Boundary Digraph and Boundary Neighbour Digraph of a graph G

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ABSTRACT

A digraph D is a pair (V, A), where V is a non-empty set whose elements are the vertices and A is the subset of the set of ordered pairs of distinct elements of V. The elements of A are called the arcs of D. A vertex v is a boundary vertex of u if d(u, v) ≤ d(u, w) for all w ∈ N(v). The boundary digraph BD(G) of a graph(digraph) G is the digraph that has the same vertex set as G and an arc from u to v exists in BD(G) if and only if v is a boundary vertex of u in G. The boundary neighbor digraph BND(G) of a graph G is the graph that has the same vertex set as G and a directed edge (arc) from u to v exists in BND(G) if and only if v is a boundary neighbor of u in G.

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A Spider is a tree with at most one vertex of degree more than two, called the center of Spider. A leg of a Spider is a path from the center to a vertex of degree one. Thus, a star with $k$ legs is a Spider of $k$ legs, each of length 1.

For disjoint graphs $G$ and $H$, the join $G + H$ has $V(G) \cup V(H)$ vertices and $E(G+H) = E(G) \cup E(H) \cup \{uv: u \in V(G) \text{ and } v \in V(H)\}$.

The eccentric digraph[5] of a digraph $G$, denoted by $ED(G)$, is the digraph with the same vertex set as $G$ with an arc from vertex $u$ to vertex $v$ in $ED(G)$ if and only if $v$ is an eccentric vertex of $u$ in $G$.

Given a positive integer $k \geq 2$, the $k^{th}$ iterated eccentric digraph of $G$ [8] is written as $ED_k(G) = ED(ED_{k-1}(G))$ where $ED_0(G) = G$.

Let $G$ be a graph. A vertex $v$ is a boundary vertex [4] of $u$ if $d(u, w) \leq d(u, v)$ for all $w \in N(v)$. A vertex $v$ is called a boundary neighbor of $u$ if $v$ is a nearest boundary of $u$.

We introduced the boundary digraph $BD(G)$ [1] of a graph $G$ and boundary digraph $BD(G)$ of a graph $G$. Let $G$ be a simple $(p, q)$ graph with vertex set $V(G)$ and edge set $E(G)$. The boundary graph of a graph $G$, denoted by $G_b(G)$ has the same set of vertices as $G$ with two vertices $u$ and $v$ being adjacent in $G_b(G)$ if and only if either $v$ is a boundary vertex of $u$ in $G$ or $u$ is a boundary vertex of $v$ in $G$.

We need the following results to study the boundary digraph of a graph $G$.

Theorem: 1.1 [9] A digraph $D$ is unilateral if and only if it has a spanning arc sequence.

Theorem: 1.2 [9] A digraph is strong if and only if it has a spanning closed arc sequence.

2. Boundary Digraph and Boundary Neighbor Digraph of a graph $G$:

In [1], we have defined the boundary digraph $BD(G)$ of a graph $G$ as follows:
The boundary digraph $BD(G)$ of a graph (digraph) $G$ is the digraph that has the same vertex set as $G$ and a directed edge (arc) from $u$ to $v$ exists in $BD(G)$ if and only if $v$ is a boundary vertex of $u$ in $G$.

Example: 2.1

$G:$

Fig. 2.1

Boundary digraph

A digraph $G$ is said to be boundary digraph if there exists a graph $H$ such that $G = BD(H)$.

Example: 2.2

$G:$

$H:$

Fig. 2.2

Here $G$ is a boundary digraph, since $BD(H) \cong G$.

Now, we define new graphs known as boundary neighbor graph and boundary neighbor digraph of a graph $G$ as follows:
The boundary neighbor graph $BN(G)$ of a graph $G$ is the graph that has the same vertex set as $G$ with two vertices $u$ and $v$ being adjacent in $BN(G)$ if and only if either $v$ is a boundary neighbor of $u$ in $G$ or $u$ is a boundary neighbor of $v$ in $G$.

The boundary neighbor digraph $BND(G)$ of a graph $G$ is the graph that has the same vertex set as $G$ and a directed edge (arc) from $u$ to $v$ exists in $BND(G)$ if and only if $v$ is a boundary neighbor of $u$ in $G$.

Example: 2.3

$G:$

$BN(G):$

Fig. 2.3

Observation: 2.1

(i) Since every vertex in a connected graph has at least one boundary vertex, in a boundary digraph, every vertex will have out degree at least one.

