Mathematical study of the effect of industrialization on the resource biomass under going harvesting and diffusion in heterogeneous habitat
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ABSTRACT
In this chapter, a Mathematical model is proposed to the study of depletion of a uniformly distributed forest biomass caused by different levels of industrialization and population in two adjoining regions of the habitat. Industrialization dependent, constant, instantaneous, and periodic emissions of pollutant into the environment are taken into consideration. Criteria for local stability, instability, and global stability of non-negative equilibrium are obtained in the absence of diffusion and in presence of diffusion. A model of a single species population living in two patch habitats with migration between them across a barrier was proposed by Freedman and Waltman [5]. The model was extended in [17,19] to include the case where animal species leaving one habitat does not necessarily reach the other habitat, the existence of a positive equilibrium as a function of barrier strengths was examined. Also Freedman [7] studied a single species diffusion model by assuming that the habitat consists of two patches and has shown that there exists a positive, monotonic, continuous non uniform steady state solution that is linearly asymptotically stable under both reservoir and no-flux boundary conditions.

Introduction
Depletion of forest resource by industrialization and rapid growth of population, particularly in the third world country is of grave concern. A typical example in this regard is the degradation of the forestry resources in the Doon Valley located in the foothills of Himalayas, Uttaranchal, India. Here the degradation of forest has been caused mainly by limestone quarries, paper, other wood-based industries and associated population growth [14]. [21] have proposed a Mathematical model for forest degradation caused by resource independent industrialization population by considering the spatial distribution of both the forest biomass as well as the density of industrialization by studying the behavior of uniform steady state solution.

It may be pointed out here that in real ecological situations when forest is degraded by industrialization distributed spatially, patchiness is caused in the forest habitat. It is worth noting that little efforts have been made to study such systems using mathematical models [6, 16, 21, and 23]. How ever in [21], has not considered the effect of patchiness caused by industrialization. Further [2-4,7] studied a single species diffusion model by assuming that the habitat consists of two adjoining patches and studied the behavior of steady state distribution and local asymptotically stable conditions.

Biotic populations are usually distributed non-uniformly in their habitat, and the distribution is often patchy, due to patchiness of the habitat which arises from a variety of mechanisms and processes under various conditions including deforestation in the case of a forest habitat. It would, thus, seem natural to study the population dynamics of a single species by including diffusion effects, in a patchy habitat. Many investigators [1,5,6,8-13] have shown that in a homogeneous habitat, the diffusion increases the stability of the system, but this may not always be true, if the habitat is patchy [7,15,17,18]. A model of a single species population living in two patch habitats with migration between them across a barrier was proposed by Freedman and Waltman [5]. The model was extended in [17,19] to include the case where animal species leaving one habitat does not necessarily reach the other habitat, the existence of a positive equilibrium as a function of barrier strengths was examined. Also Freedman [7] studied a single species diffusion model by assuming that the habitat consists of two patches and has shown that there exists a positive, monotonic, continuous non uniform steady state solution that is linearly asymptotically stable under both reservoir and no-flux boundary conditions.

Mathematical Model
We consider a forest habitat $0 \leq s \leq L_2$ linearly distributed, where forest resources are depleted by different levels of industrialization in the above adjoining regions $0 \leq s \leq L_4$ and $L_4 \leq s \leq L_2$ where $L_0 = 0$ and $L_4$ is the interface of two regions. Let $R_i(s, t)$ and $I_i(s, t)$, $i(=1,2)$ be respectively the densities of resource biomass and industrialization (population) pressure at location $s$ and time $t$ in the above mentioned $i$th regions [see figure 1]. It is assumed that $R_i(s, t)$ grows logistically in both the region with the same intrinsic growth rate and carrying capacity, i.e. in absence of industrialization leading to a uniform spatial distribution. Since the levels of industrialization are assumed to be different in these two regions, the growth rate $\alpha_i$ and carrying capacity...
Keeping all these in view, our proposed mathematical model is given by the following system of partial differential equations:

\[
\frac{\partial R_i}{\partial t} = r R_i - \frac{\beta_i R_i^2}{c} - \beta_i I_i + R_i P_i(s, t) + d_i \frac{\partial^2 R_i}{\partial s^2} \tag{1.1}
\]

\[
\frac{\partial I_i}{\partial t} = \alpha_i I_i - \alpha_i I_i^2 + D_i \frac{\partial^2 I_i}{\partial s^2} \tag{1.2}
\]

