(1,2)*-FG-Closed and (1,2)*-FG-Open Maps in Fuzzy Bitopological Spaces

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1. Introduction
Malghan [2] introduced the concept of generalized closed maps in topological spaces. Devi [1] introduced and studied sg-closed maps and gs-closed maps. Recently, Sheik John [6] defined ω-closed maps and studied some of their properties. In this paper, we introduce (1,2)*-fg-closed maps, (1,2)*-fg-open maps, (1,2)*-fg*-closed maps and (1,2)*-fg*-open maps in fuzzy bitopological spaces and obtain certain characterizations of these classes of maps. In last section, we introduce (1,2)*-fg*-homeomorphisms and prove that the set of all (1,2)*-fg*-homeomorphisms forms a group under the operation composition of functions.

1.2 Preliminaries

Definition 1.2.1
A map \( f: (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2) \) is called
(i) \( (1,2)*\)-g-closed [5] if \( f(V) \) is \( (1,2)*\)-g-closed in \( Y \), for every \( \tau_{1,2}\)-closed set \( V \) of \( X \).
(ii) \( (1,2)*\)-sg-closed [4] if \( f(V) \) is \( (1,2)*\)-sg-closed in \( Y \), for every \( \tau_{1,2}\)-closed set \( V \) of \( X \).
(iii) \( (1,2)*\)-gs-closed [4] if \( f(V) \) is \( (1,2)*\)-gs-closed in \( Y \), for every \( \tau_{1,2}\)-closed set \( V \) of \( X \).
(iv) \( (1,2)*\)-ψ-closed [3] if \( f(V) \) is \( (1,2)*\)-ψ-closed in \( Y \), for every \( \tau_{1,2}\)-closed set \( V \) of \( X \).

We introduce the following definitions

Definition 1.2.2
A map \( f: (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2) \) is called
(i) \( (1,2)*\)-fg-closed if \( f(V) \) is \( (1,2)*\)-fg-closed in \( Y \), for every \( \tau_{1,2}\)-closed set \( V \) of \( X \).
(ii) \( (1,2)*\)-fgs-closed if \( f(V) \) is \( (1,2)*\)-fgs-closed in \( Y \), for every \( \tau_{1,2}\)-closed set \( V \) of \( X \).
(iii) \( (1,2)*\)-fψ-closed if \( f(V) \) is \( (1,2)*\)-fψ-closed in \( Y \), for every \( \tau_{1,2}\)-closed set \( V \) of \( X \).

1.3 (1,2)*-fg-CLOSED MAPS

Definition 1.3.1
A map \( f: (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2) \) is said to be \( (1,2)*\)-fg-closed if the image of every \( \tau_{1,2}\)-closed set in \( X \) is \( (1,2)*\)-fg-closed in \( Y \).

Proposition 1.3.2
For any \( A \subseteq X \),
(i) \( (1,2)*\)-g-cl(A) is the smallest \( (1,2)*\)-fg-closed set containing \( A \).
(ii) \( A \) is \( (1,2)*\)-fg-closed if and only if \( (1,2)*\)-g-cl(A) = A.

Proposition 1.3.3
For any two subsets \( A \) and \( B \) of \( X \),
(i) If \( A \subseteq B \), then \( (1,2)*\)-g-cl(A) \( \subseteq \) \( (1,2)*\)-g-cl(B).
(ii) \( (1,2)*\)-g-cl(A \( \cap \) B) \( \subseteq \) \( (1,2)*\)-g-cl(A) \( \cap \) \( (1,2)*\)-g-cl(B).

Proposition 1.3.4
A map \( f: (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2) \) is \( (1,2)*\)-fg-closed if and only if \( (1,2)*\)-g-cl(f(A)) \( \leq \) \( f(\tau_{1,2}\)-cl(A)) for every subset \( A \) of \( X \).

