Inf – J and Sup - Uj Compositions Between Fuzzy Relations

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Abstract
In this paper inf – j composition (where j refers to a t- conorm) and sup- uj composition are defined. Relation between inf- j and sup- uj compositions are established. Theorems are proved that express the basic properties of inf- j and sup- uj composition.

Keywords
Inf- j, Sup- uj composition.

Introduction
Usually fuzzy relation equations are dealt using sup- i composition where i is a continuous t- norm. this is done by Zadeh [6,7] where he introduce compositional rule of interference with the help of sup- min composition. L.A. Zadeh [8] in 1965 introduced fuzzy set in his seminal paper. Goguen [4] in 1967 generalizes the concept of fuzzy sets defining them in terms of maps from a non empty set to a partially ordered set. Brown [2] in 1971 shows that Zadehs basics results carry over to the maps from a non empty set to a lattice. Sanchez [5] suggested the composition inf- j operation. The properties of this composition have been investigated in [1,2]. However in our paper we have defined inf- j along with sup- uj composition some of its properties are characterized.

Preliminaries
Fuzzy Set
Let be a non empty set. A fuzzy set in is characterized by its membership function and is interpreted as the degree of membership of element in fuzzy set for each .

T- Norm
A fuzzy intersection/t- norm i is a binary operation on the unit interval that satisfies at least the following four axioms for all .

(i) \( i(a, 1) = a \) (boundary condition)
(ii) \( b \leq d \) implies \( i(a, b) \leq i(a, d) \) (monotonicity)
(iii) \( i(a, b) = i(b, a) \) (commutativity)
(iv) \( i[a, i(b, d)] = i[i(a, b), d] \) (associativity)

Three of the most important requirements are expressed by the following axioms

(v) \( i \) is a continuous function (continuity)
(vi) \( i(a, a) \leq a \) (sub idempotency)
(vii) \( a_1 < a_2 \) and \( b_1 < b_2 \) implies \( i(a_1, b_1) < i(a_2, b_2) \) (strict monotonicity)

T- Conorm
A fuzzy union/t-conorm j is a binary operation on the unite interval that satisfies at least the following four axioms for all .

(i) \( j(a, 0) = a \) (boundary condition)
(ii) \( b \leq d \) implies \( j(a, b) \leq j(a, d) \) (monotonicity)
(iii) \( j(a, b) = j(b, a) \) (commutativity)
(iv) \( j[a, j(b, d)] = j[j(a, b), d] \) (associativity)

The most important additional requirements for fuzzy unions are expressed by the following axioms

(v) \( j \) is a continuous function (continuity)
(vi) \( j(a, a) > a \) (super idempotency)
(vii) \( a_1 < a_2 \) and \( b_1 < b_2 \) implies \( u(a_1, b_1) < u(a_2, b_2) \) (strict monotonicity)
Sup-i Composition

Sup-i composition of binary fuzzy relations, where i refers to a t-norm, generalize the standard max-min composition. Given a particular t-norm i and two fuzzy relations \( P(X, Y) \) and \( Q(Y, Z) \), the sup-i composition of \( P \) and \( Q \) is a fuzzy relation \( P \circ Q \) on \( X \times Z \) defined by

\[
[P \circ Q](x, z) = \sup_{y \in Y} [i(P(x, y), Q(y, z))] \quad \text{for all } x \in X, z \in Z.
\]

Definition

Given a continuous t-norm \( i \), let \( w_i(a, b) = \sup\{x \in [0,1] \mid i(a, x) \leq b\} \) \( \bullet \) (2)

For every \( a, b \in [0,1] \). This operation referred to as operation \( w_i \). While t-norm \( i \) may be interpreted as logical conjunction, the corresponding operation \( w_i \) may be interpreted as logical implication.

Definition

Given a t-norm \( i \) and the associated operation \( w_i \), the inf-i composition, \( w_i \), of fuzzy relation \( P(x, y) \) and \( Q(y, z) \) is defined by the equation

\[
[w_i(P \circ Q)](x, z) = \inf_{y \in Y} w_i[P(x, y), Q(y, z)]
\]

for all \( x \in X, z \in Z \).

Definition (ELIE SANCHEZ[4])

Let \( P(x, y) \) be a fuzzy relation, the fuzzy relation \( P^{-1}(y, x) \), the inverse or transpose of \( P \), is defined by

\[
P^{-1}(y, x) = P(x, y) \quad \text{for all } (y, x) \in Y \times X
\]

3. Inf-j Compositions of Fuzzy Relations

Inf-j composition of fuzzy relations, where \( j \) refers to a t-conorm, generalize the standard min-max composition.