(ii) Odd cycles are a class of graphs for which $BD(G) \cong G$.

(iii) Two isomorphic graphs have their boundary digraph isomorphic but the converse need not be true always.

Example: 2.4

$G:$

$G_1:$

$G_2:$

Fig. 2.4

$G, G_2$ are non isomorphic graphs, but $BD(G_1) = BD(G_2)$.

(iv) All the eccentric vertices of a connected graph $G$ are boundary vertices also, but not conversely. Hence, always eccentric digraph of a connected graph $G$ is a spanning subgraph of the boundary digraph of $G$.

(v) If the eccentric vertices are the only boundary vertices, then $ED(G) = BD(G)$.

(vi) $K_{1,n}$ is not a boundary digraph of any other graph.

(vii) If $G$ is disconnected, eccentric vertices are not boundary vertices.

(viii) If $G$ is the complete graph $K_n$, then $BD(G) = G^*$.
(x) A non-trivial boundary digraph has no vertex of out degree zero.

(xi) If boundary vertices are at least two in \( G \), then \( d'(v) > 0 \) for at least two vertices in \( B(G) \).

(xii) There exists at least one bidirectional edge in \( B(G) \).

(xiii) \( B(G) \) is a sub graph of \( B(G) \) and \( B(N(G)) \) is a sub graph of \( G_n \).

(xiv) \( B(G) = B(D(G)) \) if and only if there exists only one boundary vertex for each vertex in \( G \).

(xv) \( B(G) \) may be connected or disconnected.

(xvi) If \( P = v_1, v_2, \ldots, v_n \) is a diametral path in \( G \), then \( v_1 \) and \( v_n \) are boundary neighbors for some vertices of \( G \). Thus \( G \) has at least two boundary neighbors.

**Theorem 2.1**

If \( G = K_n \), then \( B(G) = B(D(G)) = K_n \).

Proof

When \( G = K_n \). Any vertex \( u \in V(G) \) is a boundary vertex of \( G \). Hence, \( B(K_n) = K_n \) and \( B(D(G)) = K_n \).

**Theorem 2.2**

Let \( W_n \), \( n \geq 4 \) be a wheel graph. Then the underlying graph of \( BD(W_n) \) is \( K_n + C_n \). Also, \( BD(W_n) = B(D(W_n)) \).

Proof

Let \( W_n \), \( n \geq 4 \) be a wheel graph with \( V(G) = \{ v_1, v_2, \ldots, v_n \} \). Let \( v \) be a central vertex of \( G \). Then, \( v \) is a boundary vertex of \( G \). Thus, \( d'(v) = 2 \) in \( BD(G) \). Hence, \( B(G) = C_n \).

**Theorem 2.3**

If \( G = C_n \), then \( BD(G) = B(D(G)) = \begin{cases} C_n \star & \text{if } n \text{ is odd} \\ \frac{n}{2} K_2 \star & \text{if } n \text{ is even} \end{cases} \)

Proof

Let \( G \) be a cycle with \( n \) vertices.

Case (i) \( n \) is odd (\( n = 2r + 1 \), where \( r \) is the radius of \( C_n \)).

Let \( v_1, v_2, \ldots, v_{2r+1}, \ldots, v_n \) be the boundary vertices of \( G \). Every vertex \( v_i \) of \( C_n \) has two boundary vertices. Thus, the out-degree of every vertex in \( B(G) \) is two. Each vertex of \( C_n \) is a boundary vertex of two vertices in \( G \). Thus, in-degree of every vertex \( v_i \) of \( B(G) \) is two in \( B(D(G)) \). Thus, we get a directed cycle \( C_n \). Hence, \( B(G) = C_n \).

Case (ii) \( n \) is even (\( n = 2r \), where \( r \) is the radius of \( C_n \)).

Every vertex of \( C_n \) has exactly one boundary vertex. Also, if \( u \) is a boundary vertex of \( v \) then \( v \) is a boundary vertex of \( u \). Therefore, we get \( BD(G) = (n/2)K_2 \star \).

**Theorem 2.4**

If \( G = P_n \), then \( BD(G) \) has \((n-2)\) transmitters.

