Strategy of finding the value function for financial problem described by non-deterministic system

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\textbf{ABSTRACT}

The purpose of this paper is to find the value function for financial problem whose dynamic is described by non-deterministic differential equation.

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\section*{Introduction}

The theory of stopping time and stopping time problems are concerned with the problems of choosing a time to take a particular action, in order to maximize an expected reward or minimize an expected cost. The nondeterministic system with jump component has great importance and many applications in different areas of science and in real life, for example in modeling asset prices in finance. The general problem of this kind of processes with jumps is considered in the literature, for example see [1], [2]. This type of problems also can be formulated with delay see [3], [4].

Consider \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, P)\) to be a complete filtered probability space satisfying the usual conditions or usual hypotheses, on which we will define the following concepts.

\textbf{Random variable}: An n-dim random variable \(X\) is a measurable function from the set of all possible outcomes \(\Omega\) to \(\mathbb{R}^n\) that is \(X : \Omega \to \mathbb{R}^n\), \(\forall \omega \in \Omega\), \(X(\omega) \in \mathbb{R}^n\).

\textbf{Random Process}: Denotes by \(X_t\), is mathematical model defined by the map \(X : \Omega \times T \to \mathbb{R}^n\) such that \(\forall t \in T, \omega \to X(\omega, t)\) is measurable.

\textbf{Sample path}: for fixed \(\omega\), the sample path of the random process \(X_t\) is the map \(t \to X(t, \omega)\). A process \(X_t\) is said to be continuous time random process if it is sample path is continuous function otherwise \(X_t\) is called discontinuous random process.

\textbf{Stopping time}: The random variable \(\tau : \Omega \to \mathbb{R}^n\) is called a stopping time if \(\tau \leq t \leq f\) for all \(t\) in \(I\). In other word random process \(T\) is called a stopping time if you can tell when it happens.

Stopping time is often defined by a stopping rule or a mechanism for deciding whether to continue or to stop a process on the basis of the present position and past events, and which will almost always lead to a decision to stop at some finite time. Stopping times usually occur in decision theory, see [5].

\textbf{Lévy process}: An adopted random process \(X_t = \{X_t : t \geq 0\}\) is said to be a Lévy process if it satisfies the following properties:

1. \(X_0 = 0\) almost surely
2. Independence of increments: For any \(0 \leq s < t\), \(X_t - X_s\) is independent of \(f_s\) [increment independent of the past].
3. Stationary increments: For any \(s < t\), \(X_t - X_s\) is equal in distribution to \(X_{t-s}\).
4. Continuity in probability: For any \(\epsilon > 0\) and \(t \geq 0\) it holds that
   \[\lim_{h \to 0} P\left(\left|X_{t+h} - X_t\right| > \epsilon\right) = 0\]

If \(X_t\) is a Lévy process then one may construct a version of \(X_t\) such that \(t \to X_t\) is almost surely right continuous with left limits (cadlag).
Radom differential equation: (or stochastic differential equation), is differential equation in which one or more of the terms is random process, resulting in a solution which is itself a random process, see [6].

Itô diffusion: Let $X_t : \Omega \times \mathbb{T} \to \mathbb{R}^n$ defined on a probability space $(\Omega, \mathcal{F}, P)$ the process $X_t$ is Itô diffusion if it satisfying a random differential equation of the form:

$$dX_t = \alpha (X_t) dt + \beta (X_t) dB_t,$$

where $B_t$ is an $m$-dimensional Brownian motion and $\alpha : \mathbb{R}^n \to \mathbb{R}^n$ and $\beta : \mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}^m$ are the drift and diffusion fields respectively. Itô diffusion named after the Japanese mathematician Kiyoshi Itô.

Infinitesimal Generator: The infinitesimal generator $A = A_x$ of $X_t$ is the operator which is defined to act on suitable functions $f : \mathbb{R}^n \to \mathbb{R}$ by:

$$Af (x) = \lim_{t \to 0} \frac{E^t[f (X_t)] - f (x)}{t} , \quad x \in \mathbb{R}^n$$

where the limit exist.

The set function $f : \mathbb{R}^n \to \mathbb{R}$ such that the limit exist at $x$ denoted by $D_A (x)$ while $D_A$ denoted to the set of functions which their limits exist for all $x \in \mathbb{R}^n$, the infinitesimal generator of random process is a partial differential operator that encodes a great of information about the process.