Proof
Suppose that \( f \) is \( (1,2)*\)-fg-closed and \( A \subseteq X \). Then \( \tau_{1,2}\)-cl(A) is \( \tau_{1,2}\)-closed in \( X \) and so \( f(\tau_{1,2}\)-cl(A)) is \( (1,2)*\)-fg-closed in \( Y \). We have \( f(A) \leq f(\tau_{1,2}\)-cl(A)) and by Propositions 1.3.2 and 1.3.3, \( (1,2)*\)-g-cl(f(A)) \( \leq \) \( (1,2)*\)-g-cl(f(\tau_{1,2}\)-cl(A))) = f(\tau_{1,2}\)-cl(A)). Conversely, let...
A be any \( \tau_{1,2} \)-closed set in \( X \). Then \( A = \tau_{1,2}\text{-cl}(A) \) and so \( f(A) = f(\tau_{1,2}\text{-cl}(A)) \geq (1,2)^{\ast}\text{-g-cl}(f(A)) \), by hypothesis. We have \( f(A) \leq (1,2)^{\ast}\text{-g-cl}(a) \). Therefore \( f(A) = (1,2)^{\ast}\text{-g-cl}(f(A)) \). That is \( f(A) \) is \( (1,2)^{\ast}\text{-fg-closed} \) by Proposition 1.3.2 and hence \( f \) is \( (1,2)^{\ast}\text{-g-closed} \).

**Proposition 1.3.5**

Let \( f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2) \) be a map such that \((1,2)^{\ast}\text{-g-cl}(f(A)) \leq f(\tau_{1,2}\text{-cl}(A))\) for every subset \( A \subset X \). Then the image \( f(A) \) of a \( \tau_{1,2} \)-closed set \( A \) in \( X \) is \((1,2)^{\ast}\text{-fg-closed}\) in \( Y \).

**Proof**

Let \( A \) be a \( \tau_{1,2} \)-closed set in \( X \). Then by hypothesis \((1,2)^{\ast}\text{-g-cl}(f(A)) \leq f(\tau_{1,2}\text{-cl}(A)) \) and so \((1,2)^{\ast}\text{-g-cl}(f(A)) \). Therefore \( f(A) \) is \((1,2)^{\ast}\text{-fg-closed}\) in \( Y \).

**Theorem 1.3.6**

A map \( f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2) \) is \((1,2)^{\ast}\text{-fg-closed}\) if and only if for each subset \( S \subset Y \) and each \( \tau_{1,2}\)-open set \( U \subset Y \) containing \( f^{-1}(S) \) there is an \((1,2)^{\ast}\text{-fg-open set V of Y such that S} \leq V \text{ and } f^{-1}(V) \leq U \).

**Proof**

Suppose \( f \) is \((1,2)^{\ast}\text{-fg-closed}\). Let \( S \subset Y \) and \( U \subset Y \) be an \( \tau_{1,2}\)-open set of \( X \) such that \( f^{-1}(S) \subset U \). Then \( V = (U) \) is an \((1,2)^{\ast}\text{-fg}\) open set containing \( S \) such that \( f^{-1}(V) \subset U \).

For the converse, let \( U \) be a \( \tau_{1,2} \)-closed set of \( X \). Then \( f^{-1}(U) \) is \((1,2)^{\ast}\text{-fg-closed} \) if and only if for each subset \( S \subset Y \) and each \( \tau_{1,2}\)-open set \( U \subset Y \) containing \( f^{-1}(S) \) there is an \((1,2)^{\ast}\text{-fg-open set V of Y such that S} \leq V \text{ and } f^{-1}(V) \leq U \).

**Example 1.3.8**

Let \( (X, \tau_1, \tau_2) \) be a fuzzy bitopological space where \( X = \{ a, b, c \} \).

\[
\tau_1 = 0, 1, \lambda = \frac{1}{a + b + c}, \quad \mu = \frac{1}{a + b + c} \quad \text{and} \quad \tau_2 = \{ 0, 1 \}.
\]

\( \tau_{12} \)-closed are \( 0, 1, \lambda' = \frac{0}{a + b + c}, \mu = \frac{1}{a + b + c} \). Then \((1,2)^{\ast}\text{-fg-closed are} \)

\[
0, 1, \lambda' = \frac{0}{a + b + c}, \mu = \frac{1}{a + b + c} \quad \text{where} \quad 0 \leq \alpha_1, \alpha_2, \alpha_3 \leq 1, \alpha_3 \neq 0.
\]

\( \sigma_1 = 0, 1, \lambda = \frac{0}{a + b + c} \quad \text{and} \quad \sigma_2 = \{ 1, 0 \} \).

\( \sigma_{12} \)-closed are \( 0, 1, \lambda' = \frac{0}{a + b + c}, \mu = \frac{1}{a + b + c} \). Then \((1,2)^{\ast}\text{-fg closed are} \)

\[
0, 1, \lambda' = \frac{0}{a + b + c}, \mu = \frac{1}{a + b + c} \quad \text{where} \quad 0 \leq \alpha_1, \alpha_2, \alpha_3 \leq 1.
\]

Let \( f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2) \) be the identity map. Then \( f \) is an \((1,2)^{\ast}\text{-fg-closed map} \).