Definition

Given a particular t-conorm \( j \) and two fuzzy relations \( P(X, Y) \) and \( Q(Y, Z) \), the inf-j composition of \( P \) and \( Q \) is a fuzzy relation \( P \circ Q \) on \( X \times Z \) defined by

\[
[P \circ Q](x, z) = \inf_{y \in Y} j[P(x, y), Q(y, z)]
\]

for all \( x \in X, z \in Z \).

Basic properties of t-conorm are expressed by the following theorem.

Theorem

For any \( a, a_i, b, d \in [0,1] \), where \( i \) takes values from an index set \( I \), operation \( u_j \) has the following properties.

(i) \( b \leq d \) implies \( j(a, b) \leq j(a, d) \) and \( j(b, a) \leq j(d, a) \)
(ii) \( j(a, j(b, d)) = j[j(a, b), d] \)
(iii) \( j(b, \sup_{i \in I} a_i) = \sup_{i \in I} j(b, a_i) \)
(iv) \( j(b, \inf_{i \in I} a_i) = \inf_{i \in I} j(b, a_i) \)
(v) \( j(\sup_{i \in I} a_i, b) \leq j(\sup_{i \in I} a_i, b) \)
(vi) \( j(\inf_{i \in I} a_i, b) = \inf_{i \in I} j(\inf_{i \in I} a_i, b) \)

Proof

(i) By definition, t-conorm \( j \) is monotonic increasing function i.e., if \( b \leq d \) then \( j(a, b) \leq j(a, d) \) also \( j \) is commutative, i.e., \( j(b, a) \leq j(d, a) \)
(ii) By definition, t-conorm \( j \) is associative i.e., \( j(j(a, b), d) = j(a, j(b, d)) \)
(iii) Now we have to prove that

\[
j(b, \sup_{i \in I} a_i) \leq \sup_{i \in I} j(b, a_i)
\]

Let \( s = \sup_{i \in I} a_i \)

\[
a_i \leq s \text{ for any } i \in I
\]

\[
j(b, a_i) \leq j(b, s) \text{ for any } i \in I
\]

\[
\Rightarrow \sup_{i \in I} j(b, a_i) \leq j(b, \sup_{i \in I} a_i)
\]
(iv) Now we have to prove that

\[
 f \left( b, \inf_{i \in I} a_i \right) = \inf_{i \in I} f(b, a_i)
\]

Let \( l = \inf_{i \in I} a_i \implies l \leq a_i \)

\[
 j(b, l) \leq j(b, a_i)
\]

\[
 j(b, l) \leq \inf_{i \in I} j(b, a_i)
\]

But \( \inf_{i \in I} j(b, a_i) \leq j(b, a_i) \)

\[
 \Rightarrow j(b, \inf_{i \in I} a_i) \leq j(b, j(b, a_i))
\]

\[
 \Rightarrow j(b, \inf_{i \in I} a_i) \leq j(j(b, b), a_i))
\]

\[
 \Rightarrow \inf_{i \in I} j(b, a_i) \leq a_i \text{ for all } i \in I
\]

\[
 \Rightarrow \inf_{i \in I} j(b, a_i) \leq \inf_{i \in I} a_i
\]

\[
 \Rightarrow j(b, \inf_{i \in I} j(b, a_i)) \leq j(j(b, b), \inf_{i \in I} a_i)
\]

\[
 \Rightarrow j(b, \inf_{i \in I} j(b, a_i)) \leq j(j(b, b), \inf_{i \in I} a_i)
\]

\[
 \Rightarrow \inf_{i \in I} j(b, a_i) \leq j(b, \inf_{i \in I} a_i)
\]

From equation (i) and (ii) we get

\[
 j(b, \inf_{i \in I} a_i) = \inf_{i \in I} j(b, a_i)
\]

(v) Now let us prove

\[
 j \left( \sup_{i \in I} a_i, b \right) \geq \sup_{i \in I} j(a_i, b)
\]

By property (iii) \( j(b, \sup_{i \in I} a_i) \geq \sup_{i \in I} j(b, a_i) \)

Since t-conorm \( j \) is commutative therefore \( j(\sup_{i \in I} a_i, b) \geq \sup_{i \in I} j(a_i, b) \)

(vi) Now we have to prove that \( j(\inf_{i \in I} a_i, b) = \inf_{i \in I} j(a_i, b) \)

By property (iv) we have \( j(b, \inf_{i \in I} a_i) = \inf_{i \in I} j(b, a_i) \)

