On $T$ - curvature tensor in Trans-Sasakian manifolds
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**ABSTRACT**

In this chapter, quasi $T$ - flat, the $\xi - T$ - flat, and $\varphi - T$ - flat trans-Sasakian manifolds are studied. It is proved that quasi $T$ - flat, the $\xi - T$ - flat, and the $\varphi - T$ - flat trans-Sasakian manifolds are $\eta$ - Einstein under the condition $\varphi(\text{grad}\alpha) = (2n - 1)(\text{grad}\beta)$.

**Keywords**

$T$ - curvature tensor, Trans-Sasakian manifold, $\eta$ - Einstein manifold.

### 1. Introduction

Oubina [12] studied a new class of almost contact Riemannian manifold known as trans-Sasakian manifold which generalizes both $\alpha$ -Sasakian and $\beta$ -Kenmotsu structure. A trans-Sasakian structure of type $(0,0),(\alpha,0)$ and $(0,\beta)$ are cosympletic [1], $\alpha$ -Sasakian [2,19] and $\beta$ -Kenmotsu [2, 10] respectively. Sasakian, $\alpha$ -Sasakian, Kenmotsu, $\beta$ -Kenmotsu are particular cases of trans-Sasakian manifold of type $(\alpha,\beta)$.

Szabo [20,21] has obtained some curvature identities and Nomizu [11] has studied some curvature properties. It is also known that a locally trans-Sasakian manifold of dimension $\geq 5$ is either cosympletic or $\alpha$ -Sasakian or $\beta$ -Kenmotsu manifold [9]. On other hand, 3 - dimensional proper trans-Sasakian manifold are constructed by Marrero [9]. De et al [5] etc. are also studied some important properties of trans-Sasakian manifold. In 2008, Tripathi and Dwivedi [22] defined and studied quasi projectively flat, $\xi$ - projectively flat and $\varphi$ - projectively flat almost contact metric manifold.

M.M. Tripathi and et al. [24] introduced the $T$ - curvature tensor which in particular cases reduces to known curvatures like conformal, concircular and projective curvature tensors and some recently introduced curvature tensors like $M$ - projective curvature tensor, $\tilde{W}$ - curvature tensor ($i = 0,...,9$) and $W$ - curvature tensors ($j = 0,1$).

M.M. Tripathi and P. Gupta [24] also found some important results on $T$ - curvature tensor in $K$ - contact and Sasakian manifolds. However, the quasi $T$ - flat, the $\xi - T$ - flat, and the $\varphi - T$ - flat trans-Sasakian manifolds are almost not discussed so far. Let $M$ be a $(2n + 1)$ - dimensional contact metric manifold. Since at each point $p \in M$, the tangent space $T_p(M)$ can be decomposed into the direct sum $T_p(M) = \varphi(T_p(M)) \oplus \{\xi_p\}$, where $\{\xi_p\}$ is the 1 -dimensional linear subspace of $T_p(M)$ generated by $\xi_p$, the $T$ - curvature tensor $T$ is a map

$$T : T_p(M) \times T_p(M) \times T_p(M) \to \varphi(T_p(M)) \oplus \{\xi_p\},$$

such that

$$T(X,Y)Z = a_0R(X,Y)Z + a_1S(Y,Z)X + a_2S(X,Z)Y + a_3S(X,Y)Z + a_4g(Y,Z)QY + a_5g(X,Z)QY + a_6g(X,Y)QZ + a_7r(g(Y,Z)X - g(X,Z)Y)$$

(1)

where $a_0,a_1,a_2,a_3,a_4,a_5,a_6$ and $a_7$ are constants and $R,S,Q$ and $r$ are the Riemannian curvature-tensor, the Ricci-tensor, the Ricci-operator and the scalar curvature of the manifold respectively. It may be natural to consider the following particular cases:

1. If $T : T_p(M) \times T_p(M) \times T_p(M) \to \{\xi_p\}$ i.e., the projection of the image of $T$ in $\varphi(T_p(M))$ is zero, such that $g(T(X,Y)Z,\varphi W) = 0$;
2. If $T : Tp(M) \times Tp(M) \times Tp(M) \rightarrow \varphi(Tp(M))$ i.e., the projection of the image of $T$ in $\{\xi_{p}\}$ is zero, such that

\[ g(T(X,Y)\xi_{p},W) = 0; \]