Where

- \(r\) = Intrinsic growth rate of resource biomass in both regions
- \(c\) = Carrying capacity of resource biomass in both regions
- \(\beta_i\) = Depletion coefficient in the \(i\)th region; \(i=1, 2\)
- \(d_i\) = Diffusion coefficient of \(R_i(s, t)\) in the \(i\)th region; \(i=1, 2\)
- \(D_i\) = Diffusion coefficient of \(I_i(s, t)\) in the \(i\)th region; \(i=1, 2\)
- \(\alpha_i\) = Growth rate of \(I_i(s, t)\) in the \(i\)th region; \(i=1, 2\)
- \(P_i(R_i(s, t))\) = Harvesting functional response function such that \(P_i(0) > 0\) for \(P_i'(R_i) < 0\) and when the habitat has carrying capacity \(C\) in the \(i\)th patch, then \(P_i(C) = 0\), \(\forall i\).

Uniform steady states

To analyse the model (1.1) and (1.2), we have taken the harvesting function in the following form:

\[
P_i(R_i(s, t)) = a R_i
\]

Using this value in (1.1), the positive uniform equilibrium point \(E_i\), \((R_i^*, I_i^*)\) is given by following equations:

\[
R_i^* = \left[\frac{r \alpha_i - \alpha_i \beta_i}{W_i \alpha_i} - \alpha_i \beta_i \right] > 0 \quad \text{if} \quad r \alpha_i > \alpha_i \beta_i \tag{2.1}
\]

\[
I_i^* = \frac{\alpha_i}{\alpha_i} > 0 \tag{2.2}
\]

Where \(W_i = \frac{r}{c} + a_i > 0\) \(\forall i = 1, 2\)

From equation (2.1) it is clear that

\[
R_i^* = \frac{r - \beta_i I_i^*}{W_i} \leq \frac{r}{W_i} \tag{3.1}
\]

We also observe that in absence of industrialization (i.e. \(\beta_i = 0\), \(i=1, 2\))

\[R_1^* = R_2^* = \frac{r}{W_i}\]

This shows that the biomass is uniformly distributed in entire habitat.

The model (1.1) and (1.2) is studied by assuming the following initial, boundary and flux-matching conditions. The model is completed by assuming some positive initial distribution for forest resource biomass and industrialization, that is,

\[
R_i(s, 0) = \chi_i(s) > 0 \quad L_i - 1 < s < L_i \quad i = 1, 2 \tag{4.1}
\]

\[
I_i(s, 0) = \delta_i(s) > 0 \quad L_i - 1 < s < L_i \quad i = 1, 2 \tag{4.2}
\]

If the region is closed then there is no diffusion of industrialization and resource biomass across the boundary, no-flux boundary condition for forest resource biomass and industrialization are

\[
\left. \frac{\partial R_i(0, t)}{\partial s} \right|_{s=0} = 0 = \left. \frac{\partial R_i(L_i, t)}{\partial s} \right|_{s=L_i}, \quad \left. \frac{\partial I_i(0, t)}{\partial s} \right|_{s=0} = 0 = \left. \frac{\partial I_i(L_i, t)}{\partial s} \right|_{s=L_i} \tag{4.3}
\]

