Eigenvalue of Sturm-Liouville problem in Neumann conditions with turning points

E.A Sazgar
University of Fanni- Herfehi Mahmodabad Iran.

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Introduction
Let us consider second-order differential equation
\[-W'' + \left(u''(\zeta^2 - 1) + q(\zeta)\right)W = 0\]
(1)

\[\zeta \in [a, b]\]
W'(a) = W'(b) = 0 \quad -a > b

W''(a) = W''(b) = 0, are called Neumann boundary conditions. The function \(f(\zeta) = \zeta^2 - 1\) is weight function and the zeros \(1\) and \(-1\) are turning points.

Differential equations with turning points play important role in branch of sciences. 2. Approximation of the solutions and Derivative of solutions

In [1] the asymptotic solutions of differential equation \(a'' = u''(\zeta)\),

\[A(u'') \sim \frac{1}{2\pi^2 u'' \eta_1^2} \left(\sum_{r=0}^{\infty} (-1)^r u_r \left(\frac{3}{2} u \eta_1^2 \zeta \right)^{2r} \right) \quad \zeta \geq 1

B(u'') \sim \frac{1}{2\pi^2 u'' \eta_2^2} \left(\sum_{r=0}^{\infty} (-1)^r u_r \left(\frac{3}{2} u \eta_2^2 \zeta \right)^{2r} \right) \quad \zeta \geq 0

A(0) = \frac{1}{3\eta_1^3 \Gamma \left(\frac{2}{3}\right)} = \frac{B(0)}{\sqrt{3}}

u(x) = \left(2x + 1\right)\left(2x + 3\right)\left(2x + 5\right)\ldots\left(6x + 1\right)

(216)^{\frac{1}{3}}

v_s = -\frac{6s + 1}{6s - 1} \quad s \geq 1

u_0 = v_0 = 1

The asymptotic solution of the standard equation of the form
\[U'' = u''(\zeta^2 - 1)U\]

(3)

For large value of the real parameter \(u\) is obtained by Olver [1].

In fact let \(U_1(u, \zeta)\) and \(U_2(u, \zeta)\) be two independent solutions of the equation (3). For \(\zeta > 0\) are of the form

\[U_1(\zeta, u) \approx u^r (2\zeta + 1^3 \eta_1^3 \Gamma \left(\frac{2}{3}\right))^{-\frac{1}{2}} u_1 \left(\frac{3}{2} u \eta_1^2 \zeta \right)^{1 (1 + 6u^{-2})}

U_2(\zeta, u) \approx u^r (2\zeta + 1^3 \eta_1^3 \Gamma \left(\frac{2}{3}\right))^{-\frac{1}{2}} u_1 \left(\frac{3}{2} u \eta_1^2 \zeta \right)^{1 (1 + 6u^{-2})}

(4)

where,

\[\eta_1^3 \approx \left[\frac{3}{2} \int (r^2 - 1)^2 \, dr \right]^{\frac{1}{2}} \quad \zeta \geq 1

\[-\left(\frac{3}{2} \zeta^2 \right)^{1 (1 - r^2)^2} \, dr \right]^{\frac{1}{2}} \quad 0 \leq \zeta \leq 1

For every value of \(u\) equation (1) has two solutions \(W_1(u, \zeta)\) and \(W_2(u, \zeta)\).
\[
\begin{align*}
W_t(u, \zeta) &= U_t(-u) - \frac{1}{2} \sqrt{2 \pi} \sum_{n=1}^\infty A_n(\zeta) \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) B_n(\zeta) \, dx,

W_u(u, \zeta) &= \frac{1}{2} \sqrt{2 \pi} \sum_{n=1}^\infty A_n(\zeta) \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) B_n(\zeta) \, dx, \quad (5)
\end{align*}
\]

Let us define

\[
K_n(u, \zeta) = \sum_{n=1}^\infty A_n(\zeta) u^{n-1} + u^{-1}(u\zeta - \sqrt{2u}) \sum_{n=1}^\infty B_n(\zeta) u^{n-1},
\]

\[
K_n(u, \zeta) = \sum_{n=1}^\infty A_n(\zeta) u^{n-1} + u^{-1}(u\zeta - \sqrt{2u}) \sum_{n=1}^\infty B_n(\zeta) u^{n-1}.
\]

On the other hand, for \( \xi > 0 \), by using the derivative of \( U_1(u, \xi) \), \( U_2(u, \xi) \) and inserting them in the derivative of \( W_t(u, \xi) \) and \( W_u(u, \xi) \), we have,

\[
W_t(u, \xi) = \sqrt{2\pi} e^{-\frac{1}{2}u^2} \left( \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) A_n(\eta_1) \sqrt{\eta_2} \right) (1 + O(u^{-1})).
\]