Let \( (Z, \eta_1, \eta_2) \) be a fuzzy bitopological space where \( Z = \{ a, b, c \} \).

\( \eta_1 = 0, 1, \lambda = \frac{0.5}{a + b + c} \quad \text{and} \quad \eta_2 = \{ 1, 0 \} \).

\( \eta_{12} \)-closed are \( 0, 1, \lambda' = \frac{0}{a + b + c}, \mu = \frac{1}{a + b + c} \). Then \((1,2)^{\ast}\text{-fg closed are} \)

\[
0, 1, \lambda' = \frac{0}{a + b + c}, \mu = \frac{1}{a + b + c} \quad \text{where} \quad 0 \leq \alpha_1, \alpha_2, \alpha_3 \leq 1, \alpha_3 \neq 0.
\]
Let \( g : (Y, \sigma_1, \sigma_2) \rightarrow (Z, \eta_1, \eta_2) \) be the identity map. Then both \( f \) and \( g \) are \((1,2)^*\)-fg-closed maps but their composition \( g \circ f \) is not \((1,2)^*\)-fg-closed. Since for the \( \tau_{12} \) closed set 
\[
\begin{align*}
0 + \frac{1}{a} + \frac{0}{b} + \frac{0}{c} = 0 + \frac{1}{a} + \frac{0}{b} + \frac{0}{c}
\end{align*}
\]
which is not and \((1,2)^*\)-fg-closed set in \( Z \).

Corollary 1.3.9

Let \( f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2) \) be \((1,2)^*\)-fg-closed and \( g : (Y, \sigma_1, \sigma_2) \rightarrow (Z, \eta_1, \eta_2) \) be \((1,2)^*\)-fg-closed and \((1,2)^*\)-fg-irresolute, then their composition \( g \circ f \) is \((1,2)^*\)-fg-closed.

Proposition 1.3.10

Let \( f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2) \) and \( g : (Y, \sigma_1, \sigma_2) \rightarrow (Z, \eta_1, \eta_2) \) be \((1,2)^*\)-fg-closed maps where \( Y \) is a \( (1,2)^*\)-g-space. Then their composition \( g \circ f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2) \) is \((1,2)^*\)-fg-closed.

Proof

Let \( \tau_{12} \)-closed set \( A \) be \((1,2)^*\)-fg-closed in \( X \). Then by hypothesis \( f(A) \) is \((1,2)^*\)-fg-closed in \( Y \). Since \( g \) is \((1,2)^*\)-fg-closed and \((1,2)^*\)-fg-irresolute by Proposition 1.3.5, \( g(f(A)) = (g \circ f)(A) \) is \((1,2)^*\)-fg-closed in \( Z \) and therefore \( g \circ f \) is \((1,2)^*\)-fg-closed.

Proposition 1.3.11

If \( f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2) \) is \((1,2)^*\)-fg-closed, \( g : (Y, \sigma_1, \sigma_2) \rightarrow (Z, \eta_1, \eta_2) \) is \((1,2)^*\)-fg-closed (resp. \((1,2)^*\)-fg-closed, \((1,2)^*\)-fg-closed, \((1,2)^*\)-fg-closed) and \( Y \) be a \((1,2)^*\)-g-space, then their composition \( g \circ f : (X, \tau_1, \tau_2) \rightarrow (Z, \eta_1, \eta_2) \) is \((1,2)^*\)-fg-closed (resp. \((1,2)^*\)-fg-closed, \((1,2)^*\)-fg-closed, \((1,2)^*\)-fg-closed and \((1,2)^*\)-fg-closed).

Proof

Similar to Proposition 1.3.10.

Proposition 1.3.12

Let \( f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2) \) be a \((1,2)^*\)-fuzzy closed map and \( g : (Y, \sigma_1, \sigma_2) \rightarrow (Z, \eta_1, \eta_2) \) be \((1,2)^*\)-fg-closed map, then their composition \( g \circ f : (X, \tau_1, \tau_2) \rightarrow (Z, \eta_1, \eta_2) \) is \((1,2)^*\)-fg-closed.

Proof

Similar to Proposition 1.3.10.

Remark 1.3.13

If \( f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2) \) is an \((1,2)^*\)-fg-closed and \( g : (Y, \sigma_1, \sigma_2) \rightarrow (Z, \eta_1, \eta_2) \) is \((1,2)^*\)-fuzzy closed, then their composition need not be an \((1,2)^*\)-fg-closed map as seen from the following example.

