Analysis of Batch Arrival Retrial G-Queue with Multi-Types of Heterogeneous Service, Feedback, Randomized J Vacations and Orbital Search

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ABSTRACT

This paper is concerned with the analysis of a single server batch arrival retrial G-queue with feedback, multi-types of service and orbital search under a randomized vacation policy. Positive customers arrive in batches according to Poisson processes. Server provides M types of heterogeneous service. If the server is idle upon the arrival of a batch, one of the customers in the batch receives any one of the types immediately and others join the orbit. Otherwise, all the customers in the batch join the orbit. After completion of service, the unsatisfied customer joins the orbit as a feedback customer. A breakdown at the busy server is represented by the arrival of a negative customer which causes the customer being in service to be lost. The repair of the failed server starts immediately. Whenever the orbit becomes empty the server takes a vacation of random length. At a vacation completion epoch, if the system is still empty, the server leaves for another vacation of same length or remains idle in the system. This pattern continues until the server finds at least one customer in the orbit or the number of vacations reaches J. At the end of Jth vacation, even if the orbit is empty the server remains in the system for new arrival. If the orbit is non-empty during the idle period, the server may search customers from the orbit. Using supplementary variable technique, various performance measures are derived. Stochastic decomposition property is established.

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length. This pattern continues until the server finds at least one customer in the orbit or number of vacations reaches J. At the end of Jth vacation, even if the orbit is empty the server remains in the system. The vacation time also follows general distribution with distribution function V(x), density function v(x), Laplace Stieltje’s transform V*(θ), nth factorial moments v(n) and conditional completion rate γ(x) = v(x)/[1-V(x)].

If the orbit is non-empty during the idle period, the server searches for the customers in the orbit with probability θ or remains idle with probability 1-θ. Various stochastic process involved in the system are independent of each other.

**Definitions and Notations**

Assuming the existence of steady state, define the following probabilities

I_n(x) is the probability that there are n customers in the system, the idle and the elapsed retrial time is x.

P_n(i)(x) is the probability that there are n customers in the system, the server is busy in i

R_n(i)(x) is the probability that there are n customers in the system, the server failed during i

V_n(j)(x) is the probability that there are n customers in the system, the server is in j

I0 is the steady state probability that the server is idle in the empty system.

Define the following probability generating functions

I(x,z) = \sum_{n=0}^{\infty} I_n(x)z^n, P_n(i)(x) = \sum_{n=0}^{\infty} P_n(i)(x)z^n, R_n(i)(x,z) = \sum_{n=0}^{\infty} R_n(i)(x)z^n

and

V_n(j)(x) = \sum_{n=0}^{\infty} V_n(j)(x)z^n, 1 \leq j \leq J

**Steady State Distributions**

The system of equations that governs the model under steady state are given below

\[\lambda^+ I_0 = \int_0^\infty V_j^0(x)\gamma(x)dx + q \sum_{j=1}^{J-1} \int_0^{\infty} V_j^0(x)\gamma(x)dx\]  

(1)

\[\frac{d}{dx} I_n(x) = - (\lambda^+ + \eta(x))I_n(x), n \geq 1\]  

(2)

\[\frac{d}{dx} P_n(i)(x) = (\lambda^+ - \lambda^- + \mu_1(x)P_n(i)(x) + (1-\delta_n0)\lambda^+ \sum_{k=1}^n C_{k}P_n-k(x), n \geq 0, i = 1,2,...,M\]  

(3)

\[\frac{d}{dx} R_n(i)(x) = -(\lambda^+ + \beta_1(x))R_n(i)(x) + (1-\delta_n0)\lambda^+ \sum_{k=1}^n C_{k}R_n-k(x), n \geq 0, i = 1,2,...,M\]  

(4)

\[\frac{d}{dx} V_n(j)(x) = -(\lambda^+ + \gamma(x))V_n(j)(x) + (1-\delta_n0)\lambda^+ \sum_{k=1}^n C_{k}V_n-k(x), n \geq 0, j = 1,2,...,J\]  