Similarly for \( W_u(u, \xi) \) we get

\[
W_t(u, \xi) = \sqrt{2\pi} e^{-\frac{1}{2}u^2} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) A_n(\eta_1) \sqrt{\eta_2}.
\]

The asymptotic behavior of \( W_t(u, \zeta), W_u(u, \xi) \) for large \( u \) and \( \xi \leq 0 \) can be determined by use of connection formula, on replacing \( \xi \) by \( -\xi \),

\[
W_t(u, -\xi) = \left( \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) A_n(\eta_1) \sqrt{\eta_2} \right) (1 + O(u^{-1})).
\]

Not that for \( \xi < 0 \), the derivative of \( W_1(u, \xi), W_2(u, \xi) \) with respect to \( \xi \) are

\[
W'_1(u, \xi) = O(u^{-1})W_1(u, \xi) + O(u^{-1})W_2(u, \xi) + O(u^{-1})W_3(u, \xi),
\]

\[
W'_2(u, \xi) = O(u^{-1})W_1(u, \xi) + O(u^{-1})W_2(u, \xi) + O(u^{-1})W_3(u, \xi).
\]

3. Asymptotic of eigenvalue for case \(-a > b\), with Neumann conditions

In this section we will study distributions of the eigenvalues of equation (1) with boundary conditions

\[
W'(a) = W'(b) = 0.
\]

If \( W(u, \xi) \) and \( W_u(u, \xi) \) be two independent solutions of equation (1) then

\[
W(u, \xi) = c_1 W_1(u, \xi) + c_2 W_2(u, \xi), \quad c_1 \neq 0, c_2 \neq 0,
\]

also is a solutions equation (1) therefore

\[
W(u, \xi) = c_1 W'_1(u, \xi) + c_2 W'_2(u, \xi).
\]

By attention to Neumann boundary conditions

\[
\begin{align*}
W'(a) &= c_1 W_1'(a) + c_2 W_2'(a) = 0, \\
W'(b) &= c_1 W_1'(b) + c_2 W_2'(b) = 0,
\end{align*}
\]

because above system having non trivial solution then the determinant coefficients must be zero therefore

\[
\begin{bmatrix}
W_1'(a) & W_2'(a) \\
W_1'(b) & W_2'(b)
\end{bmatrix} = 0.
\]

In this case the eigenvalues of equation (1) are the zeros of \( \Delta(u) = 0 \)

where,

\[
\Delta(u) = \begin{bmatrix}
W_1'(u, a) & W_2'(u, a) \\
W_1'(u, b) & W_2'(u, b)
\end{bmatrix}
\]

Since \( a < -1 \), \( W_1(u, a) \) is trigonometry and \( b > 1 \), \( W_2'(u, b) \) is algebraic.

\[
W_1'(a, a) W_2'(a, b) = \left( \sin \left( \frac{\pi u}{2} \right) + O(u^{-1}) \right) W_1(u, a) \sqrt{u + \sqrt{u^2 + \frac{4}{3}}} + \left( \cos \left( \frac{\pi u}{2} \right) + O(u^{-1}) \right) W_2(u, a) \sqrt{u - \sqrt{u^2 + \frac{4}{3}}}.
\]

\[
W_1'(a, b) W_2'(a, a) = \left( \sin \left( \frac{\pi u}{2} \right) + O(u^{-1}) \right) W_1(u, a) \sqrt{u + \sqrt{u^2 + \frac{4}{3}}} + \left( \cos \left( \frac{\pi u}{2} \right) + O(u^{-1}) \right) W_2(u, a) \sqrt{u - \sqrt{u^2 + \frac{4}{3}}}.
\]

\[
\left\{ \begin{array}{l}
\Gamma_1 = abu^2 - au^2 e^{-\frac{1}{2}u^2} + bu e^{-\frac{1}{2}u^2} - 4ue^{-\frac{1}{2}u^2}, \\
\Gamma_2 = abu^2 - au^2 e^{-\frac{1}{2}u^2} + bu e^{-\frac{1}{2}u^2} - 4ue^{-\frac{1}{2}u^2}, \\
\Gamma_3 = abu^2 - au^2 e^{-\frac{1}{2}u^2} + bu e^{-\frac{1}{2}u^2} - 4ue^{-\frac{1}{2}u^2}, \\
\Gamma_4 = abu^2 - au^2 e^{-\frac{1}{2}u^2} + bu e^{-\frac{1}{2}u^2} - 4ue^{-\frac{1}{2}u^2},
\end{array} \right.
\]

We know from \( \Delta(u) = 0 \).