(3)

3. If $T : \varphi(Tp(M)) \times \varphi(Tp(M)) \times \varphi(Tp(M)) \rightarrow \{\xi_{p}\}$ i.e., when $C$ is restricted to $\varphi(Tp(M)) \times \varphi(Tp(M)) \times \varphi(Tp(M))$, the projection of the image of $T$ in $\varphi(Tp(M))$ is zero, such that

\[ g(T(\varphi X, \varphi Y)\varphi Z, \varphi W) = 0; \]

(4)

A Riemannian manifold, satisfying the cases 1, 2 and 3 is called quasi $-T$ flat, the $\xi - T$ flat, and the $\varphi - T$ flat respectively. The present work is organized as under Section 2 contains necessary details about trans-Sasakian manifolds. Some basic results are also given in section under Section 2. In Section 3, quasi $-T$ - symmetric trans-Sasakian manifold is discussed. Section 4 contains $\xi - T$ - flat trans-Sasakian manifold and section 5 contains $\varphi - T$ - flat trans-Sasakian manifolds.

1. Preliminaries

In this section, we recall some general definitions and basic formulas which will be used later on. For this, we recommend the reference [2]. A $(2n+1)$ dimensional Riemannian manifold $(\mathcal{M}, g)$ is said to be an almost contact metric manifold if there exist a $(1,1)$ tensor field $\varphi$, a unique global non-vanishing structural vector field $\xi$ (called the vector field) and a 1-form $\eta$ such that

\[ \varphi^{2}X = -X + \eta(X)\xi; \]

(a) $\eta(\xi) = 1$, (b) $g(X, \xi) = \eta(X)$,

(5)

(c) $\eta(\varphi X) = 0$, (d) $\varphi\xi = 0$,

\[ g(\varphi X, \varphi Y) = g(X,Y) - \eta(X)\eta(Y), \]

(6)

\[ d\eta(X,Y) = g(X,\varphi Y), \quad \eta \varphi = 0, \]

(7)

Such a manifold is called contact manifold if $\eta \wedge (d\eta)^{n} \neq 0$, where $n$ is $n^{\text{th}}$ exterior power. For contact manifold we also have $d\eta = \Phi$, where $\Phi(X,Y) = g(\varphi X,Y)$ is called fundamental $2$ -form on $\mathcal{M}$. An almost contact metric manifold is called trans-Sasakian manifold of type $(\alpha, \beta)$ if

\[ (\nabla_{X}\varphi)(Y) = \alpha(g(X,Y)\xi - \eta(X)Y) + \beta(g(\varphi X,Y)\xi - \eta(X)\varphi Y), \]

(9)

for all vector field $X, Y$ on $\mathcal{M}$, where $\alpha$ and $\beta$ are some smooth real valued functions. Trans-Sasakian manifolds of type $(1,0)$ and $(0,1)$ are called Sasakian and Kenmotsu manifold respectively.

An almost contact metric manifold $\mathcal{M}$ is said to be $\eta$ -Einstein if its Ricci-tensor $S$ is of the form

\[ S(X,Y) = A g(X,Y) + B \eta(X)\eta(Y), \]

(10)

where $A$ and $B$ are smooth functions on $\mathcal{M}$. A $\eta$ -Einstein manifold becomes Einstein if $B = 0$.

If $\{e_{1}, e_{2}, \ldots, e_{2n}, \xi\}$ is a local orthonormal basis of vector fields in a $(2n+1)$ - dimensional almost contact metric manifold $\mathcal{M}$, then $\{\varphi e_{1}, \varphi e_{2}, \ldots, \varphi e_{2n}, \xi\}$ is also a local orthonormal basis. It is easy to verify that

\[ \sum_{i=1}^{2n} g(e_{i}, e_{i}) = \sum_{i=1}^{2n} g(\varphi e_{i}, \varphi e_{i}) = 2n \]

(11)

\[ \sum_{i=1}^{2n} g(e_{i}, Y)S(X, e_{i}) = \sum_{i=1}^{2n} g(\varphi e_{i}, Y)S(X, \varphi e_{i}) = S(X,Y) - S(X, \xi)\eta(Y) \]