Proof

Let \( v_1, v_2, \ldots, v_n \) be the vertices of \( P_n \). The vertices \( v_1 \) and \( v_n \) are the pendant vertices. The non-pendant vertices of \( P_n \) have two boundary vertices \( v_1 \) and \( v_n \) and the non-pendant vertices are not boundary vertices. Thus, the \((n-2)\) non-pendant vertices have indegree zero and outdegree 2. Hence, \( BD(G) \) has \((n-2)\) transmitters.

**Theorem 2.5**

Let \( G \) be a graph \( K_{2n}-M \), where \( M \) is a 1-factor, then \( BD(G) = nK_2 \star \).

Proof

Let \( G = K_{2n}-M \), where \( M \) is a 1-factor. \( G \) is a unique eccentric point graph. Thus in this graph, eccentric vertices are as same as the boundary vertices. Then \( BD(G) = nK_2 \star \).

**Theorem 2.6**

Let \( G \) be a graph \( K_{m,n} \), then \( BD(G) = K_m \star \cup K_n \star \).

Proof

Let \( G \) be a graph \( K_{m,n} \) with vertex set \( V(G) = V_1 \cup V_2 \). \( |V_i(G)|=m \) vertices form a complete digraph, since every vertex \( u \in V_1(G) \) have their boundary vertices in \( V_1(G) \) and vice versa. By the definition of \( BD(G) \), we have two components \( K_m \) and \( K_n \) of \( BD(G) \).

**Corollary 2.6**

The boundary digraph of a complete multipartite digraph is a disjoint union of complete digraphs more precisely, \( BD(K_{n_1, n_2, \ldots, n_k}) = K_{n_1} \cup K_{n_2} \cup \ldots \cup K_{n_k} \).

**Theorem 2.7**

(i) If \( G = C_n \), then \( BD(G) = C_n \star \) and \( BD(G) \) is symmetric.

(ii) If \( G = P_n \), then \( BD(G) = P_n \star \) and \( BD(G) \) is symmetric.

Proof

(i) Let \( G = C_n \). In \( C_n \), every vertex \( v_i \in V(BD(G)) \) has two boundary vertices \( v_{i-1}, v_{i+1} \), \( d'(v_i) = 2 \) in \( BD(G) \). Also, every vertex \( v \) is a boundary vertex of exactly two vertices of \( G \). Therefore, \( d'(v_i) = 2 \) in \( BD(G) \). Hence, \( BD(G) = C_n \star \).

(ii) Let \( G = P_n \) with \( V(G) = \{ v_1, v_2, v_3, \ldots, v_n \} \). The non-adjacent vertices of \( v_i \) in \( G \) are the boundary vertices of \( v_i \) in \( BD(G) \). In \( BD(G) \), for every vertex \( v_i \), \( d'(v_i) = d'(v_{i+1}) = 2 \) for \( i = 2, 3, \ldots, n-1 \) and \( d'(v_1) = d'(v_n) = 2 \) for \( i = 1, n \). Hence, \( BD(G) = P_n \star \).

**Remark 2.1**

(i) When \( n \) is even, \( BD(C_n) = (n/2)K_2 \star \) and \( BD(C_n) = C_n \star \).

(ii) When \( n \) is odd, \( BD(C_n) = BD(C_n) = C_n \star \).

**Theorem 2.8**

If \( G = S_{m,n} \), then \( BD(G) \) has \((m+n)\) vertices and \( m+n \) vertices of \( BD(G) \) form a complete symmetric sub graph, where \( S_{m,n} = K_m + K_1 + K_1 + K_n \).

Proof

Let \( G \) be a bistar \( S_{m,n} \) with \( m+n \) pendant vertices, \( u \) and \( v \) are the central vertices. The \( m+n \) pendant vertices form a complete symmetric sub graph in \( BD(G) \) and the central vertices \( u \) and \( v \) have out degree \( m+n \) in \( BD(G) \) and \( d'(v) = d'(u) = 0 \). Hence, \( u \) and \( v \) are transmitters in \( BD(G) \).

**Theorem 2.9**

If \( G = K_1, n \), then \( BD(K_1, n) \) is unilateral.

Proof

Consider \( G = K_1, n \) with vertex set \( V(G) = \{ v, v_1, v_2, \ldots, v_n \} \). Let \( v \) be a central vertex of \( G \) and \( v \) has \( n \) boundary vertices in \( G \). Thus, the out-degree of a vertex \( v \) is \( n \) and in-degree is zero in \( BD(G) \). The pendant vertices of \( G \) form a complete digraph in \( BD(G) \). Thus, \( v, v_1, v_2, \ldots, v_n \) is a spanning arc sequence in \( BD(G) \). By Theorem 1.1, \( BD(G) \) is unilateral. The underlying graph of \( BD(K_1, n) \) is \( K_1 + K_n = K_{n+1} \).