One can show that any compactly-supported $C^2$ (twice differentiable with continuous second derivative) function $f$ lies in $D_A$ and that

$$Af (x) = \sum_i b_i (x) \frac{\partial f}{\partial x_i} (x) + \frac{1}{2} \sum_{i,j} \left( \sigma(x) \sigma(x)^T \right)_{i,j} \frac{\partial^2 f}{\partial x_i \partial x_j} (x),$$

or, in terms of the gradient and scalar and Frobenius inner products,

$$Af (x) = b (x) \cdot \nabla_x f (x) + \frac{1}{2} \left( \sigma(x) \sigma(x)^T \right) : \nabla_x \nabla_x f (x).$$

Itô processes: Let $B_t$ be 1-dimensional Brownian motion on $(\Omega, \mathcal{F}, P)$. An Itô process (also called random integral) is random processes $X_t$ on $(\Omega, \mathcal{F}, P)$ adopted to $f_t$ of the form:

$$dX_t = u (s,w) dt + v (s,w) dB_t,$$

Or in integral form:

$$X_t = X_0 + \int_0^t u (s,w) ds + \int_0^t v (s,w) dB_s.$$

Theorem (Itô formula): Ito’s formula is the fundamental theorem of stochastic calculus, just as one speaks of the fundamental theorem of ordinary integral/differential calculus.

Let $X_t$ be an Itô processes given by $dX_t = u_t dt + v_t dB_t$, let $g (x,t) \in C^2 \left( [0, \infty) \times \mathbb{R} \right)$, then $Y_t = g (t, X_t)$ is again Itô processes and

$$dY_t = \frac{\partial g (t, X_t)}{\partial t} dt + \frac{\partial g (t, X_t)}{\partial x} dX_t + \frac{1}{2} \frac{\partial^2 g (t, X_t)}{\partial x^2} (dX_t)^2.$$

For the proof see [7].

Assume that on the complete probability space $(\Omega, \mathcal{F}, P)$, the dynamics of the asset price $X(t)$ follows the differential equation with jump of the form:

$$dX(t) = \mu X(t) dt + \sigma X(t) dB_t + \int_0^\infty h(X_t,y)\tilde{N}(dt,dy) \quad (1)$$

With initial condition

$$X(0) = x > 0 \quad (2)$$
Where \( \mu(X), \sigma(X) \) and \( h(X) \) are real valued functions and \( \{\mathcal{B}\} \) is 1-dimensional Brownian motion with respect to \( \{\mathcal{F}_t\} \) and \( \dot{N}(dt, dy) \) is the Compensated Poisson random measure given by:

\[
\dot{N}(dt, dy) = N(dt, dy) - dt \pi(dy)
\]

Where \( \pi(dy) \) is the Levy measure associated to \( N \) and for integrability reason assume that:

\[
\int_{-1}^{\infty} (1 \wedge y^2) \pi(dy) < \infty \quad (3)
\]

\[
\int_{-1}^{\infty} y \pi(dy) < \infty \quad (4)
\]

we also assume that the size of a jump is greater than \(-1\), so that \( X(t) \) remain non-negative for all \( t \geq 0 \) a.s.

Suppose that \( g : \mathbb{R}^2 \to \mathbb{R} \) is continuous function such that:

\[
g(X_t) = e^{-\rho t} (X_t - c)
\]  

(5)

**Statement of the Problem**

Our considered problem is to find the value function \( \varphi \) and stopping time \( \tau \) such that the following supremum is attained:

\[
\Phi(s, x) = \sup_{\tau} E^{(s, x)} \left[ g(X_{\tau}) \cdot 1_{\{\tau < \infty\}} \right] = E^{(s, x)} \left[ e^{-\rho \tau} (X_{\tau} - c) \cdot 1_{\{\tau < \infty\}} \right] \quad (6)
\]

The supremum is being taken over all stopping time \( \tau \) where \( \tau^* \) is the optimal stopping time. The price \( X_t \) is modeled by the geometric Levy process, \( c > 0 \) is the transaction coast and \( \rho > 0 \) is the discounting factor.

If we can find a function \( \varphi \) then we have found a solution \( \varphi = \Phi \) and the stopping time \( \tau \).