Example 1.3.14

Let \( (X, \tau_1, \tau_2) \) be a fuzzy bitopological space where \( X = \{a, b, c\} \).

\[
\begin{align*}
\tau_1 &= 0,1, \lambda = \frac{1}{a} + \frac{0}{b} + \frac{0}{c}, \\
\mu &= \frac{0}{a} + \frac{1}{b} + \frac{1}{c} \\
\tau_{12} &= \{0,1\}
\end{align*}
\]

\( \sigma_1 \)-closed are
\[
\begin{align*}
0,1, \lambda' = \frac{0}{a} + \frac{1}{b} + \frac{1}{c}, \\
\mu' = \frac{0}{a} + \frac{1}{b} + \frac{1}{c} \\
\text{Then, } (1,2)^*-\text{fg closed are}
\end{align*}
\]

\[
\begin{align*}
0,1, \lambda' = \frac{0}{a} + \frac{1}{b} + \frac{0}{c}, \\
\mu' = \frac{0}{a} + \frac{1}{b} + \frac{1}{c}, \\
\text{where } 0 \leq \alpha_1, \alpha_2, \alpha_3 \leq 1, \alpha_2 \neq 0.
\end{align*}
\]

Let \( (Y, \sigma_1, \sigma_2) \) be a fuzzy bitopological space where \( Y = \{a, b, c\} \).

\[
\begin{align*}
\sigma_1 &= 0,1, \lambda = \frac{1}{a} + \frac{0}{b} + \frac{0}{c}, \\
\mu &= \frac{0}{a} + \frac{1}{b} + \frac{1}{c} \\
\sigma_{12} &= \{0,1\} \\
\text{Then, } (1,2)^*-\text{fg closed are}
\end{align*}
\]

\[
\begin{align*}
0,1, \lambda' = \frac{0}{a} + \frac{1}{b} + \frac{1}{c}, \\
\mu' = \frac{0}{a} + \frac{1}{b} + \frac{1}{c} \\
\text{where } 0 \leq \alpha_1, \alpha_2, \alpha_3 \leq 1.
\end{align*}
\]

Let \( f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2) \) be the identity map. Then \( f \) is an \((1,2)^*\)-fg-closed map.

Let \( (Z, \eta_1, \eta_2) \) be a fuzzy bitopological space where \( Z = \{a, b, c\} \).

\[
\begin{align*}
\eta_1 &= 0,1, \lambda = \frac{1}{a} + \frac{0}{b} + \frac{1}{c}, \\
\mu &= \frac{0}{a} + \frac{1}{b} + \frac{1}{c} \\
\eta_2 &= \{0,1\}.
\end{align*}
\]
Let \( f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2) \) and \( g : (Y, \sigma_1, \sigma_2) \rightarrow (Z, \eta_1, \eta_2) \) be two maps such that their composition \( g \circ f : (X, \tau_1, \tau_2) \rightarrow (Z, \eta_1, \eta_2) \) is an \((1,2)^*\)-fuzzy closed map. Then the following statements are true.

(i) If \( f \) is \((1,2)^*\)-fuzzy continuous and surjective, then \( g \) is \((1,2)^*\)-fuzzy closed.

(ii) If \( g \) is \((1,2)^*\)-irresolute and injective, then \( f \) is \((1,2)^*\)-fuzzy closed.

(iii) If \( f \) is \((1,2)^*\)-fuzzy continuous, surjective, and \((X, \tau) \) is a \((1,2)^*\)-\(T_\alpha\)-space, then \( g \) is \((1,2)^*\)-fuzzy closed.

(iv) If \( g \) is strongly \((1,2)^*\)-fuzzy continuous and injective, then \( f \) is \((1,2)^*\)-fuzzy closed.

**Proof**

(i) Let \( A \) be a \( \sigma_1, \sigma_2\)-closed set of \( Y \). Since \( f \) is \((1,2)^*\)-fuzzy continuous, \( f^1(A) \) is \( \tau_1, \tau_2\)-closed in \( X \) and since \( g \circ f \) is \((1,2)^*\)-fuzzy closed, \( (g \circ f)(A) \) is \((1,2)^*\)-fuzzy closed in \( Z \). That is \( g(A) \) is \((1,2)^*\)-fuzzy closed in \( Z \), since \( f \) is surjective. Therefore \( g \) is \( (1,2)^*\)-fuzzy closed map.