(12)

for all $X, Y \in T(M)$. In view of (8) and (12), we have

\[ \sum_{i=1}^{2n} g(e_{i}, \varphi Y)S(\varphi X, e_{i}) = \sum_{i=1}^{2n} g(\varphi e_{i}, \varphi Y)S(\varphi X, \varphi e_{i}) = S(\varphi X, \varphi Y) \]

(13)

If $\mathcal{M}$ is a trans-Sasakian manifold, then it is known that

\[ R(X, \xi)\xi = (\alpha^{2} - \beta^{2} - \alpha\beta)(X - \eta(X)\xi), \]

(14)

for all $X \in T(M)$.

\[ S(X, \xi) = (2n(\alpha^{2} - \beta^{2}) - \xi\beta)\eta(X) - (2n-1)\beta \alpha \]

and

\[ \eta(X). (\alpha^{2} - \beta^{2} - \alpha\beta)(X - \eta(X)\xi), \]

(15)
\[ S(\xi, \xi) = 2n(\alpha^2 - \beta^2 - \varepsilon \beta), \]  

From (16), we have
\[ \sum_{i=1}^{2n} S(e_i, e_i) = \sum_{i=1}^{2n} S(\varphi \phi_i, \varphi \phi_i) = r - 2n(\alpha^2 - \beta^2 - \varepsilon \beta) \]  

where \( r = \text{trace}(Q) \) is the scalar curvature. In a trans-Sasakian manifold, we have
\[ g(R(\xi, Y)Z, \xi) = (\alpha^2 - \beta^2 - \varepsilon \beta) g(\varphi Y, \varphi Z), \]  

for all \( Y, Z \in T(M) \). Consequently,
\[ \sum_{i=1}^{2n} g(R(e_i, Y)Z, e_i) = \sum_{i=1}^{2n} g(R(\varphi \phi_i, Y)Z, \varphi \phi_i) \]
\[ = S(Y, Z) - (\alpha^2 - \beta^2 - \varepsilon \beta) g(\varphi Y, \varphi Z), \]  

We begin with the following lemma:

**Lemma 1** The quasi-\( T \)-flat trans-Sasakian manifold takes the form of (10) under the condition
\[ \varphi(\text{grad} \beta) = (2n-1)(\text{grad} \beta). \]

**Proof.** Let \( M \) be a \( (2n+1) \)-dimensional trans-Sasakian manifold.

We write the curvature tensor \( T \) in its (0,4) form as follows
\[ g(T(X, Y)Z, W) = a_0 g(R(X, Y)Z, W) \]
\[ + a_1 S(Y, Z)g(X, W) + a_2 S(X, Z)g(Y, W) \]
\[ + a_3 S(X, Y)g(Z, W) + a_4 g(Y, Z)S(X, W) \]
\[ + a_5 g(X, Z)S(Y, W) + a_6 g(X, Y)S(Z, W) \]
\[ + a_7 g(Y, Z)g(X, W) - g(X, Z)g(Y, W)) \]  

From (21), we have
\[ g(T(\varphi X, Y)Z, \varphi W) = a_0 g(R(\varphi X, Y)Z, \varphi W) \]
\[ + a_1 S(\varphi X, Z)g(\varphi X, \varphi W) + a_2 S(\varphi X, Z)g(\varphi X, \varphi W) \]
\[ + a_3 S(\varphi X, Y)g(Z, \varphi W) + a_4 g(Y, Z)S(\varphi X, \varphi W) \]
\[ + a_5 g(\varphi X, Z)S(Y, \varphi W) + a_6 g(\varphi X, Y)S(Z, \varphi W) \]
\[ + a_7 g(\varphi X, Z)g(\varphi X, \varphi W) - g(\varphi X, Z)g(\varphi X, \varphi W)) \]  