Since t-conorm \( j \) is commutative therefore \( j(\inf_{i \in I} a_i, b) = \inf_{i \in I} j(a_i, b) \)

**Theorem**

Let \( P(X, Y), P_j(X, Y), Q(Y, Z) \) and \( Q_j(Y, Z) \) be fuzzy relations. Then

(i) \( Q_1 \subseteq Q_2 \text{ then } P \circ Q_1 \subseteq P \circ Q_2 \) and \( Q_j \circ R \supseteq Q_j \circ R \)

(ii) \( (P \circ Q)^j = P_j \circ R = P \circ (Q \circ R) \)

(iii) \( P_j \left( \bigcup_{i \in I} Q_i \right) = \bigcup_{i \in I} P \circ Q_i \)

(iv) \( P_j \left( \bigcap_{i \in I} Q_i \right) = \bigcap_{i \in I} P \circ Q_i \)

(v) \( \left( \bigcup_{i \in I} P_i \right) \circ Q = \bigcup_{i \in I} P_i \circ Q \)

(vi) \( \left( \bigcap_{i \in I} P_i \right) \circ Q = \bigcap_{i \in I} P_i \circ Q \)

(vii) \( (P_j \circ Q)^{-1} = Q^{-1}_j \circ P^{-1} \)

(i) Since \( Q_1 \subseteq Q_2 \).
Let \[ [P \circ Q_j](x, z) = \inf_{y \in \mathbb{Z}} j[P(x, y), Q_1(y, z)] \subseteq \inf_{y \in \mathbb{Z}} j[P(x, y), Q_2(y, z)] = [P \circ Q_2](x, z) \]

Therefore \[ P \circ Q_1 \subseteq P \circ Q_2 \]

Similarly \[ Q_1 \circ R \subseteq Q_2 \circ R \]

(ii) By definition of inf-j composition,
\[ [(P \circ Q_j) \circ R](x, v) = \inf_{z \in \mathbb{Z}} j[(P \circ Q_j)(x, z), R(z, v)] \]
\[ = \inf_{z \in \mathbb{Z}} \inf_{y \in \mathbb{Y}} [j[P(x, y), Q_j(y, z)], R(z, v)] \]
\[ = \inf_{z \in \mathbb{Z}} \inf_{y \in \mathbb{Y}} [j[P(x, y), Q(y, z)], R(z, v)] \]
\[ = \inf_{z \in \mathbb{Z}} j[P(x, y), \inf_{y \in \mathbb{Y}} j(Q(y, z), R(z, v))] \]
\[ = [P \circ (Q \circ R)](x, z) \]

(iii) We know that \[ Q_i \subseteq \bigcup_{i \in I} Q_i \]

Then by property (i) we have \[ P \circ Q_i \subseteq P \circ \left( \bigcup_{i \in I} Q_i \right) \]

Therefore \[ P \circ \left( \bigcup_{i \in I} Q_i \right) \supseteq \bigcup_{i \in I} (P \circ Q_i) \]

(iv) We know that \[ \bigcap_{i \in I} Q_i \subseteq Q_i \]

Then by property (i) have \[ P \circ \left( \bigcap_{i \in I} Q_i \right) \subseteq P \circ Q_i \]

\[ \Rightarrow P \circ \left( \bigcap_{i \in I} Q_i \right) \subseteq (P \circ Q_i) \]

But \[ \bigcap_{i \in I} (P \circ Q_i) \subseteq P \circ Q_i \]

Then by property (i) we have \[ P^{-1} \circ \left( \bigcap_{i \in I} (P \circ Q_i) \right) \subseteq P^{-1} \circ (P \circ Q_i) \]

By property (ii) we have \[ P^{-1} \circ \left( \bigcap_{i \in I} (P \circ Q_i) \right) \subseteq (P^{-1} \circ P) \circ Q_i \]

Since \[ P^{-1} \subseteq P^{-1} \circ P \quad \text{(j is super idempotent)} \]

Then \[ \Rightarrow \bigcap_{i \in I} (P \circ Q_i) \subseteq Q_i \]

\[ \Rightarrow \bigcap_{i \in I} (P \circ Q_i) \subseteq \bigcap_{i \in I} Q_i \]

\[ P^{-1} \circ \left( \bigcap_{i \in I} (P \circ Q_i) \right) \subseteq (P^{-1} \circ P) \circ \left( \bigcap_{i \in I} Q_i \right) \]
\[ P^{-1}\left(\bigcap_{i \in I} (P_i \circ Q_i)\right) \subseteq P^{-1}\left(\bigcap_{i \in I} Q_i\right) \]

\[ \bigcap_{i \in I} (P_i \circ Q_i) \subseteq P_i\left(\bigcap_{i \in I} Q_i\right) \]