(5)

with boundary conditions

\[I_n(0) = \sum_{i=1}^M \left( \delta J_1^i(0)\mu_1(x)dx + \delta J_0^i(0)\mu_1(x)dx \right) + \left( \sum_{i=1}^M \int_0^\infty R_j^i(x)\beta_1(x)dx + \int_0^\infty V_j^i(x)\gamma(x)dx \right) \right\}, n \geq 1\]  

(6)

\[P_0(i)(0) = P_0^i(0) = \left[ \lambda^+ c_1 I_0 + \int_0^\infty I_1(x)\eta(x)dx + \left( \sum_{i=1}^M \int_0^\infty R_j^i(x)\beta_1(x)dx + \int_0^\infty V_j^i(x)\gamma(x)dx \right) \right\]  

(7)

\[P_0^i(0) = \left[ \lambda^+ c_1 I_0 + \int_0^\infty I_1(x)\eta(x)dx + \left( \sum_{i=1}^M \int_0^\infty R_j^i(x)\beta_1(x)dx + \int_0^\infty V_j^i(x)\gamma(x)dx \right) \right\], n \geq 1, i = 1,2,...,M\]  

(8)

\[R_n^i(0) = \lambda^- \int_0^\infty P_n^i(x)dx, n \geq 0, i = 1,2,...,M\]  

(9)

\[V_n^j(0) = \left[ \int_0^\infty V_n^j(x)\gamma(x)dx, n = 0, j = 2,3,...,J \right\]  

(10)
\[
V_n^{(1)}(0) = \left\{ \frac{M}{\sum_{i=1}^{\infty} \int_{0}^{\infty} p^{(i)}_{n}(x)u_{i}(x)dx + \int_{0}^{\infty} r^{(i)}_{n}(x)\beta_{i}(x)dx} \right\} n = 0
\]
\[
\sum_{n=1}^{\infty} \int_{0}^{\infty} i_{n}(x)dx + \sum_{n=1}^{\infty} \sum_{i=1}^{M} \int_{0}^{\infty} p^{(i)}_{n}(x)dx + \int_{0}^{\infty} r^{(i)}_{n}(x)dx = \sum_{n=1}^{\infty} \sum_{i=1}^{M} \int_{0}^{\infty} v^{(i)}_{n}(x)dx = 1
\]

Multiplying equations (2) to (11) by \( z^{n} \) and summing over all possible values of \( n \), we get the following partial differential equations
\[
\left[ \frac{\partial}{\partial x} + (\lambda^{+} + \eta(x)) \right] i(x, z) = 0
\]
\[
\left[ \frac{\partial}{\partial x} + (\lambda^{+} + \lambda^{-} - \lambda^{+} C(z) + \mu_{1}(x)) \right] p^{(i)}(x, z) = 0, i = 1, 2, ..., M
\]
\[
\left[ \frac{\partial}{\partial x} + (\lambda^{+} - \lambda^{+} C(z) + \beta_{1}(x)) \right] r^{(i)}(x, z) = 0, i = 1, 2, ..., M
\]
\[
\left[ \frac{\partial}{\partial x} + (\lambda^{+} - \lambda^{+} C(z) + \gamma(z)) \right] v_{j}(x, z) = 0, j = 1, 2, ..., J
\]

Solving equation (5) at \( n=0 \), we have
\[
V_{0}^{(j)}(x) = V_{0}^{(j)}(0)e^{-\lambda^{+}x}[1 - V(x)], j = 1, 2, ..., J
\]

Multiplying equation (20) by \( \gamma(x) \) and integrating with respect to \( x \) from 0 to \( \infty \), we have
\[
\int_{0}^{\infty} V_{0}^{(j)}(x)\gamma(x)dx = \int_{0}^{\infty} V_{0}^{(j)}(0)e^{-\lambda^{+}x}(1 - V(x))\gamma(x)dx
\]
\[
= V_{0}^{(j)}(0)v^{\ast}(\lambda^{+}), j = 1, 2, ..., J
\]

Equation (10) gives
\[
V_{0}^{(j)}(0) = q\sqrt{V_{0}^{(j-1)}(0)}v^{\ast}(\lambda^{+}, j = 2, 3, ..., J
\]

From equations (10) and (11) it is clear that \( V_{0}^{(0)}(0, z) = V_{0}^{(0)}(0) \).