(ii) Let \( B \) be a \( \tau_1, \tau_2\)-closed set of \( X \). Since \( g \circ f \) is \((1,2)^*\)-fuzzy closed, \( (g \circ f)(B) \) is \((1,2)^*\)-fuzzy closed in \( Z \). Since \( g \) is \((1,2)^*\)-irresolute, \( f^1((g \circ f)(B)) \) is \((1,2)^*\)-fuzzy closed in \( Y \). That is \( f(B) \) is \((1,2)^*\)-fuzzy closed in \( Y \), since \( g \) is injective. Thus \( f \) is \((1,2)^*\)-fuzzy closed.

(iii) Let \( C \) be a \( \sigma_1, \sigma_2\)-closed set of \( Y \). Since \( f \) is \((1,2)^*\)-fuzzy continuous, \( f^1(C) \) is \((1,2)^*\)-fuzzy closed in \( X \). Since \( X \) is a \((1,2)^*\)-\(T_\alpha\)-space, \( f^1(C) \) is \( \tau_1, \tau_2\)-closed in \( X \) and so as in (i), \( g \) is \((1,2)^*\)-fuzzy closed map.

(iv) Let \( D \) be a \( \tau_1, \tau_2\)-closed set of \( X \). Since \( g \circ f \) is \((1,2)^*\)-fuzzy closed, \( (g \circ f)(D) \) is \((1,2)^*\)-fuzzy closed in \( Z \). Since \( g \) is strongly \((1,2)^*\)-fuzzy continuous, \( (g \circ f)(D) \) is \( \sigma_1, \sigma_2\)-closed in \( Y \). That is \( f(D) \) is \( \sigma_1, \sigma_2\)-closed in \( Y \), since \( g \) is injective. Therefore \( f \) is \((1,2)^*\)-fuzzy closed map.

In the next theorem we show that \((1,2)^*\)-fuzzy normality is preserved under \((1,2)^*\)-fuzzy continuous, \((1,2)^*\)-fuzzy closed maps.

**Theorem 1.3.15**

A set \( A \) of \( X \) is \((1,2)^*\)-fuzzy-open if and only if \( F \subseteq \tau_{1,2}\)-int\( (A) \) whenever \( F \) is \((1,2)^*\)-fuzzy closed and \( F \subseteq A \).

**Theorem 1.3.16**

A set \( A \) of \( X \) is \((1,2)^*\)-fuzzy normal if and only if \( F \subseteq \tau_{1,2}\)-int\( (A) \) whenever \( F \) is \((1,2)^*\)-fuzzy closed and \( F \subseteq A \).

**Theorem 1.3.17**

If \( f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2) \) is a \((1,2)^*\)-fuzzy continuous, \((1,2)^*\)-fuzzy closed map from a \((1,2)^*\)-fuzzy normal space \( X \) onto a space \( Y \), then \( Y \) is \((1,2)^*\)-fuzzy normal.

**Proof**

Let \( A \) and \( B \) be two disjoint \( \sigma_1, \sigma_2\)-closed subsets of \( Y \). Since \( f \) is \((1,2)^*\)-fuzzy continuous, \( f^1(A) \) and \( f^1(B) \) are disjoint \( \tau_{1,2}\)-closed sets of \( X \). Since \( X \) is \((1,2)^*\)-fuzzy normal, there exist disjoint \( \tau_{1,2}\)-open sets \( U \) and \( V \) of \( X \) such that \( f^1(A) \subseteq U \) and \( f^1(B) \subseteq V \). Since \( f \) is \((1,2)^*\)-fuzzy closed, by Theorem 1.3.6, there exist disjoint \((1,2)^*\)-fuzzy open sets \( G \) and \( H \) in \( Y \) such that \( A \subseteq G, B \subseteq H, f^1(G) \subseteq U \) and \( f^1(H) \subseteq V \). Since \( U \) and \( V \) are disjoint, \( \sigma_1, \sigma_2\)-int\( (G) \) and \( \sigma_1, \sigma_2\)-int\( (H) \) are disjoint \( \sigma_1, \sigma_2\)-open sets in \( Y \). Since \( A \) is \( \sigma_1, \sigma_2\)-closed, \( A \) is \((1,2)^*\)-fuzzy closed and therefore we have by Theorem 1.3.16, \( A \subseteq \sigma_1, \sigma_2\)-int\( (G) \). Similarly \( B \subseteq \sigma_1, \sigma_2\)-int\( (H) \) and hence \( Y \) is \((1,2)^*\)-fuzzy normal.