From equations (i) and (ii)
\[ P_i\left(\bigcap_{i \in I} Q_i\right) = \bigcap_{i \in I} (P_i \circ Q_i) \]

(v)
\[ \bigcup_{i \in I} P_i \subseteq \bigcup_{i \in I} (P_i \circ Q) \]

We know that
\[ P_i \subseteq \bigcup_{i \in I} P_i \text{ for all } i \in I \]

Then by property (i)
\[ P_i \circ Q \subseteq \left(\bigcup_{i \in I} P_i\right) \circ Q \]

\[ \bigcup_{i \in I} (P_i \circ Q) \subseteq \left(\bigcup_{i \in I} P_i\right) \circ Q \]

(vi)
\[ \bigcap_{i \in I} P_i \subseteq \bigcap_{i \in I} (P_i \circ Q) \]

But we know that
\[ \bigcap_{i \in I} P_i \subseteq P_i \text{ for all } i \in I \]

Then by property (i)
\[ \left(\bigcap_{i \in I} P_i\right) \circ Q \subseteq P_i \circ Q \]

\[ \left(\bigcap_{i \in I} P_i\right) \circ Q \subseteq \bigcap_{i \in I} (P_i \circ Q) \]

But
\[ \bigcap_{i \in I} (P_i \circ Q) \subseteq P_i \circ Q \text{ for all } i \in I \]

Then by property (i) we have
\[ \left(\bigcap_{i \in I} (P_i \circ Q)\right) \circ Q^{-1} \subseteq (P_i \circ Q) \circ Q^{-1} \]

\[ \Rightarrow \left(\bigcap_{i \in I} (P_i \circ Q)\right) \circ Q^{-1} \subseteq P_i \circ (Q \circ Q^{-1}) \]

Because \( Q^{-1} \subseteq (Q \circ Q^{-1}) \) is a super idempotent
\[ \Rightarrow \bigcap_{i \in I} (P_i \circ Q) \subseteq P_i \text{ for all } i \in I \]

\[ \Rightarrow \left(\bigcap_{i \in I} (P_i \circ Q)\right) \subseteq \bigcap_{i \in I} P_i \]

\[ \Rightarrow \left(\bigcap_{i \in I} (P_i \circ Q)\right) \circ Q^{-1} \subseteq \left(\bigcap_{i \in I} P_i\right) \circ (Q \circ Q^{-1}) \]

\[ \Rightarrow \left(\bigcap_{i \in I} (P_i \circ Q)\right) \circ Q^{-1} \subseteq \left(\bigcap_{i \in I} P_i\right) \circ Q \]

\[ \Rightarrow \bigcap_{i \in I} (P_i \circ Q) \subseteq \left(\bigcap_{i \in I} P_i\right) \circ Q \]

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\[ \Rightarrow \bigcap_{i \in I} (P_i \circ Q) \subseteq \left(\bigcap_{i \in I} P_i\right) \circ Q \]
From equations (I) and (II) we get

\[ \bigcap_{i \in I} (P_i \circ Q) = \left( \bigcap_{i \in I} P_i \right) \circ Q \]

(vii) \[ (P \circ Q)^{-1} = Q^{-1} \circ P^{-1} \]

Here \[ (P \circ Q)^{-1}(z, x) = (P \circ Q)(x, z) \]

= \inf_{y \in Y} j(P(x, y), Q(y, z))

= \inf_{y \in Y} j(P(z, y), P^{-1}(y, x))

= \left( Q^{-1} \circ P^{-1} \right)(z, x)

\[(P \circ Q)^{-1} = Q^{-1} \circ P^{-1} \]

Sup-\(u_i\) Composition of Fuzzy Relations

Definition

Given a t-conorm \( j \), let \( u_j(a, b) = \inf \{ X \in [0,1] \mid j(a, x) \geq b \} \) \hspace{1cm} \text{.........(6)}

For every \( a, b \in [0,1] \)

**Theorem**

The operation \( u_j \) satisfies the following properties.