Applying equations (22) repeatedly for \( j = J, J-1, ... \) we get
\[
V^{(j)}(0, z) = \frac{V_{0}^{(j)}(0)}{\sqrt{v^{\ast}(\lambda^{+})}}^{J-1}, j = 1, 2, ..., J-1
\]

Substituting equation (22) and (23) in equation (1) and after some algebraic manipulations, we get
\[
V_{0}^{(j)}(0) = \frac{\lambda^{+}1_{0}}{v^{\ast}(\lambda^{+})} \left[ 1 + \frac{q(1 - q\sqrt{v^{\ast}(\lambda^{+})})^{J-1}}{(q\sqrt{v^{\ast}(\lambda^{+})})^{J-1}(1 - q\sqrt{v^{\ast}(\lambda^{+}))}} \right]
\]

Solving the partial differential equations (13), (14), (15) and (16) we get respectively
\[
I(x, z) = I(0, z)exp[- \lambda^{+}x][1 - A(x)]
\]
\[
P^{(i)}(x, z) = P^{(i)}(0, z)exp[-(\lambda^{+} + \lambda^{-} - \lambda^{+} C(z))[1 - B(x), i = 1, 2, ..., M
\]
\[
R^{(i)}(x, z) = R^{(i)}(0, z)exp[-(\lambda^{+} - \lambda^{+} C(z)))[1 - R(x)], i = 1, 2, ..., M
\]
\[
V^{(i)}(x, z) = V^{(i)}(0, z)exp[-(\lambda^{+} - \lambda^{+} C(z))][1 - V(x)], j = 1, 2, ..., J
\]

Substituting the expression of \( P^{(i)}(x, z) \) in equation (19), we obtain
\[
R^{(i)}(0, z) = \lambda^{+}P^{(i)}(0, z)(1 - B^{1}_{i}(g(z)))/g(z), i = 1, 2, ..., M
\]

where, \( g(z) = \lambda^{+} + \lambda^{-} - \lambda^{+} C(z) \)

Using equations (25), (26), (27) and (28) in equation (17) and (18), we get
\begin{equation}
I(0,z) = \sum_{i=1}^{M} \left( V^{(j)}(0, z) V^* (h(z)) + \sum_{i=1}^{M} \left( \delta(z - 1) + 1 \right) P^{(i)}(0, z) B_i^* (g(z)) + \delta R^{(i)}(0, z) R_i^* (h(z)) \right) - \lambda^+ I_0 - \sum_{j=1}^{J} V^{(j)}(0)
\end{equation}

\begin{equation}
P^{(i)}(0, z) = \frac{p_i}{z} \int_{C(z)} + \int_{0, z} \left[ A^* (\lambda^+) + C(z)(1 - A^* (\lambda^+)) \right] + \frac{J}{1} \sum_{j=1}^{J} V^{(j)}(0, z) V^* (h(z)) + \sum_{j=1}^{M} R^{(i)}(0, z) R_i^*(h(z)) \right], i = 1, 2, \ldots, M
\end{equation}

where, \( h(z) = \lambda^+ - \lambda^+ C(z) \)

Using equations (23), (24), (29) and (31) in equation (30) and simplifying we obtain

\begin{equation}
I(0, z) = I_0 \left[ \left| C(z) + QV^* (h(z)) \right| \sum_{i=1}^{M} \left( \delta(z - 1) + 1 \right) g(z) B_i^* (g(z)) + \lambda^{-} \delta(1 - B_i^* (g(z)) R_i^* (h(z))) \right] + \frac{J}{1} \sum_{j=1}^{J} V^{(j)}(0, z) V^* (h(z)) + \sum_{j=1}^{M} R^{(i)}(0, z) R_i^*(h(z)) \right], i = 1, 2, \ldots, M
\end{equation}

and

\begin{equation}
Q = \frac{V^* (\lambda^+^*)}{\left| (qV^* (\lambda^+^*) \right| J - 1 + \left| (qV^* (\lambda^+^*) \right| J - 1 - 1]
\end{equation}