for all \( X, Y, Z, W \in T(M) \). If \( \{e_1, e_2, \ldots, e_{2n}, \xi\} \) is a local orthonormal basis of vector fields in \( M \). Now on putting \( X = W = e_i \) and taking summation over \( 1 \leq i \leq 2n \) in (22) we have
\[ \sum_{i=1}^{2n} g(T(\varphi \phi_i, Y)Z, \varphi \phi_i) \]
\[ = \sum_{i=1}^{2n} (a_0 g(R(\varphi \phi_i, Y)Z, \varphi \phi_i) + a_1 S(Y, Z)g(\varphi \phi_i, \varphi \phi_i) \]
\[ + a_2 S(\varphi \phi_i, Z)g(Y, \varphi \phi_i) + a_3 S(\varphi \phi_i, Y)g(\varphi \phi_i, \varphi \phi_i) \]
\[ + a_4 g(Y, Z)S(\varphi \phi_i, \varphi \phi_i) + a_5 g(\varphi \phi_i, Z)S(Y, \varphi \phi_i) \]
\[ + a_6 g(\varphi \phi_i, Y)S(Z, \varphi \phi_i) \]
\[ + a_7 g(\varphi \phi_i, Z)g(\varphi \phi_i, \varphi \phi_i) - g(\varphi \phi_i, Z)g(Y, \varphi \phi_i) \]  

for all \( Y, Z \in T(M) \). Using (11), (12), (17) and (19), we have
\[\sum_{i=1}^{2n} g(T(\varphi e_i, Y)Z, \varphi e_i)\]
\[= a_0(S(Y, Z) - (\alpha^2 - \beta^2 - \xi \beta) g(\varphi Y, \varphi Z)) + 2na_1S(Y, Z) + a_2(S(Y, Z) - S(Z, \xi) \eta(Y)) + a_3(S(Y, Z) - S(Y, \xi) \eta(Z)) + a_4 g(Y, Z)(r - 2n(\alpha^2 - \beta^2 - \xi \beta)) + a_5(S(Y, Z) - S(Y, \xi) \eta(Z)) + a_6(S(Y, Z) - S(Z, \xi) \eta(Y)) + a_7 r(2n-1)g(Y, Z) - \eta(Y) g(Y, Z)\]

(24)

If \( M \) satisfies (2), then from (24), we have
\[0 = a_0(S(Y, Z) - (\alpha^2 - \beta^2 - \xi \beta) g(\varphi Y, \varphi Z)) + 2na_1S(Y, Z) + a_2(S(Y, Z) - S(Z, \xi) \eta(Y)) + a_3(S(Y, Z) - S(Y, \xi) \eta(Z)) + a_4 g(Y, Z)(r - 2n(\alpha^2 - \beta^2 - \xi \beta)) + a_5(S(Y, Z) - S(Y, \xi) \eta(Z)) + a_6(S(Y, Z) - S(Z, \xi) \eta(Y)) + a_7 r(2n-1)g(Y, Z) - \eta(Y) \eta(Y)\]

(25)

If \( \varphi(\text{grad} \alpha) = (2n-1)(\text{grad} \beta) \), then \( \xi \beta = 0 \) [5], and by using (7) and (13) in (25) we get
\[(a_0 + 2na_1 + a_2 + a_3 + a_4 + a_5 + a_6)S(Y, Z) = (a_0 + a_6)g(Y, Z) + (a_0 + 2na_1 + a_2 + a_3 + a_4) \eta(Y) \eta(Z) + (a_0 + 2na_1 + a_2 + a_3 + a_4 + a_6) r g(Y, Z) - (a_0 + (2n-1)a_7) \eta(Y) \eta(Z) + a_7 r(2n-1)g(Y, Z) - \eta(Y) \eta(Z)\]

(26)

If \( a_0 + 2na_1 + a_2 + a_3 + a_4 + a_6 \neq 0 \), from (26) we have
\[S(Y, Z) = \left\{ \begin{array}{l}
(a_0 + 2na_1 + a_2 + a_3 + a_4 + a_6)r g(Y, Z) \\\n+ 2n(\alpha^2 - \beta^2)(a_2 + a_3 + a_4 + a_6 - a_0(\alpha^2 - \beta^2) + a_7) \eta(Y) \eta(Z)
\end{array} \right\}
\]

(27)

we can rewritten (26), as
\[S = A_0 g \times A_1 \eta \otimes \eta\]

(28)

where
\[A_0 = \frac{(a_0 + 2na_1 + a_2 + a_3 + a_4 + a_6)r}{(a_0 + 2na_1 + a_2 + a_3 + a_4 + a_6)}\]
\[\& \quad A_1 = \frac{2n(\alpha^2 - \beta^2)(a_2 + a_3 + a_4 + a_6) - a_0(\alpha^2 - \beta^2) + a_7 r}{(a_0 + 2na_1 + a_2 + a_3 + a_4 + a_6)}\]