\[
\sin \left( \frac{\pi u}{2} \right) + O(u^{-1}) \right) W_1(u, a) W_2(u, b) \Gamma_1 + \left( \cos \left( \frac{\pi u}{2} \right) + O(u^{-1}) \right) W_1(u, a) W_2(u, b) \Gamma_2,
\]

\[
= \left( \sin \left( \frac{\pi u}{2} \right) + O(u^{-1}) \right) W_1(u, a) W_2(u, b) \Gamma_1 + \left( \cos \left( \frac{\pi u}{2} \right) + O(u^{-1}) \right) W_1(u, a) W_2(u, b) \Gamma_2.
\]

In fact, form (9) we have obtained

\[
\tan \left( \frac{\pi u}{2} \right) = \frac{W_1(u, a) W_2(u, b) \gamma_1 + W_1(u, a) W_2(u, b) \gamma_2}{W_1(u, a) W_2(u, b) \gamma_1 + W_1(u, a) W_2(u, b) \gamma_2}.
\]

By attention that,

\[
\begin{align*}
W_1(u, \xi) &\approx U_1(u, \xi) K_n(\xi, \xi), \\
W_2(u, \xi) &\approx U_2(u, \xi) K_n(\xi, \xi), \\
\Gamma_1 &= U_1(u, a) U_2(u, b) K_n(a, b), \\
\Gamma_2 &= U_1(u, a) U_2(u, b) K_n(a, b),
\end{align*}
\]

\[
\begin{align*}
U_1(u, a) U_2(u, b) K_n(a, b) &= 2\pi^2 \left[ \left( \frac{1}{2} \right)^{\frac{1}{2}} \left( \frac{1}{2} \right)^{\frac{1}{2}} \right] \lambda(\eta_1, \eta_2) \left( \frac{1}{2} \right)^{\frac{1}{2}} \left( \frac{1}{2} \right)^{\frac{1}{2}} \lambda(\eta_1, \eta_2), \\
\lambda(\eta_1, \eta_2) &= \frac{1}{\sqrt{\eta_1}} \lambda(\eta_2, \eta_1).
\end{align*}
\]

In calculating and summarizing equation (20) we define the notations,

\[
N_1 = \left( \sqrt{\pi} \Gamma_1 \frac{1}{2} + \frac{1}{2} \right)^{\frac{1}{2}} e^{-\frac{1}{2}}, \quad N_2 = \left( \frac{\eta_1 - \eta_2}{(a-1)(b-1)} \right)^{\frac{1}{2}}.
\]
\[
U_i(a, u) = \frac{\pi}{2} - \frac{\lambda_i \Gamma(u)}{\lambda_i \Gamma(u) + 1} (1 + O(a^{-1})).
\]

From (14) we can write

\[
\tan \left( \frac{\pi u}{2} \right) = \frac{\lambda_i \Gamma(u) e^{\alpha u}}{\lambda_i \Gamma(u) e^{\beta u}} (1 + O(a^{-1})).
\]

and

\[
\alpha = 3u \left( \frac{\eta_n^2}{3} + \eta_n^2 \right), \alpha \to \infty \Rightarrow e^{-\alpha} \to o,
\]

and

\[
\beta = 3u \left( \frac{\eta_n^2}{3} - \eta_n^2 \right), u \to \infty, \beta \to \infty, \Rightarrow e^{-\beta} \to 0.
\]

Therefore from (14) we will have,

\[
\tan \left( \frac{\pi u}{2} \right) = \frac{\lambda_i \Gamma(u) e^{\alpha u}}{\lambda_i \Gamma(u) e^{\beta u}} (1 + O(a^{-1})).
\]

and

\[
x = \frac{\lambda_i \Gamma(u) e^{\alpha u}}{\lambda_i \Gamma(u) e^{\beta u}} = \frac{\lambda_i}{\lambda_i} \times G_1 \times e^{\alpha u} = \frac{\lambda_i}{\lambda_i} \times G_1 e^{\alpha u}.
\]

\[
\lambda_i = 1 - \frac{2\pi}{2} \sinh \left( \frac{\pi u}{3} \right) \sinh \left( \frac{\pi u}{3} \right) + \frac{2\pi}{2} \left( \sinh \left( \frac{\pi u}{3} \right) \sinh \left( \frac{\pi u}{3} \right) + \frac{1}{u \lambda_i} \right) + O(a^{-1}).
\]

\[
\tan \left( \frac{\pi u}{2} \right) = 1 - \frac{2\pi}{2} \sinh \left( \frac{\pi u}{3} \right) \sinh \left( \frac{\pi u}{3} \right) + \frac{2\pi}{2} \left( \sinh \left( \frac{\pi u}{3} \right) \sinh \left( \frac{\pi u}{3} \right) + \frac{1}{u \lambda_i} \right) + O(a^{-1}).
\]

References