Using equations (32) in equation (31) and on solving we have

\begin{equation}
P^{(i)}(0, z) = I_0 \lambda^+ p_i [1 - B_i^* (g(z))] A^* (\lambda^+) (C(z) - 1) + QV^* (h(z)) + A^* (\lambda^+) + C(z)(1 - A^* (\lambda^+)) \right| \left| (1 - 0) V^* (h(z)) - 1 \right| Q - 1]
\end{equation}

Using equations (33) in equation (29) and simplifying we get

\begin{equation}
R^{(i)}(0, z) = I_0 \lambda^+ - \lambda^+ p_i [1 - B_i^* (g(z))] A^* (\lambda^+) (C(z) - 1) + QV^* (h(z)) + A^* (\lambda^+) + C(z)(1 - A^* (\lambda^+)) \right| \left| (1 - 0) V^* (h(z)) - 1 \right| Q - 1]
\end{equation}

By defining the partial probability generating functions \( \chi(z) = \sum_{j=1}^{J} \chi(x, z) dx \), we can find the orbit size probability generating functions as follows.

The partial probability generating function of the orbit size when the server is idle is

\begin{equation}
I(z) = I_0 \left[ (1 - A^* (\lambda^+)) \right] \left[ C(z) + QV^* (h(z)) \right] \sum_{i=1}^{M} p_i [1 - (\delta(z - 1) + 1)] g(z) B_i^* (g(z)) + \lambda^{-} \delta(1 - B_i^* (g(z)) R_i^* (h(z))) \right] + \frac{J}{1} \sum_{j=1}^{J} V^{(j)}(0, z) V^* (h(z)) + \sum_{j=1}^{M} R^{(i)}(0, z) R_i^*(h(z)) \right], i = 1, 2, \ldots, M
\end{equation}

The partial probability generating function of the orbit size when the server is busy in type i service is

\begin{equation}
P^{(i)}(z) = I_0 \lambda^+ p_i [1 - B_i^* (g(z))] A^* (\lambda^+) (C(z) - 1) + QV^* (h(z)) + A^* (\lambda^+) + C(z)(1 - A^* (\lambda^+)) \right| \left| (1 - 0) V^* (h(z)) - 1 \right| Q - 1]
\end{equation}

The partial probability generating function of the orbit size when the server failed during type i service is under repair is given by

\begin{equation}
R^{(i)}(z) = I_0 \lambda^- p_i [1 - B_i^* (g(z))] A^* (\lambda^+) (C(z) - 1) + QV^* (h(z)) + A^* (\lambda^+) + C(z)(1 - A^* (\lambda^+)) \right| \left| (1 - 0) V^* (h(z)) - 1 \right| Q - 1]
\end{equation}

The partial probability generating function of the orbit size when the server is on vacation is

\begin{equation}
V^{(j)}(z) = I_0 \left[ \frac{Q \left| 1 - V^* (h(z)) \right|}{(1 - C(z))} \right], 1 \leq j \leq J
\end{equation}

Using the normalizing condition (12), \( I_0 \) can be obtained as

\begin{equation}
I_0 = \frac{\lambda^{-} \lambda^{-} v_q I + \lambda^{-} A^* (\lambda^+) - 0Q(1 - A^* (\lambda^+)) (1 - \delta) \sum_{i=1}^{M} p_i B_i^* (\lambda^-)}{(\lambda^+ \lambda^+ v_q I + \lambda^{-} A^* (\lambda^+) - 0Q(1 - A^* (\lambda^+)) (1 - \delta) \sum_{i=1}^{M} p_i B_i^* (\lambda^-)}
\end{equation}