(i) \( j(a, b) \geq d \text{ iff } u_j(a, d) \leq b \)

(ii) \( u_j(u_j(a, b), b) \leq a \)

(iii) \( u_j(j(a, b), d) = u_j(a, u_j(b, d)) \)

(iv) \( a \leq b \text{ implies } u_j(a, d) \geq u_j(b, d) \) and \( u_j(d, a) \leq u_j(d, b) \)

(v) \( u_j(\sup_{i \in I} a_i, b) \leq \inf_{i \in I} u_j(a_i, b) \)

(vi) \( u_j(\inf_{i \in I} a_i, b) = \sup_{i \in I} u_j(a_i, b) \)

(vii) \( u_j(b, \inf_{i \in I} a_i) \leq \inf_{i \in I} u_j(b, a_i) \)

(viii) \( u_j(b, \sup_{i \in I} a_i) \leq \sup_{i \in I} u_j(b, a_i) \)

(ix) \( j(a, u_j(a, b)) \geq b \) for any \( a, b, d \in [0,1] \), where \( i \in I \)

**Proof**

(i) Now let us prove \( j(a, b) \geq d \text{ iff } u_j(a, d) \leq b \)

If \( j(a, b) \geq d \)

Then \( b \in \{ x \mid j(a, x) \geq d \} \)

\[ b \geq \inf \{ x \mid j(a, x) \geq d \} = u_j(a, d) \]

\[ \Rightarrow u_j(a, d) \leq b \]

if \( u_j(a, d) \leq b \)

then \( j(a, b) \geq j[a, u_j(a, d)] \)

\[ = j[a, \inf \{ x \mid j(a, x) \geq d \}] \]

\[ = \inf \{ j(a, x) / j(a, x) \geq d \} \]

\[ \geq d \]

\[ \Rightarrow j(a, d) \geq d \]

(ii) We have to prove \( u_j(u_j(a, b), b) \leq a \) \hspace{1cm} \text{i.e., } j(u_j(a, b), a) \geq b \text{ by (i)}

Let \( j(u_j(a, b), a) = j(\inf \{ x \mid j(a, x) \geq b \}, a) \)

\[ = \inf \{ j(x, a) / j(a, x) \geq b \} \]

\[ = \inf \{ j(a, x) / j(a, x) \geq b \} \]

\[ \geq b \]

\[ \Rightarrow j(u_j(a, b), a) \geq b \]
\( \therefore u_j(u_j(a, b), b) \leq a \)

(iii) Now we have to prove that 
\[ u_j[j(a, b), d] = u_j[a, u_j(b, d)] \]

Let 
\[ u_j[a, u_j(b, d)] = \inf\{x/j(a, x) \geq u_j(b, d)\} \]
\[ = \inf\{x/x \geq u_j[(a, b), d]\} \]
\[ = u_j[j(a, b), d] \]

Since by property 1
\[ j(a, x) \geq u_j(b, d) \]
\[ \iff j[b, j(a, x)] \geq d \]
\[ \iff j[j(a, b), x] \geq d \]
\[ \iff x \geq u_j[j(a, b), d] \]

(iv) \( a \leq b \) implies 
\[ u_j(a, d) \geq u_j(b, d) \] and 
\[ u_j(d, a) \leq u_j(d, b) \]

Let 
\[ u_j(d, a) = \inf\{x/j(d, x) \geq a\} \]
\[ = \inf\{x/x/j(d, x) \geq a\} \]
\[ = u_j[d, b] \]

\( \therefore u_j(d, a) \leq u_j(d, b) \)

Next we have to prove that 
\[ u_j(a, d) \geq u_j(b, d) \]

i.e., 
\[ u_j(b, d) \leq u_j(a, d) \]

i.e., 
\[ u_j(b, u_j(a, d)) \geq d \]

Let 
\[ (b, u_j(a, d)) = j(b, \inf\{x/j(a, x) \geq d\}) \]
\[ = j(\inf\{x/j(a, x) \geq d\}, b) \]
\[ \geq j(\inf\{x/j(a, x) \geq d\}, a) \]
\[ = \inf\{x/j(a, x) \geq d\} \]
\[ \geq d. \]

\( \therefore j(b, u_j(a, d)) \geq d \)

\( u_j(b, d) \leq u_j(a, d) \)

i.e., 
\[ u_j(a, d) \geq u_j(b, d) \]

(v) now let us prove that 
\[ u_j\left[sup\{a_i, b\}\right] \leq \inf\{u_j(a, b)\} \]

Let 
\[ s = sup\{a_i\} \]
then 
\[ s \geq a_i \]

\[ \Rightarrow \]
\[ a_i \leq s \]

\[ \Rightarrow \]
\[ u_j(a_i, b) \geq u_j(s, b) \]
for any \( i \in l \) (by property (iv))

\[ \Rightarrow \]
\[ \inf\{u_j(a, b)\} \geq u_j(s, b) \]
for any \( i \in l \)

i.e., 
\[ \inf\{u_j(a, b)\} \geq u_j(s, b) \]