**Definition 1.3.18**

A map \( f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2) \) is said to be an \((1,2)^*\)-fuzzy-open map if the image \( f(A) \) is \((1,2)^*\)-fuzzy-open in \( Y \) for each \( \tau_{1,2}\)-open set \( A \) in \( X \).
Proof
(i) $\Rightarrow$ (ii). Suppose $f$ is (1,2)*-fg-open. Let $A \subseteq X$. Then $\tau_{1,2}$-int$(A)$ is $\tau_{1,2}$-open in $X$ and so $f(\tau_{1,2}$-int$(A))$ is (1,2)*-fg-open in $Y$. We have $f(\tau_{1,2}$-int$(A)) \subseteq f(A)$. Therefore by Proposition 1.3.2, $f(\tau_{1,2}$-int$(A)) \subseteq (1,2)*$-g-int$(f(A))$.

(ii) $\Rightarrow$ (iii). Suppose (ii) holds. Let $x \in X$ and $U$ be an arbitrary $\tau_{1,2}$-neighborhood of $x$ in $X$. Then there exists an $\tau_{1,2}$-open set $G$ such that $x \in G \subseteq U$. By assumption, $f(G) = f(\tau_{1,2}$-int$(G)) \subseteq (1,2)*$-g-int$(f(G))$. This implies $f(G) = (1,2)*$-g-int$(f(G))$. By Proposition 1.3.2, we have $f(G)$ is (1,2)*-fg-open in $Y$. Further, $f(x) \in f(G) \subseteq f(U)$ and so (iii) holds, by taking $W = f(G)$.

(iii) $\Rightarrow$ (i). Suppose (iii) holds. Let $U$ be any $\tau_{1,2}$-open set in $X$, $x \in U$ and $f(x) = y$. Then $y \in f(U)$ and for each $y \in f(U)$, by assumption there exists an (1,2)*-g-neighborhood $W_y$ of $y$ in $Y$ such that $W_y \subseteq f(U)$. Since $W_y$ is an (1,2)*-g-neighborhood of $y$, there exists an (1,2)*-fg-open set $V_y$ in $Y$ such that $y \in V_y \subseteq W_y$. Therefore, $f(U) = \bigcup \{V_y : y \in f(U)\}$ is an (1,2)*-fg-open set in $Y$. Thus $f$ is an (1,2)*-fg-open map.

**Theorem 1.3.21**

A map $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is (1,2)*-fg-open if and only if for any subset $S$ of $Y$ and for any $\tau_{1,2}$-closed set $F$ containing $f^{-1}(S)$, there exists an (1,2)*-fg-closed set $K$ of $Y$ containing $S$ such that $f^{-1}(K) \subseteq F$.

**Proof**

Similar to Theorem 1.3.6.

**Corollary 1.3.22**

A map $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is (1,2)*-fg-open if and only if $f^{-1}((1,2)*$-g-cl$(B)) \subseteq \tau_{1,2}$-cl$(f^{-1}(B))$ for each subset $B$ of $Y$.

**Proof**

Suppose that $f$ is (1,2)*-fg-open. Then for any $B \subseteq Y$, $f^{-1}(B) \subseteq \tau_{1,2}$-cl$(f^{-1}(B))$. By Theorem 1.3.21, there exists an (1,2)*-fg-closed set $K$ of $Y$ such that $B \subseteq K$ and $f^{-1}(K) \subseteq \tau_{1,2}$-cl$(f^{-1}(B))$. Therefore, $f^{-1}((1,2)*$-g-cl$(B)) \subseteq f^{-1}(K) \subseteq f^{-1}(\tau_{1,2}$-cl$(f^{-1}(B)))$, since $K$ is an (1,2)*-fg-closed set in $Y$.

Conversely, let $S$ be any subset of $Y$ and $F$ be any $\tau_{1,2}$-closed set containing $f^{-1}(S)$. Put $K = (1,2)*$-g-cl$(S)$. Then $K$ is an (1,2)*-fg-closed set and $S \subseteq K$. By assumption, $f^{-1}(K) = f^{-1}((1,2)*$-g-cl$(S)) \subseteq \tau_{1,2}$-cl$(f^{-1}(S)) \subseteq F$ and therefore by Theorem 1.3.21, $f$ is (1,2)*-fg-open.

Finally in this section, we define another new class of maps called (1,2)*-fg*-closed maps which are stronger than (1,2)*-fg-closed maps.

**Definition 1.3.23**

A map $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to be (1,2)*-fg*-closed if the image $f(A)$ is (1,2)*-fg-closed in $Y$ for every (1,2)*-fg-closed set $A$ in $X$.