**Performance Measures**

**Probability that the server is idle in the non-empty system is given by**

\begin{equation}
I = \lim_{z \to 1} \frac{(1 - A^* (\lambda^+)) (1 - \delta) \sum_{i=1}^{M} p_i B_i^* (\lambda^-) + \lambda^+ \lambda^- m_1 (1 - B_i^* (\lambda^-)) \sum_{i=1}^{M} p_i B_i^* (\lambda^-) + \lambda^+ \lambda^{-} m_1 (1 - B_i^* (\lambda^-)) \sum_{i=1}^{M} p_i B_i^* (\lambda^-) + \delta \sum_{i=1}^{M} p_i B_i^* (\lambda^-)}{(\lambda^+ \lambda^+ v_q I + \lambda^{-} A^* (\lambda^+) - 0Q(1 - A^* (\lambda^+)) (1 - \delta) \sum_{i=1}^{M} p_i B_i^* (\lambda^-)}
\end{equation}
Probability that the server is busy is given by
\[
P = \lim_{z \to l_1} \frac{\sum_{i=1}^{M} \lambda^i m_1 p_i (1 - B_1^{*}(\lambda^{-}))}{\lambda^{-} [1 - \delta \sum_{i=1}^{M} p_i B_i^{*}(\lambda^{-})]}.
\] (41)

Probability that the server is under repair is given by
\[
R = \lim_{z \to l_1} \frac{\sum_{i=1}^{M} \beta_i (l) p_i (1 - B_i^{*}(\lambda^{-}))}{[1 - \delta \sum_{i=1}^{M} p_i B_i^{*}(\lambda^{-})]}.
\] (42)

Probability that the server is on vacation is given by
\[
V = \lim_{z \to 1} \frac{\lambda^i v_1 (Q\lambda^{-} [1 - m_1 (1 - A^{*}(\lambda^+) + z^i m_1 (1 - A^{*}(\lambda^+))] - \lambda^m m_1 [1 - \delta \sum_{i=1}^{M} p_i B_i^{*}(\lambda^{-})] - \lambda^m m_1 [1 - \delta \sum_{i=1}^{M} p_i B_i^{*}(\lambda^{-})] + \lambda^m m_1 p_i (l - B_i^{*}(\lambda^{-})))]}{[1 - \delta \sum_{i=1}^{M} p_i B_i^{*}(\lambda^{-})]}
\] (43)

Probability generating function of the number of customers in the system is given by
\[
(1 + \lambda^i q_1) (\sum_{i=1}^{M} p_i B_i^{*}(g(z))) = \frac{1}{z - \lambda^i v_1 (Q\lambda^{-} [1 - m_1 (1 - A^{*}(\lambda^+) + z^i m_1 (1 - A^{*}(\lambda^+))] - \lambda^m m_1 [1 - \delta \sum_{i=1}^{M} p_i B_i^{*}(\lambda^{-})] - \lambda^m m_1 [1 - \delta \sum_{i=1}^{M} p_i B_i^{*}(\lambda^{-})] + \lambda^m m_1 p_i (l - B_i^{*}(\lambda^{-})))]}{[1 - \delta \sum_{i=1}^{M} p_i B_i^{*}(\lambda^{-})]}
\]

Mean number of customers in the orbit \(L_0\) is given by
\[
L_0 = I_0 + \sum_{i=1}^{M} p_i (j) + R (j) + \frac{1}{z - 1} V (j)
\] (44)

where,
\[
\text{Nr}^{*}(l) = -I_0 \lambda \lambda^i m_1 [A^{*}(\lambda^+) - \theta(Q(1 - A^{*}(\lambda^+))) - \lambda^i v_1 Q(1 - \delta \sum_{i=1}^{M} p_i B_i^{*}(\lambda^{-}))]
\] (45)