**Theorem 2.10**

A digraph \( BD(G) \) is strong if \( G = C_n \) or \( G = C_n \), \( n \) is odd.
from u to the last vertex of Q and an arc sequence from that vertex to v. By Theorem 1.2 BD(G) is strong.

(ii) Assume G = \( \overrightarrow{C_n} \) by theorem 2.7, BD(\( \overrightarrow{C_n} \)) = \( C_n^* \). The proof is similar to proof of case(i).

**Theorem: 2.11**

If G is a graph with radius 1, then radius of BD(G) is one.

**Proof**

Let G be a graph with radius one. The center vertex u has p−1 boundary vertices. Thus the out degree of u is p−1 in BD(G). Hence, the eccentricity of u is one in BD(G).

**Theorem: 2.12**

If a connected graph G has a pendant vertex, then in degree of that vertex is p−1 in BD(G).

**Proof**

Let G be a connected graph with pendant vertex u. u is a boundary vertex of all other vertices of G. By the definition of BD(G), in degree of u is p−1.

**Theorem: 2.13**

If T is a tree with m pendant vertices, then BD(T) has p−m transmitters and m vertices having in-degree p−1 and out-degree m−1.

**Proof**

Let T be a tree with m pendant vertices. These pendant vertices are only the boundary vertices. Thus the pendant vertices are boundary vertex of other vertices of T. Hence, the in-degree of pendant vertex of T is p−1 and the out-degree of pendant vertex is m−1 in BD(T). A non pendant vertex of T has in degree zero and out degree m > 0 in BD(T). Therefore, BD(T) has p−m transmitters.

**Theorem: 2.14**

If \( G = C_{2n+1} \), then BD(G) is Hamiltonian and Eulerian.

**Proof**

Let \( G = C_{2n+1} \). By Theorem 2.10, BD(G) has a spanning closed arc sequence. Thus, BD(G) is Hamiltonian. By Theorem 2.3, in BD(G), \( d^- (v_i) = d^+ (v_i) = 2 \) for all \( v_i \in V(BD(G)) \), i = 1, 2, 3, ..., 2n+1. Thus, BD(G) is Eulerian.

**Theorem: 2.15**

Let \( G = H^* \), then BD(G) has n transmitters and n vertices form a complete sub graph.

**Proof**

Consider \( G = H^* \). \( |V(H)| = n \) and \( |V(G)| = 2n \). By the definition of boundary digraph, the non pendant vertices of G are transmitters. The pendant vertices of G form a complete symmetric sub graph on n vertices.

**Theorem: 2.16**

Let G be a graph with r(G) > 1. If G has a pendant vertex, then ED(G) ≠ BD(G).

**Proof**

Suppose \( v \) is a pendant vertex of G and u is its support vertex. Then there is an arc from u to v in BD(G), but it is not in ED(G). Hence, ED(G) ≠ BD(G).

**Theorem: 2.17**

If there is a triangle with a vertex of degree two or there exist a clique in G with r vertices, having a vertex of degree r−1, then ED(G) ≠ BD(G).

**Proof**

Let G be a connected graph. Consider G has a triangle with a vertex of degree two. Let u, v, w form a triangle with \( d(w) = 2 \). Then w is a boundary vertex of u and v. But w is not an eccentric vertex of u and v and hence ED(G) ≠ BD(G).

Similarly, if there exist a clique in G with r vertices and having a vertex of degree r−1 in that clique, then BD(G) ≠ ED(G).

**Remark: 2.2**

The converse of the theorem 2.16 is not true. For example,

**Fig: 2.5**

Here ED(G) ≠ BD(G). But G has no pendant vertex.

**Theorem: 2.18**

If G is a self-centered unique eccentric point graph, then BD(G) = (n/2)K_{2}\*.

**Proof**

Assume G is a self-centered unique eccentric point graph with n vertices. Every vertex is an eccentric vertex in G. Each vertex has exactly only one boundary vertex which is also the eccentric vertex. Hence, BD(G) = (p/2)K_{2}\* and its underlying graph is \( G_b(G) = (p/2)K_2 \).