**Remark 1.3.24**

Since every $\tau_{1,2}$-closed set is an (1,2)*-fg-closed set we have (1,2)*-fg*-closed map is an (1,2)*-fg-closed map. The converse is not true in general as seen from the following example.

**Example 1.3.25**

Let $(Y, \sigma_1, \sigma_2)$ be a fuzzy bitopological space where $Y = \{a, b, c\}$.

$$\sigma_1 = 0,1, \lambda = \frac{1}{a} + \frac{0}{b} + \frac{0}{c}$$

and $\sigma_2 = [0,1]$.

$\sigma_{12}$-closed are

\[ 0,1, \lambda' = \frac{0}{a} + \frac{1}{b} + \frac{1}{c} \]

Then (1,2)*-fg closed are

\[ 0,1, \lambda' = \frac{0}{a} + \frac{1}{b} + \frac{1}{c} \] where $0 \leq \alpha_1, \alpha_2, \alpha_3 \leq 1$.

Let $(Z, \eta_1, \eta_2)$ be a fuzzy bitopological space where $Z = \{a, b, c\}$.

$$\eta_1 = 0,1, \lambda = \frac{0.5}{a} + \frac{0}{b} + \frac{0}{c}$$

and $\eta_2 = [0,1]$.

$\eta_{12}$-closed are

\[ 0,1, \lambda' = \frac{0}{a} + \frac{0.5}{b} + \frac{1}{c} \]

Then (1,2)*-fg closed are

\[ 0,1, \lambda' = \frac{0}{a} + \frac{0.5}{b} + \frac{1}{c} \] where $0 \leq \alpha_1, \alpha_2, \alpha_3 \leq 1, \alpha_3 \neq 0$.

Let $g : (Y, \sigma_1, \sigma_2) \to (Z, \eta_1, \eta_2)$ be the identity map. Then $g$ is (1,2)*-fg closed map but not (1,2)*-fg*-closed map.

Since

\[ \frac{1}{a} + \frac{0}{b} + \frac{0}{c} \]

is (1,2)*-fg-closed set in $X$, but its image under $g$ is

\[ \frac{1}{a} + \frac{0}{b} + \frac{0}{c} \]

which is not (1,2)*-fg-closed set in $Z$.

**Proposition 1.3.26**

A map $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is (1,2)*-fg*-closed if and only if $(1,2)*$-g-cl$(f(A)) \subseteq f(1,2)*$-g-cl$(A)$ for every subset $A$ of $X$.

**Proof**

Similar to Proposition 1.3.4.
Analogous to $(1,2)^*fg^*$-closed map we can also define $(1,2)^*fg^*$-open map.

**Proposition 1.3.27**

For any bijection $f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$, the following statements are equivalent:

(i) $f$ is $(1,2)^*fg^*$-irresolute.

(ii) $f$ is $(1,2)^*fg^*$-open.

(iii) $f$ is $(1,2)^*fg^*$-closed map.

**Proof**

Similar to Proposition 1.3.19.

**Proposition 1.3.28**

If $f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ is $(1,2)^*fg^*$-irresolute and $(1,2)^*fg^*$-closed, then it is an $(1,2)^*fg^*$-closed map.

**Proof**

The proof follows from Proposition 1.3.7.

**1.4. $(1,2)^*fg^*$-Homeomorphisms**

The notion of $(1,2)^*fg^*$-homeomorphisms plays a very important role in fuzzy bitopological spaces. By definition, an $(1,2)^*fg^*$-homeomorphism between two fuzzy bitopological spaces $(X, \tau_1, \tau_2)$ and $(Y, \sigma_1, \sigma_2)$ is a bijective map $f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ when $f$ and $f^{-1}$ are $(1,2)^*fg^*$-continuous.

We introduce the following definition:

**Definition 1.4.1**

A bijection $f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ is said to be

(i) $(1,2)^*fg^*$-homeomorphism if $f$ is both $(1,2)^*fg^*$-continuous and $(1,2)^*fg^*$-open.

(ii) $(1,2)^*fg^*$-homeomorphism if both $f$ and $f^{-1}$ are $(1,2)^*fg^*$-irresolute.

We denote the family of all $(1,2)^*fg^*$-homeomorphisms of a fuzzy bitopological space $(X, \tau_1, \tau_2)$ onto itself by $(1,2)^*fg^*h(X)$.

**Theorem 1.4.2**

Let $f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ be a bijective $(1,2)^*fg^*$-continuous map. Then the following are equivalent:

(i) $f$ is an $(1,2)^*fg^*$-open map.