Therefore \( M \) is an \( \eta \)-Einstein manifold. In particular, \( M \) becomes an Einstein manifold provided, in (28), we get the required lemma. In view of lemma 1, we can state that:

**Theorem 1** A quasi \( T \)-flat trans-Sasakian manifold is \( \eta \)-Einstein if \( \varphi(\text{grad} \alpha) = (2n-1)(\text{grad} \beta) \).

3. \( \xi - T \)-flat trans-Sasakian manifolds
We begin with the following lemma:

**Lemma 2** The flat trans-Sasakian manifold takes the form of (10) under the condition

\[ \varphi(\text{grad}\alpha) = (2n-1)\text{grad}\beta. \]

**Proof.** Let \( M \) be \( (2n+1) \)-dimensional trans-Sasakian manifold. Putting \( Z = \xi \) in (1), we have

\[
g(T(X,Y)\xi, W) = a_0 g(R(X,Y)\xi, W) + a_1 S(Y,\xi) g(X, W) \\
+ a_2 S(X,\xi) g(Y, W) + a_3 S(Y, \xi) g(\xi, W) \\
+ a_4 g(Y, \xi) S(X, W) + a_5 g(X, \xi) S(Y, W) \\
+ a_6 g(X, Y) S(\xi, W) + a_7 r(g(Y, \xi) g(X, W) - g(X, \xi) g(Y, W)) \\
\text{for all } X, Y, W \in T(M). 
\]

If \( \{e_1, e_2, \ldots, e_{2n}, \xi\} \) is a local orthonormal basis of vector fields in \( M \). Now on putting \( Z = \xi \) and taking summation over \( 1 \leq i \leq 2n \) in (29), we have

\[
\sum_{i=1}^{2n} g(T(X, \xi) \xi, W) \\
= \sum_{i=1}^{2n} (a_0 g(R(X, \xi) \xi, W) + a_1 S(\xi, \xi) g(X, W) \\
+ a_2 S(X, \xi) g(\xi, W) + a_3 S(\xi, \xi) g(\xi, W) \\
+ a_4 g(\xi, \xi) S(X, W) + a_5 g(X, \xi) S(\xi, W) \\
+ a_6 g(\xi, \xi) S(\xi, W) + a_7 r(g(\xi, \xi) g(X, W) - g(X, \xi) g(\xi, W)) \\
\text{for all } Y, Z \in T(M). \text{ Using (8), (15), (16) and (18), we have}
\]

\[
\sum_{i=1}^{2n} T(X, \xi, \xi, W) \\
= \sum_{i=1}^{2n} (a_0 ((\alpha^2 - \beta^2 - \xi \beta)(g(X, W) - \eta(X) \eta(W))) \\
+ a_1 ((2n(\alpha^2 - \beta^2) - \xi \beta) \eta(X) - (2n-1) X \beta - (\varphi X) \alpha) g(X, W) \\
+ a_2 2n(\alpha^2 - \beta^2 - \xi \beta) \eta(X) \eta(W) + a_3 2n(\alpha^2 - \beta^2 - \xi \beta) \eta(X) \eta(W) \\
+ a_4 S(X, W) + a_5 2n(\alpha^2 - \beta^2 - \xi \beta) \eta(X) \eta(W) \\
+ a_6 2n(\alpha^2 - \beta^2 - \xi \beta) \eta(X) \eta(W) + a_7 r(g(X, W) - \eta(X) \eta(W)) \\
\text{for all } Y, Z \in T(M) \text{. Using (3), then from (31), we have}
\]

\[
\sum_{i=1}^{2n} a_0 ((\alpha^2 - \beta^2 - \xi \beta)(g(X, W) - \eta(X) \eta(W))) \\
+ a_1 ((2n(\alpha^2 - \beta^2) - \xi \beta) \eta(X) - (2n-1) X \beta - (\varphi X) \alpha) g(X, W) \\
+ a_2 2n(\alpha^2 - \beta^2 - \xi \beta) \eta(X) \eta(W) + a_3 2n(\alpha^2 - \beta^2 - \xi \beta) \eta(X) \eta(W) \\
+ a_4 S(X, W) + a_5 2n(\alpha^2 - \beta^2 - \xi \beta) \eta(X) \eta(W) \\
+ a_6 2n(\alpha^2 - \beta^2 - \xi \beta) \eta(X) \eta(W) + a_7 r(g(X, W) - \eta(X) \eta(W)) = 0
\]