(vi) we have to prove that 
\[ u_j\left[inf\{a_i, b\}\right] = sup\{u_j[a, b]\} \]

Let 
\[ l = \inf\{a_i\} \]
then 
\[ a_i \geq l \Rightarrow l \leq a_i \]

\[ u_j(l, b) \geq u_j(a_i, b) \]
for any \( i \in l \) (by property 4)

Hence 
\[ u_j(l, b) \geq sup\{u_j[a, b]\} \]

\[ \therefore u_j(l, b) \geq sup\{u_j[a, b]\} \] (i)

Since 
\[ sup\{u_j[a, b]\} \geq u_j(a_i, b) \]
for all \( i \in l \)

\[ j(a_i, sup\{u_j[a, b]\}) \geq b \]
for all \( i \in l \)

\[ j(l, sup\{u_j[a, b]\}) = inf\{j(a_i, sup\{u_j[a, b]\}) \geq b\} \]

Again by property (i)
\[ u_j(l, b) \leq sup\{u_j[a, b]\} \] (ii)

From equations (i) and (ii)
\[ u_j\left[inf\{a_i, b\}\right] = sup\{u_j[a, b]\} \]
(vii) $u_j \left( b, \inf_{i \in I} a_i \right) \leq \inf_{i \in I} u_j(b, a_i)$

Let $l = \inf_{i \in I} a_i \Rightarrow l \leq a_i$

$u_j(b, l) \leq u_j(b, a_i)$ for any $i \in I$ (by prop. 4)

Hence $u_j(b, l) \leq \inf_{i \in I} u_j(b, a_i)$

$u_j \left( b, \inf_{i \in I} a_i \right) \leq \inf_{i \in I} u_j(b, a_i)$

(viii) $u_j \left( b, \sup_{i \in I} a_i \right) = \sup_{i \in I} u_j(b, a_i)$

Let $s = \sup_{i \in I} a_i \Rightarrow s \geq a_i \Rightarrow a_i \leq s$

$u_j(b, a_i) \leq u_j(b, s)$ for any $i \in I$ (by prop. 4)

$\sup_{i \in I} u_j(b, a_i) \leq u_j(b, s)$ for any $i \in I$  

…………………………..(i)

On the other hand

Since $\sup_{i \in I} u_j(b, a_i) \geq u_j(b, a_{i_0})$ for all $i_0 \in I$

By property (i) we have

$\sup_{i \in I} \left[ b, \sup_{i \in I} u_j(b, a_i) \right] \geq a_{i_0}$

$\sup_{i \in I} \left[ b, \sup_{i \in I} u_j(b, a_i) \right] \geq s$

$u_j(b, s) \leq \sup_{i \in I} u_j(b, a_i)$

$u_j \left( b, \sup_{i \in I} a_i \right) \leq \sup_{i \in I} u_j(b, a_i)$

……………………………………..(ii)

From equations (i) and (ii) we get

$u_j \left( b, \sup_{i \in I} a_i \right) = \sup_{i \in I} u_j(b, a_i)$

(xi) $f[a, u_j(a, b)] \geq b$

Let $f[a, u_j(a, b)] = f[a, \inf \{ j(a, x) / j(a, x) \geq b \}]$

$= \inf \{ j(a, x) / j(a, x) \geq b \}$

$\geq b$

$\therefore f[a, u_j(a, b)] \geq b$

Hence the theorem.

**Definition**

Given a $t$-conorm $j$ and the associated operation $u_j$, the sup-$u_j$ composition, $u_j$ of fuzzy relation $P(X, Y)$ and $Q(Y, Z)$ is defined by the equation

$$(P \circ Q)(x, z) = \sup_{y \in Y} u_j[P(x, y), Q(y, z)]$$

………..(7)

For all $x \in X, z \in Z$.

Basic properties of the sup-$u_j$ composition are expressed by the following theorems.

**Theorem**

Let $P(X, Y), Q(Y, Z), R(X, Z)$ and $S(Z, V)$ be fuzzy relations. Then

1. The following properties are equivalent
   
   (i) $P \circ Q \supseteq R$  
   
   (ii) $Q \supseteq P^{-1} \circ R$  
   
   (iii) $P \supseteq (Q \circ R^{-1})^{-1}$
(2) \( u_j \) \( P \circ (Q \circ S) = (P \circ Q) \circ S \)

**Proof**

First let us prove (i) \( \Rightarrow \) (ii)

Assume that \( P \circ Q \supseteq R \)