And finally considering the continuity and the flux-matching conditions at the interface \(S = L_i\) for \(R_i(s, t)\) and \(I_i(s, t)\)

\[
R_i(L_i, t) = R_{2-i}(L_i, t), \quad I_i(L_i, t) = I_{2-i}(L_i, t) \quad \forall t \geq 0 \tag{4.4}
\]

\[
I_i(L_i, t) = I_{2-i}(L_i, t), \quad D_i \left. \frac{\partial R_i}{\partial s} \right|_{s=L_i} = D_{2-i} \left. \frac{\partial R_{2-i}}{\partial s} \right|_{s=L_i}, \quad D_i \left. \frac{\partial I_i}{\partial s} \right|_{s=L_i} = D_{2-i} \left. \frac{\partial I_{2-i}}{\partial s} \right|_{s=L_i} \quad \forall t \geq 0 \tag{4.5}
\]

Linear and Non-linear stability analysis

**Theorem 1** The steady state solution of the system (1.1) and (1.2) with conditions (4.1-4.5) is locally asymptotically stable if the following conditions holds

(i) \(A_i = r - 2a R_i^* - \beta_i I_i^* \leq 0\)

(ii) \(B_i = \alpha_i - 2a \alpha \beta_i I_i^* \leq 0\)

(iii) \(\beta_i R_i^* \geq 4A_i B_i\)

**Proof.**

Let \(R_i(s, t) = R_i^* + m_i(s, t)\) and \(I_i(s, t) = I_i^* + n_i(s, t)\)

\[m_i(s, t) \text{ and } n_i(s, t) \text{ are the small perturbations around equilibrium states.}\]

Using (5.1) and (5.2), the linearised system of differential equations for the equilibrium point \(E_i\) is given by (6.1) and (6.2) as follows:

\[
\frac{\partial m_i}{\partial t} = m_i \left[ r - 2a R_i^* - \beta_i I_i^* \right] - m_i (\beta_i R_i^*) + d_i \frac{\partial^2 m_i}{\partial s^2} \tag{6.1}
\]

\[
\frac{\partial n_i}{\partial t} = n_i \left[ \alpha_i - 2a \alpha \beta_i I_i^* \right] + D_i \frac{\partial^2 n_i}{\partial s^2} \tag{6.2}
\]

Now consider the following Liapunov function

\[
V(s, t) = \frac{1}{2} \sum_{i=1}^{2} \int_{L_{i-1}}^{L_i} (m_i^2 + n_i^2) ds
\]

Its time derivative is given by

\[
\frac{\partial V}{\partial t} = \sum_{i=1}^{2} \int_{L_{i-1}}^{L_i} \left( m_i \frac{\partial m_i}{\partial t} + n_i \frac{\partial n_i}{\partial t} \right) ds
\]

Now using (7.1) and (7.2) in (8.1) we get:
\[ \frac{\partial V}{\partial t} = \sum_{i=1}^{L} \left[ \frac{m_{i}}{R_{i} + m_{i}} \frac{ \partial^{2} m_{i}}{\partial s^{2}} ds + \sum_{j=1}^{L} D_{ij} \frac{ n_{j}}{k_{ij} + n_{j}} \frac{ \partial^{2} n_{j}}{\partial s^{2}} ds \right] + \sum_{j=1}^{L} \frac{D_{ij}}{k_{ij} + n_{j}} \frac{ \partial^{2} n_{j}}{\partial s^{2}} ds \]

(7.2)

Using boundary and flux-matching conditions for forest resource and industrialization density (4.3)-(4.5), we get

\[ \frac{\partial V}{\partial t} = \sum_{i=1}^{L} \frac{m_{i}}{R_{i} + m_{i}} \frac{ \partial^{2} m_{i}}{\partial s^{2}} ds = -\sum_{i=1}^{L} \frac{D_{ij}}{k_{ij} + n_{j}} \frac{ \partial^{2} n_{j}}{\partial s^{2}} ds \]