If \( \varphi(\text{grad}\alpha) = (2n-1)(\text{grad}\beta) \), then \( \xi \beta = 0 \), which implies that

\[
a_4 S(X, W) \\
= -(a_0 (\alpha^2 - \beta^2) + a_7) g(X, W) \\
+ (a_0 (\alpha^2 - \beta^2) \eta(X) \eta(W) \\
- 2na_1 (\alpha^2 - \beta^2) g(X, W) \\
- 2na_2 (\alpha^2 - \beta^2) g(X, W) \\
- 2na_3 (\alpha^2 - \beta^2) \eta(X) \eta(W)
\]
\[-2na_i(\alpha^2 - \beta^2)\eta(X)\eta(W)\]
\[-2na_0(\alpha^2 - \beta^2)\eta(X)\eta(W)\]
\[+ a_7 r\eta(X)\eta(W)\]

If \(a_4 \neq 0\) then from (33), we get
\[S(X,W) = \left(\frac{(a_0 + 2na_i)(\alpha^2 - \beta^2) + a_7 r}{a_4}\right)g(X,W)\]
\[\left(-\frac{a_0 + 2n(a_2 + a_3 + a_5 + a_6)(\alpha^2 - \beta^2) - a_7 r}{a_4}\right)\eta(X)\eta(W)\]

we can rewritten (34), as
\[S = A_5 g + A_6 \eta \otimes \eta\]

where
\[A_5 = -\frac{(a_0 + 2na_i)(\alpha^2 - \beta^2) + a_7 r}{a_4}\]
\[& A_3 = -\left(-\frac{a_0 + 2n(a_2 + a_3 + a_5 + a_6)(\alpha^2 - \beta^2) - a_7 r}{a_4}\right)\]

in (35), we get the required lemma.

In view of lemma 2, we can state that:

**Theorem 2** A Ricci tensor quasi \(\xi - T\) - flat trans-Sasakian manifold is \(\eta -\) Einstein if \(\phi(\text{grad}\alpha) = (2n - 1)(\text{grad}\beta)\).

4. \(\phi - T\) - flat trans-Sasakian manifold

**Lemma 3** The \(\phi - T\) - flat trans-Sasakian manifold takes the form of (10) under the condition \(\phi(\text{grad}\alpha) = (2n - 1)\text{grad}\beta\).

**Proof.** Let \(M\) be a \(2n + 1\) - dimensional trans-Sasakian manifold. From (1), we have
\[T(\phi X, \phi Y, \phi Z, \phi W)\]
\[= a_0 R(\phi X, \phi Y, \phi Z, \phi W) + a_1 S(\phi Y, \phi Z)g(\phi X, \phi W)\]
\[+ a_2 S(\phi X, \phi Z)g(\phi Y, \phi W) + a_3 S(\phi X, \phi Y)g(\phi Z, \phi W)\]
\[+ a_4 g(\phi Y, \phi Z)S(\phi X, \phi W) + a_5 g(\phi X, \phi Z)S(\phi Y, \phi W)\]
\[+ a_6 g(\phi X, \phi Y)S(\phi Z, \phi W)\]
\[+ a_7 r(g(\phi Y, \phi Z)g(\phi X, \phi W) - g(\phi X, \phi Z)g(\phi Y, \phi W)\]