**Corollary 2.18:**

If G is a Hypercube graph \( Q_n \), then BD(G) = 2^{n-1}K_{2}\*.

**Proof**

Let G be a Hypercube graph \( Q_n \). \( Q_n \) is a self centered unique eccentric point graph. In this graph, the boundary vertices are the eccentric vertices. By Theorem 2.18, BD(G) = ED(G) = (2^{n/2})K_{2}\* = 2^{n-1}K_{2}\*.

**Theorem: 2.19**

If G is a Cocktail party graph \( H_{m,n} \), then BD(G) = nK_{2}\*.

**Proof**

Let G be a Cocktail party graph \( H_{m,n} \). The boundary vertices are the eccentric vertices. BD(G) = ED(G) = nK_{2}\*.

**Theorem: 2.20**

If G is a disconnected graph with k components, then

(i) BD(G) is also a disconnected digraph.

(ii) ED(G) is connected and ED(G) is a k-partite graph.

**Proof:**

(i) Let G be a disconnected graph with two components \( G_1 \) and \( G_2 \). Here the boundary vertices of \( G_1 \) are in \( G_1 \) and vice versa. Thus BD(G_1) and BD(G_2) are the components of BD(G). Hence BD(G) is a disconnected graph with two components. Similarly if G has k components, then BD(G) also has k components.

(ii) Consider a connected graph G has k components. Let \( G_1, G_2, ..., G_k \) be the components of G. For each vertex \( v \in V(G_i) \), e(v) = V(G_i) \cup V(G_{i+1}) \cup ... \cup V(G_{k-1}) \cup V(G_k). \) Thus all the vertices of \( G_i \) are adjacent to all the vertices of other components of G in ED(G). There is no arcs between any two vertices of \( G_i \) in ED(G). Therefore, ED(G) is a k-partite graph.
Theorem: 2.21
For a path \( P_m \),
(i) \( BN(P_m) = \overline{K}_{n-1} + K_1 + K_1 + \overline{K}_{n-1} \) if \( m \) is even and \( m = 2n \).
(ii) \( BN(P_m) = G' \) if \( n \) is odd and \( m = 2n+1 \), where \( G' \) is given in figure 2.6.

\[
G': \quad \begin{array}{c}
\text{Fig: 2.6}
\end{array}
\]

BND(G) is a digraph whose underlying graph is \( BN(P_m) \).

Proof
Let \( G = P_m \) be a path with \( n \) vertices \( v_1, v_2, v_3, \ldots, v_n \).

Case(i): \( m \) is even (\( m = 2n \))

\[ V(G) = \{ v_1, v_2, v_3, \ldots, v_n \} \]

\[ \text{V(G)} = \{ v_1, v_2, v_3, \ldots, v_n \} \text{, } v_1 \text{ and } v_2 \text{ are the boundary vertices of } P_n. \]

V(G) is a boundary neighbour of \( v_2, v_3, \ldots, v_n \).

Case(ii): \( m \) is odd (\( m = 2n+1 \))

\[ V(G) = \{ v_1, v_2, v_3, \ldots, v_n \} \]

v1 and v2 are the boundary vertices of \( P_n \).

Thus, \( BN(G) = K_{n-1} + K_1 + K_1 + \overline{K}_{n-1} \).

Remark: 2.3
(i) A symmetric path \( P \) of length \( n \) is an induced sub graph of \( BN(G) \).
(ii) \( BN(G) \) has \( n(m-1) \) transmitters.

Theorem: 2.23
Let \( G \) be a connected graph with \( m \) vertices. Then \( BN(G_{Kn}) = mK_{n+1} \) and \( BN(G_{Kn}) = mK_{n+1}^* \).

Proof
Consider a connected graph \( G \) with vertex set \( |V(G)| = m \).

If \( G \) is a spider with \( m+1 \) vertices, then \( BN(G) = (K_{n+1}) - u \), where \( u \) is a vertex which is attached with \( u \) in \( K_{n+1} - K_1 \), where \( u \) is the central vertex of \( G \).

Corollary: 2.24
For any connected graph \( G \),
(i) \( BN(G_{mK}) = P_m + K_1 \).

(ii) n symmetric arc sequence with length \( m \) in \( BN(G_{mK}) \).

Theorem: 2.25
If \( G \) is a spider with \( 2n+1 \) vertices, then \( BN(G) = (K_{n+1}) - K \), where \( u \) is a vertex which is attached with \( u \) in \( K_{n+1} - K_1 \), where \( u \) is the central vertex of \( G \).