(8.1)

\[ \frac{\partial V}{\partial t} = \sum_{i=1}^{L} \frac{n_{i}}{k_{i} + n_{i}} \frac{ \partial^{2} n_{i}}{\partial s^{2}} ds = -\sum_{j=1}^{L} D_{ij} \frac{ \partial^{2} n_{j}}{\partial s^{2}} ds \]

(8.2)

Using Sylvester’s criteria for (8.2) and choosing

(i) \( A_{i} = r - 2W_{i} R_{i}^{*} - \beta R_{i}^{*} \leq 0 \)

(ii) \( B_{i} = \alpha_{l} - 2 \alpha_{0} I_{i}^{*} \leq 0 \)

(iii) \( \beta_{i}^{*} R_{i}^{*} \leq 4A_{i} B_{i} \)

It is shown that \( \frac{\partial V}{\partial t} \) is negative definite and hence it is proved that \( E_{i} \) is locally asymptotically stable.

**Theorem-2** The steady state solution of the system (1.1) and (1.2) with conditions (4.1-4.5) is globally asymptotically stable if the following condition \( \beta_{i}^{*} < 4\alpha_{0} W_{i} \) holds.

**Proof:** Using the same transformation as taken in (5.1) and (5.2), the nonlinearised system of differential equations for the equilibrium point \( E_{i} \) is given below by (9.1) and (9.2)

\[ \frac{\partial m_{i}}{\partial t} = (R_{i}^{*} + m_{i})[ -W_{i} m_{i} - \beta R_{i}^{*} ] + d_{i} \frac{\partial^{2} m_{i}}{\partial s^{2}} \]

(9.1)

\[ \frac{\partial n_{i}}{\partial t} = (I_{i}^{*} + n_{i})[ -\alpha_{l} n_{i} ] + D_{ij} \frac{\partial^{2} n_{j}}{\partial s^{2}} \]

(9.2)

Now consider the following Liapunov function

\[ V(s,t) = \sum_{i=1}^{L} \left[ \frac{m_{i} - R_{i}^{*}}{1 + m_{i}} \ln \left( 1 + \frac{m_{i}}{R_{i}^{*}} \right) + \frac{n_{i} - I_{i}^{*}}{1 + n_{i}} \ln \left( 1 + \frac{n_{i}}{I_{i}^{*}} \right) \right] ds \]

Its time derivative is given by

\[ \frac{\partial V}{\partial t} = \sum_{i=1}^{L} \frac{m_{i}}{R_{i} + m_{i}} \frac{ \partial m_{i}}{\partial t} + \frac{n_{i}}{I_{i}^{*} + n_{i}} \frac{ \partial n_{i}}{\partial t} \frac{ \partial V}{\partial s} \]

(10)

Now using (9.1) and (9.2) in (10) we get:

\[ \frac{\partial V}{\partial t} = \sum_{i=1}^{L} \left[ \frac{m_{i}}{R_{i} + m_{i}} \frac{ \partial m_{i}}{\partial t} + \frac{n_{i}}{I_{i}^{*} + n_{i}} \frac{ \partial n_{i}}{\partial t} \right] ds \]

\[ \frac{\partial m_{i}}{\partial t} = \sum_{i=1}^{L} \left[ \frac{m_{i}}{R_{i} + m_{i}} \frac{ \partial m_{i}}{\partial t} + \frac{n_{i}}{I_{i}^{*} + n_{i}} \frac{ \partial n_{i}}{\partial t} \right] ds \]

\[ \frac{\partial n_{i}}{\partial t} = \sum_{i=1}^{L} \left[ \frac{m_{i}}{R_{i} + m_{i}} \frac{ \partial m_{i}}{\partial t} + \frac{n_{i}}{I_{i}^{*} + n_{i}} \frac{ \partial n_{i}}{\partial t} \right] ds \]

Using boundary and flux-matching conditions for forest resource and industrialization density (4.3)-(4.5), we get

\[ \frac{\partial V}{\partial t} = \sum_{i=1}^{L} \frac{m_{i}}{R_{i} + m_{i}} \frac{ \partial m_{i}}{\partial s} ds + \sum_{j=1}^{L} D_{ij} \frac{ n_{j}}{k_{ij} + n_{j}} \frac{ \partial n_{j}}{\partial s} ds \]

