\overline{G}$ has two components and each vertex $v$ of $\overline{G}$ constitutes a split majority dominating set of $\overline{G}$. Therefore $d_{sm}(\overline{G}) = p$ and $d_{sm}(G) = 2$. In this case, $d_{sm}(\overline{G}) + d_{sm}(G) \leq p + 2$. We prove the another result in the similar fashion.
References