Then by definition of inf- \( j \) composition we have

\[
\inf_{y \in Y} j(P(x, y), Q(y, z)) \supseteq R(x, z)
\]

\[
\Rightarrow j(P(x, y), Q(y, z)) \supseteq R(x, z) \quad \forall x \in X, y \in Y, z \in Z
\]

\[
\Rightarrow u_j(P_j^{-1}(y, x), R(x, z)) \subseteq Q(y, z) \quad \forall x \in X, y \in Y, z \in Z
\]

\[
\Rightarrow \sup_{y \in Y} u_j(P_j^{-1}(y, x), R(x, z)) \subseteq Q(y, z)
\]

\[
\Rightarrow (P_j^{-1} \circ R)(y, z) \subseteq Q(y, z)
\]

\[
Q \supseteq P_j^{-1} \circ R
\]

This proves (i) implies (ii)

Now we have to prove (ii) implies (iii)

Let

\[
Q \supseteq P_j^{-1} \circ R
\]

Then by definition of the sup- \( u_j \) composition we have

\[
\sup_{y \in Y} u_j(P_j^{-1}(y, x), R(x, z)) \subseteq Q(y, z)
\]

\[
\Rightarrow u_j(P_j^{-1}(y, x), R(x, z)) \subseteq Q(y, z) \quad \forall x \in X, y \in Y, z \in Z
\]

\[
\Rightarrow j(P(x, y), Q(y, z)) \supseteq R(x, z) \quad \forall x \in X, y \in Y, z \in Z
\]

\[
\Rightarrow j(Q_j^{-1}(z, y), P_j^{-1}(y, x)) \supseteq R_j^{-1}(z, x)
\]

\[
\Rightarrow u_j(Q_j^{-1}(z, y), R_j^{-1}(z, x)) \subseteq P_j^{-1}(y, x)
\]

\[
\Rightarrow \sup_{z \in Z} u_j(Q_j^{-1}(z, y), R_j^{-1}(z, x)) \subseteq P_j^{-1}(y, x)
\]

\[
\Rightarrow \left( Q \circ R_j^{-1} \right)(y, x) \subseteq P_j^{-1}(y, x)
\]

\[
\Rightarrow \left( Q \circ R_j^{-1} \right)^{-1}(x, y) \subseteq P(x, y)
\]

This proves (ii) implies (iii)

Now we have to prove (iii) implies (i)

Let us assume that

\[
P \supseteq (Q \circ R_j^{-1})^{-1}
\]

\[
\Rightarrow P_j^{-1} \supseteq Q \circ R_j^{-1}
\]

Then by equivalent preposition (8) \& (9) we have

\[
Q_j^{-1} \circ P_j^{-1} \supseteq R_j^{-1}
\]

\[
\Rightarrow P \circ Q \supseteq R \quad \text{This proves (iii) implies (i)}
\]

2. Now we have to prove that

\[
P \supseteq \left( \sup_{y \in Y} u_j \left( (P \circ Q) \circ S \right)(x, y) \right)
\]

\[
\left[ P \circ \left( Q \circ S \right) \right](x, v) = \sup_{y \in Y} \left[ P(x, y), \left( Q \circ S \right)(y, v) \right]
\]

By equivalent preposition in (1) we have
\[ j \left[ P^{-1}(y,x), \left( P \circ \left( Q \circ S \right) \right) \right](x,v) \supseteq \left( Q \circ S \right)(y,v) \]

\[ \sup_{z \in Z} u_{j} \left( Q(y,z), S(z,v) \right) \]

\[ j \left[ P^{-1}(y,x), \left( P \circ \left( Q \circ S \right) \right) \right](x,v) \supseteq u_{j} \left( Q(y,z), S(z,v) \right) \forall z \in Z \]

\[ u_{j} \left[ P(x,y), u_{j} \left( Q(y,z), S(z,v) \right) \right] \subseteq \left[ P \circ \left( Q \circ S \right) \right](x,v) \]

\[ j \left[ (P(x,y), Q(y,z), S(z,v)) \right] \subseteq \left[ P \circ \left( Q \circ S \right) \right](x,v) \]

\[ j \left[ j(P(x,y), Q(y,z)) \right]^{i}, \left[ P \circ \left( Q \circ S \right) \right](x,v) \supseteq S(z,v) \]

\[ u_{j} \left[ \left( P \circ Q \right)^{-1}(x,v), S^{-1}(v,z) \right] \subseteq j(P(x,y), Q(y,z)) \forall x \in X, y \in Y, z \in Z, v \in V \]

\[ \inf_{y \in Y} j(P(x,y), Q(y,z)) \]