(11.1)

And

\[ \frac{\partial V}{\partial t} = \sum_{i=1}^{L} \frac{n_{i}}{k_{i} + n_{i}} \frac{ \partial n_{i}}{\partial s} ds \]

(11.2)

Using Sylvester’s criteria that \( \frac{\partial V}{\partial t} \) is negative definite provided that the following condition is satisfied,

\[ \beta_{i}^{*} < 4\alpha_{0} W_{i} \]

Hence \( E_{i} \) is globally asymptotically stable if \( \beta_{i}^{*} < 4\alpha_{0} W_{i} \).

**Table-1**

<table>
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<tr>
<th>Parameters</th>
<th>Patch-1</th>
<th>Figure</th>
<th>Patch-2</th>
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<td>( r_{0} )</td>
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**Table-2**

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<td>( R_{1}^{*} )</td>
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<td>( I_{1}^{*} )</td>
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**MODEL IN HOMOGENEOUS HABITAT**

**MODEL WITHOUT DIFFUSION**

Here the forest resource biomass and industrialization (population) are uniformly distributed throughout the habitat \( (0 \leq s \leq L_{2}) \) with no diffusion. Let \( R \) and \( I \) denote the forest resource and Industrialization densities respectively at the location \( s \) and at any time \( t \). Then our model takes the following form

\[ \frac{\partial R}{\partial t} = rR - r_{0} R^{2} - \beta RI - aR^{2} \]

(12.1)

\[ \frac{\partial I}{\partial t} = cI - \alpha_{0} I^{2} \]

(12.2)

**Uniform equilibrium states**

The uniform positive equilibrium point \( E^{*}(R^{*}, I^{*}) \) becomes

\[ R^{*} = \frac{r_{0} - \alpha_{0} \beta}{W_{0}} > 0 \quad \text{if} \quad \alpha_{0} > \alpha \beta \]

(13.1)

\[ I^{*} = \frac{\alpha}{\alpha_{0}} > 0 \]

(13.2)

Where \( W = \frac{r_{0}}{c} + a \)

In this case the initial and boundary conditions of the forestry resource biomass and Industrialization become

\[ (s,0) = \chi(s) > 0 \quad , \quad L_{0} < s < L_{2} \]

(14.1)

\[ I(s,0) = \delta(s) > 0 , \quad L_{0} < s < L_{2} \]

(14.2)

And

\[ \frac{\partial I}{\partial t} = \frac{\partial R}{\partial t} \]

(14.3)
Linear and Non-linear stability analysis

Theorem 3 The steady state solution of the system (12.1) and (12.2) with conditions (14.1-14.3) is locally asymptotically stable if the following conditions holds

(i) \( A = r - 2WR^2 - \beta I^* \leq 0 \)
(ii) \( B = \alpha - 2\alpha_0 I^* \leq 0 \)
(iii) \( \beta^2 R^{*2} \leq 4AB \)

Proof

Let \( R = R^* + m(s,t) \)
\( I = I^* + n(s,t) \)

Where \( m(s,t) \) and \( n(s,t) \) are the small perturbations around equilibrium states. Using these, the linearised system of differential equations for the equilibrium point \( E^* \) is given by (16.1) and (16.2) as follows:

\[
\frac{\partial m}{\partial t} = m(-2WR^2 - \beta I^*) - n(\beta R^*)
\]
(16.1)

\[
\frac{\partial n}{\partial t} = n(\alpha - 2\alpha_0 I^*)
\]
(16.2)

Now consider the following Liapunov function

\[ V(s,t) = \frac{1}{2} \int_{t_0}^{t} (m^2 + n^2) ds \]