for all \(X, Y, Z, W \in T(M)\). If \(\{e_1, e_2, \ldots, e_{2n}, \xi\}\) is a local orthonormal basis of vector fields in a \(2n + 1\) - dimensional almost contact manifold \(M\), then \(\{\phi e_1, \phi e_2, \ldots, \phi e_{2n}, \xi\}\) is local orthonormal basis. From (36), we have
\[\sum_{i=1}^{2n} T(\phi e_i, \phi Y, \phi Z, \phi e_i)\]
\[= \sum_{i=1}^{2n} a_0 R(\phi e_i, \phi Y, \phi Z, \phi e_i) + a_1 S(\phi Y, \phi Z)g(\phi e_i, \phi e_i)\]
\[+ a_2 S(\phi e_i, \phi Z)g(\phi Y, \phi e_i) + a_3 S(\phi e_i, \phi Y)g(\phi Z, \phi e_i)\]
\[+ a_4 g(\phi Y, \phi Z)S(\phi e_i, \phi e_i) + a_5 g(\phi e_i, \phi Z)S(\phi Y, \phi e_i)\]
\[+ a_6 g(\phi e_i, \phi Y)S(\phi Z, \phi e_i)\]
\[+ a_7 r(g(\phi Y, \phi Z)g(\phi e_i, \phi e_i) - g(\phi e_i, \phi Z)g(\phi Y, \phi e_i)\]

Using (11), (13), (19) and (37), we have...
\[
\sum_{i=1}^{2n} T(e_i, \varphi Y, \varphi Z, \varphi e_i) \\
= \sum_{i=1}^{2n} \left[ a_0 (S(\varphi Y, \varphi Z) - (\alpha^2 - \beta^2 - \xi \beta)(g(\varphi Y, \varphi Z))
+ 2na_4 S(\varphi Y, \varphi Z) + a_2 S(\varphi Y, \varphi Z) + a_3 S(\varphi Y, \varphi Z)
+ a_5 S(\varphi Y, \varphi Z) + a_6 S(\varphi Y, \varphi Z) + a_7 r(2ng(\varphi Y, \varphi Z) - g(\varphi Y, \varphi Z)) \right]
\]

If \( M \) is \( \varphi - T - \) flat, then from (38), we have
\[
0 = (a_0 + 2na_4 + a_2 + a_3 + a_5 + a_6) S(\varphi Y, \varphi Z)
+ (-a_0(\alpha^2 - \beta^2) + (r - 2n)(\alpha^2 - \beta^2)a_4 + (2n - 1)a_7 r) g(\varphi Y, \varphi Z)
\]
If \( a_0 + 2na_4 + a_2 + a_3 + a_5 + a_6 \neq 0 \), then from (39), we get
\[
S(\varphi Y, \varphi Z) = \left[ a_0(\alpha^2 - \beta^2) - (r - 2n)(\alpha^2 - \beta^2)a_4 + (2n - 1)a_7 r \right] (a_0 + 2na_4 + a_2 + a_3 + a_5 + a_6) g(\varphi Y, \varphi Z)
\]
we can rewritten (40), as
\[
S(\varphi Y, \varphi Z) = A_1 g(\varphi Y, \varphi Z)
\]
where
\[
A_1 = \left[ a_0(\alpha^2 - \beta^2) - (r - 2n)(\alpha^2 - \beta^2)a_4 + (2n - 1)a_7 r \right] (a_0 + 2na_4 + a_2 + a_3 + a_5 + a_6)
\]
and Using (4) and (41) in (38), we get
\[
a_6 R(\varphi X, \varphi Y, \varphi Z, \varphi W)
= - (a_1 A_1 + a_4 A_4 + a_2 r) g(\varphi Y, \varphi Z) g(\varphi X, \varphi W)
- (a_2 A_4 + a_4 A_1 - a_7 r) g(\varphi X, \varphi Z) g(\varphi Y, \varphi W)
- (a_4 A_4 + a_6 A_4) g(\varphi X, \varphi Y) g(\varphi Z, \varphi W)
\]
in (43), we get the required lemma.

In view of lemma 3, we can state that:

**Theorem 3** A quasi \( \varphi - T - \) flat trans-Sasakian manifold is \( \eta - \) Einstein if \( \varphi(\text{grad} \alpha) = (2n - 1)(\text{grad} \beta) \).

**5. A Trans-Sasakian manifold with respect to the \( \varphi - T - \) flat**

**Lemma 4** The Ricci tensor of trans-Sasakian manifold takes the form of (10) under the condition \( \varphi - T - \) flat and \( \varphi(\text{grad} \alpha) = (2n - 1)(\text{grad} \beta) \).