\[ u_{j} \left[ \left[ P \circ \left( Q \circ S \right) \right](x,v), S^{-1}(v,z) \right] \subseteq \left( P \circ Q \right)(x,z) \]

\[ j \left[ \left[ P \circ \left( Q \circ S \right) \right]^{-1}(v,x), \left( P \circ Q \right) \right](x,z) \supseteq S^{-1}(v,z) \]

\[ j \left[ \left( P \circ Q \right)^{-1}(v,x), \left( P \circ \left( Q \circ S \right) \right) \right](x,v) \supseteq S(z,v) \]

\[ u_{j} \left[ \left( P \circ Q \right)(x,z), S(z,v) \right] \subseteq \left[ P \circ \left( Q \circ S \right) \right](x,v) \]

\[ \sup_{z \in Z} u_{j} \left[ \left( P \circ Q \right)(x,z), S(z,v) \right] \subseteq \left[ P \circ \left( Q \circ S \right) \right](x,v) \]

\[ \left[ \left( P \circ Q \right)^{u \circ S}(x,v) \right] \subseteq \left[ P \circ \left( Q \circ S \right) \right](x,v) \]

By the definition of \( u_{j} \) composition we have

\[ \left[ \left( P \circ Q \right)^{u \circ S}(x,v) = \sup_{z \in Z} \left[ \left( P \circ Q \right)(x,z), S(z,v) \right] \right] \]

\[ \sup_{z \in Z} \left[ \left( P \circ Q \right)(x,z), S(z,v) \right] \]

\[ j \left[ \left( P \circ Q \right)^{-1}(v,x), \left[ P \circ \left( Q \circ S \right) \right] \right](x,v) \supseteq S(z,v) \Rightarrow j \left[ \left[ \left( P \circ Q \right)^{u \circ S} \right](x,v), \left( P \circ Q \right) \right](x,z) \supseteq S^{-1}(v,z) \]

\[ u_{j} \left[ \left[ \left( P \circ Q \right)^{u \circ S} \right](x,v), S^{-1}(v,z) \right] \subseteq \left( P \circ Q \right)(x,z) \]

\[ u_{j} \left[ \left[ \left( P \circ Q \right)^{u \circ S} \right](x,v), S^{-1}(v,z) \right] \subseteq \inf_{y \in Y} j(P(x,y), Q(y,z)) \]

\[ j \left[ \left[ \left( P \circ Q \right)^{u \circ S} \right](x,v), \left[ j(P(x,y), Q(y,z)) \right] \supseteq S^{-1}(v,z) \right] \]
\[
\Rightarrow j\left[ j(P(x, y), Q(y, z)) \right] \supseteq \left[ (P \circ Q) \circ S \right](x, v) \supseteq S(z, v)
\]
\[
\Rightarrow u\left[ j(P(x, y), Q(y, z)), S(z, v) \right] \subseteq \left[ (P \circ Q) \circ S \right](x, v)
\]
\[
\Rightarrow u\left[ P(x, y), u_j(Q(y, z), S(z, v)) \right] \subseteq \left[ (P \circ Q) \circ S \right](x, v)
\]
\[
\Rightarrow j\left( P^{-1}(y, x), P^u(Q \circ S) \right)(x, v) \supseteq u_j(Q(y, z), S(z, v)) \quad \forall x \in X, y \in Y, z \in Z, v \in V
\]
\[
\Rightarrow j\left( P^{-1}(y, x), P^{u_j}(Q \circ S) \right)(x, v) \supseteq \sup_{z \in Z} u_j(Q(y, z), S(z, v))
\]
\[
\Rightarrow j\left( P^{-1}(y, x), \left[ (P \circ Q) \circ S \right](x, v) \right) \supseteq \left( Q \circ S \right)(y, v)
\]
\[
\Rightarrow u_j\left[ P(x, y), (Q \circ S)(y, v) \right] \subseteq \left[ (P \circ Q) \circ S \right](x, v)
\]
\[
\Rightarrow \sup_{y \in Y} u_j\left[ P(x, y), (Q \circ S)(y, v) \right] \subseteq \left[ (P \circ Q) \circ S \right](x, v)
\]
\[
\Rightarrow P^{u_j}\left[ Q \circ S \right](x, v) \subseteq \left[ (P \circ Q) \circ S \right](x, v)
\]

From equations (i) and (ii) we get
\[
P^{u_j}\left[ Q \circ S \right] = \left[ (P \circ Q) \circ S \right]
\]

This proves (2)

**Conclusion**

An attempt is made to prove important theorems using the newly defined inf- j and sup- u_j compositions. This can be further developed to solve problems with fuzzy relational equations.

**References**