Its derivative is given by

\[ \frac{\partial V}{\partial t} = \frac{1}{2} \int_{t_0}^{t} \left( m \frac{\partial m}{\partial t} + n \frac{\partial n}{\partial t} \right) ds \]
(17)

Now using (16.1) and (16.2) in (17) we get:

\[
\frac{\partial V}{\partial t} = \int_{t_0}^{t} \left( m \frac{\partial m}{\partial t} + n \frac{\partial n}{\partial t} \right) ds
\]

Using Sylvester’s criteria and choosing

(i) \( A = r - 2WR^2 - \beta I^* \leq 0 \)
(ii) \( B = \alpha - 2\alpha_0 I^* \leq 0 \)
(iii) \( \beta^2 R^{*2} \leq 4AB \)

It is shown that \( \frac{\partial V}{\partial t} \) is negative definite and hence it is proved that \( E^* \) is locally asymptotically stable.

Theorem 4

The steady state solution of the system (12.1) and (12.2) with conditions (14.1-14.3) is global asymptotically stable if the following condition \( \beta^2 2 < 4\alpha \alpha_0 W_i^* \) holds.

Proof: Using the same transformation as taken in (15.1) and (15.2), the non linearised system of differential equations for the equilibrium point \( E^* \) is given below by (18.1) and (18.2)

\[
\frac{\partial m}{\partial t} = (R^* + m)(-2WR^2 - \beta I^*)
\]
(18.1)

\[
\frac{\partial n}{\partial t} = (I^* + n)(\alpha - 2\alpha_0 I^*)
\]
(18.2)

Now consider the following Liapunov function

\[ V(s,t) = \int_{t_0}^{t} \left[ m - R^* \ln \left( 1 + \frac{m}{R^*} \right) \right] + \left[ n - I^* \ln \left( 1 + \frac{n}{I^*} \right) \right] ds \]

Its derivative is given by

\[
\frac{\partial V}{\partial t} = \int_{t_0}^{t} \left[ \frac{\partial m}{\partial t} \frac{\partial m}{\partial t} + \frac{\partial n}{\partial t} \frac{\partial n}{\partial t} \right] ds
\]

Now using (20.1) and (20.2) in (21) we get:

\[
\frac{\partial V}{\partial t} = -\int_{t_0}^{t} \left[ \alpha I^* \ln \left( 1 + \frac{n}{I^*} \right) \right] ds
\]

It is then found using Sylvester’s criteria that \( \frac{\partial V}{\partial t} \) is negative definite proving \( E^* \) is global asymptotically stable in the region \((0, L_2)\) provided \( \beta^2 2 < 4\alpha_0 W_i^* \).

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Model with diffusion

Now we study the behavior of the uniform steady state solution of the model (1.1) and (1.2) with diffusion in a single homogeneous habitat. In this case we consider that both the Industrialization (population) and forest resource is not spatially uniformly distributed. Here \( R(s,t), I(s,t), d_i, D, \beta, \) becomes \( R(s,t), I(s,t), d, D, \beta \) respectively ; \( \forall s \in [0, L_2]. \) Then the model in this case can be written as

\[
\frac{\partial R}{\partial t} = rR - \frac{rR^2}{c} - \beta RI - R \frac{\partial^2 R}{\partial s^2} + d \frac{\partial^2 R}{\partial s^2}
\]
(19.1)

\[
\frac{\partial I}{\partial t} = \alpha I - \alpha_0 I^2 + D \frac{\partial^2 I}{\partial s^2}
\]
(19.2)

Uniform equilibrium states

To analyse the model (19.1) and (19.2), we have taken the harvesting function in the following form:

Using this value in (19.1), we get positive equilibrium points

\[
R^* = \frac{r_0 \alpha - \alpha \beta}{W_0} > 0 \quad \alpha_0 > \alpha \beta
\]
(20.1)
\[ I^* = \frac{\alpha}{\sigma_0} > 0 \quad (20.2) \]

Where \( W = \frac{\mu}{\alpha} + \alpha > 0 \)