Corollary: 2.26
If \( G \) is a spider with \( km+1 \) vertices, then \( BN(G) = (K_{km+1}) - K \), where \( u \) is a vertex which is attached with \( u \) in \( K_{km+1} - K_1 \), where \( u \) is the central vertex of \( G \).

Theorem: 2.27
If \( G \) is a spider with \( 2n+1 \) vertices, then \( BN(G) = (K_{n+1}) - u \), where \( u \) is a vertex which is attached with \( u \) in \( K_{n+1} - K_1 \), where \( u \) is the central vertex of \( G \).
Proof
When $G$ is a wounded spider having one wounded leg $e = uv$, let $u$ be a central vertex. By the definition of $BN(G)$, vertex $v$ is a boundary neighbor of the pendant vertices of $G$ and the central vertex $u$. The boundary vertex of other support vertices is their corresponding pendant vertices in $G$. Thus the support vertex of $G$ other than $u$ become a pendant vertex in $BN(G)$ and the pendant vertex $v$ of wounded leg $e$ in $G$ become a central vertex in $BN(G)$. The central vertex of $G$ is adjacent to $v$ only in $BN(G)$. Hence, $BN(G)$ is a wounded spider.

Corollary 2.27
If $G$ is a wounded spider, then $BN(G)$ is a graph of radius 2 and diameter 4.

Proof
The proof follows from the Theorem 2.27.

Remark 2.4
If $G$ is a wounded spider with $m$ wounded legs and $n$ non wounded legs, then
(i) $K_{m,n}$ is an induced sub graph of $BN(G)$.
(ii) $K_{1,m}$ is an induced sub graph of $BN(G)$.
(iii) $BN(G)$ has $n$ pendant vertices.

Theorem 2.28
If G is a wounded spider with $n$ non wounded legs, then $BND(G)$ has $n+1$ transmitters.

Proof
Assume $G$ has $m$ wounded legs and $n$ non wounded legs. The central vertex of $G$ and $m$ pendant vertices have $n$ boundary neighbors. The support vertices of $G$ has only one boundary neighbor. The support vertex $v_i$ and central vertex $u$ of $G$ are not a boundary neighbor of any other vertex. Therefore, $d^+(v_i) = d^+(u) = 0$ and $d^+(v_i) > 0$, $d^+(u) > 0$. This implies, $BND(G)$ has $n+1$ transmitters.

Iterated Boundary Digraph
The Boundary Digraph $BD(G)$ of a digraph $G$ is the digraph that has the same vertex set as $G$ and the arc set defined as follows there is an arc from $u$ to $v$ if and only if $v$ is a boundary vertex of $u$.

Corollary 2.29
For every digraph $G$ there exist smallest integer numbers $p > 0$ and $t \geq 0$ such that $BD^3(G) \cong BD^{p+t}(G)$, where $\cong$ denotes graph isomorphism.

Iterated Boundary Neighbor Digraph
The Boundary Neighbor Digraph $BND(G)$ of a digraph $G$ is the digraph that has the same vertex set as $G$ and the arc set defined as follows: there is an arc from $u$ to $v$ if and only if $v$ is a boundary neighbor of $u$.

An example of a graph and its boundary neighbor digraph is given in Example 2.3. Note that arcs of graphs are drawn as directed edges with arrows.

Given a positive integer $k \geq 2$, the $k^{th}$ iterated boundary neighbor digraph of $G$ is written as $BND^k(G) = BND(BND^{k-1}(G))$ where $BND^0(G) = G$. The following example illustrates these definitions showing graph $G$ and its iterated boundary digraphs $BD(G)$, $BD^2(G)$, $BD^3(G)$ and $BD^4(G)$. Note that in this case, $BD^3(G) = BD^4(G)$.

An interesting line of investigation concerns the iterated sequence of boundary digraphs. For every digraph $G$ there exist smallest integer numbers $p > 0$ and $t \geq 0$ such that $BD^i(G) \cong BD^{p+t}(G)$, where $\cong$ denotes graph isomorphism.

Conclusion
In this paper, some properties of boundary digraph of a graph $G$, boundary neighbor digraph of a graph $G$ and Boundary neighbor digraph of a graph $G$ are discussed. Iterated boundary digraph and Iterated boundary neighbor digraph are studied.

References