Probability generating function of the number of customers in the system is given by
\[
P_i(z) = \sum_{i=1}^{M} p_i B_i^{*}(g(z))(A^{*}(\lambda^+) + C(z)(1 - A^{*}(\lambda^+))(Q(1 - (1 - 0)V^*(h(z))) - 0QV^*(h(z)) + A^{*}(\lambda^+) (1 - C(z)))/(1 - C(z))D(z)
\] (46)

Mean number of customers in the orbit \(L_0\) is given by
\[
L_0 = I_0 + \sum_{i=1}^{M} p_i B_i^{*}(g(z)(h(z) - \delta g(z)))(A^{*}(\lambda^+) + C(z)(1 - A^{*}(\lambda^+))(Q(1 - (1 - 0)V^*(h(z))) - 0QV^*(h(z)) + A^{*}(\lambda^+) (1 - C(z)))/(1 - C(z))D(z)
\] (47)
$L_n = P_n(\lambda) = \left[ Dr^\nu(1) N_r(1) - N_r^\nu(1) Dr^\nu(1) \right] \frac{3 Dr^\nu(1)}{\delta}$

where,

$N_r^\nu(1) = -2\lambda^{-} m_1 (A^+ (\lambda^{-}) - \delta Q(1-A^+ (\lambda^{-}))) + \lambda^+ v_1 Q [1 - \delta \sum_{i=1}^{M} p_i B_i^+ (\lambda^-)]$  \hspace{1cm} (47)

$N_r(1) = I_0 \left( -m_1 A^+ (\lambda^{-}) + m_1 (1-A^+ (\lambda^{-}) \delta Q - \lambda^+ v_1 Q (1-\delta - \lambda^+ m_1 \sum_{i=1}^{M} p_i B_i^+(\lambda^-)) \right) + \lambda^+ v_1 Q [1 - \delta \sum_{i=1}^{M} p_i B_i^+(\lambda^-)]$

Reliability Indices

Let $A(t)$ be the pointwise availability of the server at time $t$, that is the probability that the server is idle or busy. The steady state availability of the server will be $A = \lim A(t)$.

The availability of the server is given by

$A = 1 - (R + V) = \frac{\\lambda^+ v_1 Q + \lambda^- A^+ (\lambda^-) - \delta Q(1-A^+ (\lambda^-))) + \lambda^+ v_1 Q \lambda^+ [1 - m_1 + m_1 A^+ (\lambda^-)]}{[1 - \delta \sum_{i=1}^{M} p_i B_i^+(\lambda^-)]}$

Steady state failure frequency of the server is

$F = \frac{\\lambda^+ m_1 \sum_{i=1}^{M} \left(1 - B_i^+(\lambda^-)\right)}{[1 - \delta \sum_{i=1}^{M} p_i B_i^+(\lambda^-)]}$

Stochastic Decomposition

Theorem: The number of customers in the system ($L_n$) can be expressed as the sum of two independent random variables, one of which is the mean number of customers ($L$) in the unreliable batch arrival G-queue with multi-optimal service and feedback and the other is the mean number of customers in the orbit ($L_0$) given that the server is idle or on vacation.

Proof: The probability generating function $\pi(z)$ of the number of customers in the Classical queuing system with negative customers, feedback, server breakdown and randomized J vacation is given by

$\pi(z) = \frac{z - \delta [z - \sum_{i=1}^{M} R_i B_i^+(\lambda^-)] - \delta (1-z)(z - \delta [z - \sum_{i=1}^{M} R_i B_i^+(\lambda^-)]) - \delta [z - \sum_{i=1}^{M} R_i B_i^+(\lambda^-)] - \delta (1-z)(z - \delta [z - \sum_{i=1}^{M} R_i B_i^+(\lambda^-)])}{[1 - \delta \sum_{i=1}^{M} p_i B_i^+(\lambda^-)]}$

The probability generating function $\chi(z)$ of the number of customers in the orbit given that the server is idle or on vacation is

$\chi(z) = \frac{I_0 + I(z) + \sum_{j=1}^{V} V_j(z)}{I_0 + 1 + V}$

References