In this case the initial and boundary conditions of the forestry resource biomass and industrialization become
\[ R(s,0) = \gamma(s) > 0 \quad , \quad L_0 < s < L_2 \quad (21.1) \]
\[ I(s,0) = \delta(s) > 0 \quad , \quad L_0 < s < L_2 \quad (21.2) \]

And
\[ \frac{\partial R(0,t)}{\partial s} = 0 = \frac{\partial R(L_2^*,0)}{\partial s} \quad , \quad \frac{\partial I(0,t)}{\partial s} = 0 = \frac{\partial I(L_2^*,0)}{\partial s} \quad (21.3) \]

**Linear and non-linear stability analysis**

**Theorem 5** The steady state solution of the system (19.1) and (19.2) with conditions (21.1-21.3) is locally asymptotically stable if it satisfies the same conditions as in theorem-3 .

**Proof:** Using the same transformation as taken in (15.1) and (15.2), the linearised system of differential equations for the equilibrium point \( E^* \) is given below by (22.1) and (22.2)
\[ \frac{\partial m}{\partial t} = m [ m^2 - 2WR^* - \beta I^*] - mR^* \quad D \frac{\partial^2 m}{\partial s^2} \quad (22.1) \]
\[ \frac{\partial n}{\partial t} = n [ \alpha - 2\alpha_0 I^*] + D \frac{\partial^2 n}{\partial s^2} \quad (22.2) \]

Now consider the following Liapunov function
\[ V(s, t) = \frac{1}{2} \int_{L_0}^{L_2} \left( m^2 + n^2 \right) ds \]

Its time derivative is given by
\[ \frac{\partial V}{\partial t} = \int_{L_0}^{L_2} \left( \frac{\partial m}{\partial t} + \frac{\partial n}{\partial t} \right) ds \quad (23.1) \]

Now using (22.1) and (22.2) in (23.1) we get:
\[ \frac{\partial V}{\partial t} = \int_{L_0}^{L_2} \left[ \left( m \frac{\partial m}{\partial t} + n \frac{\partial n}{\partial t} \right) \right] ds \]
\[ + D \int_{L_0}^{L_2} \left( \frac{\partial^2 m}{\partial s^2} \right)^2 + D \int_{L_0}^{L_2} \left( \frac{\partial^2 n}{\partial s^2} \right)^2 \]

Using boundary and flux-matching conditions for forest resource and industrialization density (21.1)-(21.3), we get
\[ \int_{L_0}^{L_2} \frac{\partial m}{\partial s} \left. \right|_{s=0} ds = - \int_{L_0}^{L_2} \left( \frac{\partial m}{\partial s} \right)^2 ds \quad (24.1) \]
\[ \int_{L_0}^{L_2} \frac{\partial n}{\partial s} \left. \right|_{s=0} ds = - \int_{L_0}^{L_2} \left( \frac{\partial n}{\partial s} \right)^2 ds \quad (24.2) \]

Using Sylvester’s criteria for (23.2) and choosing
(i) \( A = r - 2WR^* - \beta I^* \leq 0 \)
(ii) \( B = \alpha - 2\alpha_0 I^* \leq 0 \)
(iii) \( \beta^2 R^2 \leq 4AB \)

It is shown that \( \frac{\partial V}{\partial t} \) is negative definite and hence it is proved that \( E^* \) is locally asymptotically stable.

**Theorem 6** The steady state solution of the system (12.1) and (12.2) with conditions (14.1-14.3) is globally asymptotically stable if it satisfies the same condition as in theorem-4 .