(ii) $f$ is an $(1,2)^*fg^*$-homeomorphism.

(iii) $f$ is an $(1,2)^*fg^*$-closed map.

**Proof**

Follows from Proposition 1.3.19.

**Proposition 1.4.3**

If $f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ and $g : (Y, \sigma_1, \sigma_2) \rightarrow (Z, \eta_1, \eta_2)$ are $(1,2)^*fg^*$-homeomorphisms, then their composition $g \circ f : (X, \tau_1, \tau_2) \rightarrow (Z, \eta_1, \eta_2)$ is also $(1,2)^*fg^*$-homeomorphism.

**Proof**

Let $U$ be $(1,2)^*fg^*$-open set in $(Z, \eta_1, \eta_2)$. Now, $(g \circ f)^{-1}(U) = f^{-1}(g^{-1}(U)) = f^{-1}(V)$, where $V = g^{-1}(U)$. By hypothesis, $V$ is $(1,2)^*fg^*$-open in $Y$ and so again by hypothesis, $f^{-1}(V)$ is $(1,2)^*fg^*$-open in $X$. Therefore, $g \circ f$ is $(1,2)^*fg^*$-irresolute.

Also, for an $(1,2)^*fg^*$-closed set $G$ in $X$, we have $(g \circ f)(G) = g(f(G)) = g(W)$, where $W = f(G)$. By hypothesis $f(G)$ is $(1,2)^*fg^*$-open in $Y$ and so again by hypothesis, $g(f(G))$ is $(1,2)^*fg^*$-open in $Z$. That is $(g \circ f)(G)$ is $(1,2)^*fg^*$-open in $Z$ and therefore $(g \circ f)^{-1}$ is $(1,2)^*fg^*$-irresolute. Hence $g \circ f$ is a $(1,2)^*fg^*$-homeomorphism.

**Theorem 1.4.4**

The set $(1,2)^*fg^*$-h$(X)$ is a group under the operation of composition of maps.

**Proof**

Define a binary operation $\ast : (1,2)^*fg^*$-h$(X) \times (1,2)^*fg^*$-h$(X) \rightarrow (1,2)^*fg^*$-h$(X)$ by $f \ast g = g \circ f$ for all $f, g \in (1,2)^*fg^*$-h$(X)$ and $o$ is the usual operation of composition of maps. Then by Proposition 1.3.1, $g \circ f \in (1,2)^*fg^*$-h$(X)$. We know that the composition of maps is associative and the identity map $I : (X, \tau_1, \tau_2) \rightarrow (X, \tau_1, \tau_2)$ belonging to $(1,2)^*fg^*$-h$(X)$ serves as the identity element. If $f \in (1,2)^*fg^*$-h$(X)$, then $f^{-1} \in (1,2)^*fg^*$-h$(X)$ such that $f \circ f^{-1} = f^{-1} \circ f = I$ and so inverse exists for each element of $(1,2)^*fg^*$-h$(X)$. Therefore, $(1,2)^*fg^*$-h$(X)$ is a group under the operation of composition of maps.

**Theorem 1.4.5**

Let $f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ be an $(1,2)^*fg^*$-homeomorphism. Then $f$ induces an $(1,2)^*fg^*$-homeomorphism from the group $(1,2)^*fg^*$-h$(X)$ onto the group $(1,2)^*fg^*$-h$(Y)$.

**Proof**

Using the map $f$, we define a map $\theta : (1,2)^*fg^*$-h$(X) \rightarrow (1,2)^*fg^*$-h$(X)$ by $\theta(h) = f \circ h \circ f^{-1}$ for every $h \in (1,2)^*fg^*$-h$(X)$. Then $\theta$ is a bijection. Further, for all $h_1, h_2 \in (1,2)^*fg^*$-h$(X)$, we have $\theta(h_1 \circ h_2) = \theta(h_1) \circ \theta(h_2) = (f \circ h_1 \circ f^{-1}) \circ (f \circ h_2 \circ f^{-1}) = \theta(h_1 \circ h_2)$.

Therefore, $\theta$ is a $(1,2)^*fg^*$-homeomorphism and so it is an $(1,2)^*fg^*$-isomorphism induced by $f$.

**Theorem 1.4.6**

$(1,2)^*fg^*$-homeomorphism is an equivalence relation in the collection of all bitopological spaces.

**Proof**

Reflexivity and symmetry are immediate and transitivity follows from Proposition 1.4.3.

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**References**