**Proof.** Let \( M \) be a \((2n + 1)\) - dimensional trans-Sasakian manifold. Now, assume that \( M \) is \( \varphi - T - \) flat. In a trans-Sasakian manifold, in view of (5), (6), (8) and (14) we can easily verify that
\[
R(\varphi^2 X, \varphi^2 Y, \varphi^2 Z, \varphi^2 W) = R(X, Y, Z, W)
- g(Y, Z)\eta(\eta(W)) + g(X, Z)\eta(\eta(W))
+ g(Y, W)\eta(\eta(Z)) - g(X, W)\eta(\eta(Y))\eta(\eta(Z))
\]
Re-Uplacing \( X, Y, Z, W \) by \( \varphi X, \varphi Y, \varphi Z, \varphi W \) respectively in (4), we get
\[
- a_0 R(\varphi^2 X, \varphi^2 Y, \varphi^2 Z, \varphi^2 W)
= (a_1 A_1 + a_4 A_4 + a_2 r) g(\varphi Y, \varphi Z) g(\varphi X, \varphi W)
+ (a_2 A_4 + a_4 A_1 - a_7 r) g(\varphi X, \varphi Z) g(\varphi Y, \varphi W)
+ (a_4 A_4 + a_6 A_4) g(\varphi X, \varphi Y) g(\varphi Z, \varphi W)
\]
From (44) and (45), we get
\[
-a_i R(X, Y, Z, W)
= (A_i a_4 + A_4 a_i + a_r) g(Y, Z) g(X, W) \\
+ (A_i a_3 + A_3 a_i - a_r) g(Y, Z) g(Y, W) \\
+ (A_i a_2 + A_2 a_i) g(X, Y) g(Z, W) \\
- (A_i a_3 + A_3 a_i) g(X, Y) \eta(Z) \eta(W) \\
- (A_i a_2 + A_2 a_i) g(Z, W) \eta(X) \eta(Y) \\
- (A_i a_3 + A_i a_3 - a_r + a_0) \eta(Y) \eta(Z) g(X, W) \\
- (A_i a_2 + A_2 a_i + a_r + a_0) \eta(X) \eta(W) g(Y, Z) \\
- (A_i a_1 + A_1 a_i + a_r + a_0) \eta(Y) \eta(W) g(X, Z) \\
- (A_i a_1 + A_1 a_i + a_r - a_0) \eta(X) \eta(Z) g(Y, W) \\
+ \sum_{i=1}^{6} a_i A_i \eta(X) \eta(Y) \eta(Z) \eta(W)
\]

for all \(X, Y, Z, W \in T(M)\). If \(\{e_1, e_2, \ldots, e_{2n}, \xi\}\) is a local orthonormal basis of vector fields in \(M\). Now on putting \(X = W = e_i\) and taking summation over \(1 \leq i \leq 2n\) in (46) we have

\[
S(Y, Z) = A_5 g(\phi Y, \phi Z) + (2n - 1) \eta(Y) \eta(Z)
\]

where

\[
A_5 = -\frac{1}{a_0} \left[(2n-1)(a_1 A_4 + a_4 A_1 + a_r) + A_1 (a_2 + a_3 + a_5 + a_6 - a_r) - a_0 - a_0 (\alpha^2 - \beta^2)\right]
\]

Using (47) and (46) in (28), we get

\[
T(X, Y, Z, \phi W)
= (a_4 + a_4)(A_5 - A_4) g(Y, Z) g(X, \phi W) \\
+ (a_4 + a_4)(A_5 - A_4) g(X, Y) g(Z, \phi W) \\
+ (a_2 + a_2)(A_5 - A_4) g(Y, Z) g(Y, \phi W) \\
+ ((a_1 + a_4)A_4 + a_r - a_0) \sum_{i=1}^{6} a_i A_i \eta(Y) \eta(Z)
\]

in (48), we get the required lemma. In view of lemma 4 we can state that:

**Theorem 4** A trans-Sasakian manifold is \(\eta\) – Einstein if \(\phi - T\) – flat and \(\phi(\text{grad} \chi) = (2n - 1)(\text{grad} \beta)\).

**Acknowledgment**

The authors are thankful to Professor S. Ahmad Ali Dean School of Applied Sciences, B.B.D. University, Lucknow, providing suggestions for the improvement of this paper.

**References**