**Proof:** Using the same transformation as taken in (15.1) and (15.2), the nonlinearised system of differential equations for the equilibrium point \( E^* \) is given below by

\[ \frac{\partial m}{\partial t} = (R^* + m) \left( Wm - \beta n \right) + D \frac{\partial^2 m}{\partial s^2} \quad (25.1) \]
\[ \frac{\partial n}{\partial t} = -\alpha_0 n(I^* + n) \left( -\alpha n \right) + D \frac{\partial^2 n}{\partial s^2} \quad (25.2) \]

Now consider the following Liapunov function
\[ V(s, t) = \int_{L_0}^{L_2} \left( \left( m - R^* \ln \left( 1 + \frac{m}{R^*} \right) \right) + \left( n - I^* \ln \left( 1 + \frac{n}{I^*} \right) \right) \right) ds \]

Its time derivative is given by
\[ \frac{\partial V}{\partial t} = \int_{L_0}^{L_2} \left( \frac{m}{R^* + m} \frac{\partial m}{\partial t} + \frac{n}{I^* + n} \frac{\partial n}{\partial t} \right) ds \quad (26) \]

Now using (25.1) and (25.2) in (26) we get:
\[ \frac{\partial V}{\partial t} = \int_{L_0}^{L_2} \left( \frac{m}{R^* + m} \frac{\partial m}{\partial t} + \frac{n}{I^* + n} \frac{\partial n}{\partial t} \right) ds + \int_{L_0}^{L_2} \left( \frac{\partial^2 m}{\partial s^2} \right)^2 ds \]

But
\[ \int_{L_0}^{L_2} \left( \frac{m}{R^* + m} \frac{\partial m}{\partial t} ds = - \int_{L_0}^{L_2} \left( \frac{\partial^2 m}{\partial s^2} \right)^2 ds \quad (27.1) \]
\[ \int_{L_0}^{L_2} \left( \frac{n}{I^* + n} \frac{\partial n}{\partial t} \right) ds = - \int_{L_0}^{L_2} \left( \frac{\partial^2 n}{\partial s^2} \right)^2 ds \quad (27.2) \]

It is then found using Sylvester’s criteria that \( \frac{\partial V}{\partial t} \) is negative definite proving \( E^* \) is globally asymptotically stable in the region \((0, L_2)\) provided \( \beta^2 < 4\alpha_0 W \).

| Table-1 |
|-----------------|-----------------|-----------------|-----------------|
| Parameters | Patch-1 | Figure | Patch-2 | Figure |
| r_0 | 0.094492378 | R* | 0.967 |
| C | 400 | D_1 | 0.7 |
| \beta_1 | 0.000004 | \beta_2 | 0.00002 |
| a_1 | 6 | a_2 | 8 |
| \alpha_1 | 8.759 | \alpha_2 | 74.97 |
| \alpha_{10} | 0.00003 | \alpha_{20} | 0.00001 |
| d_1 | 0.5 | d_2 | 0.6 |
| D_1 | 0.7 | D_2 | 0.8 |

**Table-2**

<table>
<thead>
<tr>
<th>Parameters</th>
<th>(Patch-1)</th>
<th>Figure</th>
</tr>
</thead>
<tbody>
<tr>
<td>R^*</td>
<td>400</td>
<td>R*</td>
</tr>
<tr>
<td>I_1</td>
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</tr>
</tbody>
</table>

**Conclusion**

In the present model, we have assumed that the density of resource biomass is governed by the logistic function with the same intrinsic growth rate and carrying capacity in the entire habitat. Further the harvesting of resource biomass and distribution of resource biomass in the patchy habitat are
assumed. The rate of depletion of forest resource biomass density due to industrialization, harvesting function and diffusion coefficients are considered to be different in each patch. It is further assumed that the density of industrialization is also governed by general logistic function in both the regions but with different growth rates and diffusion coefficients. The linear and non linear stability analysis is carried out for the positive uniform equilibrium state in homogeneous as well as patchy habitat by using Liapunov direct method. It is found that the positive uniform equilibrium state is both linear (local) and non linear (global) asymptotically stable under some conditions involving parameters in each case. It is shown that the equilibrium level of the resource biomass in two patches decreases as the density of industrialization or the rate of depletion due to industrialization increases.

REFERENCES