**Verification theorem**

Consider that \( Q = \mathbb{R}^+ \times \mathbb{R} \)

Suppose we can find appositive function \( \varphi \) such that \( \varphi(s, x) \in C^{1,2} \) and satisfy the following conditions:

a) \( \varphi(s, x) \geq g(s, x) \) in \( Q \)

\[
\forall (s, x) \in Q \Rightarrow L \varphi(s, x) = \frac{\partial \varphi}{\partial s} + \mu(x) \frac{\partial \varphi}{\partial x} + \frac{1}{2} \sigma^2(x) \frac{\partial^2 \varphi}{\partial x^2}
\]

\[
+ \int_{-1}^\infty \left[ \varphi(s, x + h(x, y)) - \varphi(s, x) - h(x, y) \frac{\partial \varphi(s, x)}{\partial x} \right] \pi(dy) \leq 0.
\]

b) Assume that for \( T_R = T \wedge R \wedge \inf \{t > 0 \mid X_t > R\} \) and \( R > 0 \),
\[
\lim_{R \to \infty} E^{(s,x)}[\varphi (s + T_R), X(T_R)] = 0,
\]
then:
\[
\varphi(s,x) \geq \Phi(s,x) \quad \forall (s,x) \in Q
\]
(7)

Define the continuation region \(D\) as:
\[
D = \{(s,x) \in Q: \varphi(x) > g(x)\}
\]
(8)
d) The family \(\varphi(s,\tau), \tau \leq \tau_D\) is uniformly integrable \(w.r.t\) \(Q\).

e) Suppose \(L\varphi(s,x) = 0\) for all \((s,x) \in D\), and let \(\tau_D = \inf \{t > 0, X_t \in D\} < \infty\)

then:
\[
\varphi(s,x) = \Phi(s,x) \quad \forall (s,x) \in Q
\]
(9)

Proof

Verify Ito formula for semi martingale (see [8]), we have

\[
\varphi(s + T_R, X(T_R)) - \varphi(s, X(0)) = \int_0^{T_R} \left( \frac{\partial \varphi}{\partial s} + \mu(x) \frac{\partial \varphi}{\partial x} + \frac{1}{2} \sigma^2(x) \frac{\partial^2 \varphi}{\partial x^2} + \right. \\
\left. \int_{-\infty}^{\infty} \left[ \varphi(s,x + h(x,y)) - \varphi(s,x) - h(x,y) \frac{\partial \varphi(s,x)}{\partial x} \right] \pi(dy) \right) dt \\
+ \int_0^{T_R} \sigma(x) \frac{\partial \varphi}{\partial x} d B(t) + \int_0^{T_R} \left[ \varphi(s,x + h(x,y)) - \varphi(s,x) \right] \left[ \sigma(x) \frac{\partial \varphi}{\partial x} d B(t) + \int_0^{T_R} \left[ \varphi(s,x + h(x,y)) - \varphi(s,x) \right] \right] N(dt, dy)
\]

\[
+ \sum_{0 < t_j < T_R} \left\{ \varphi(t_j, x_{j-}) - \varphi(t_j, x_{j}) - \frac{\partial \varphi}{\partial x} \left( s + t_j, x_{j} \right) \Delta x_{i,j} \right\}
\]
(10)

the sum is taken over all jumping times \(t_j \in [0, T_R]\) and \(\Delta x_{i,j} = x_{i,j} - x_{i,j-}\).

When we apply the Dynkin’s formula to (10) (see [2]), we get

\[
E^{(s,x)}[\varphi(s + T_R, X(T_R))] = \varphi(s,x) + \int_0^{T_R} \left[ L\varphi(s,x) dt - g(s,x) \right] \\
+ \sum_{0 < t_j < T_R} \left\{ \varphi(t_j, X_{j-}) - \varphi(t_j, X_{j}) - \frac{\partial \varphi}{\partial x} \left( s + t_j, X_{j} \right) \Delta X_{i,j} \right\}
\]
(11)

from condition (b) we have
By using mean value property on the terms under the summation above we can see that:

\[
\Delta \phi(t_j, X_{t_j}) = \frac{\partial \phi}{\partial x}(s + t_j, \hat{X}_{t_j}) \Delta X_{t_j}
\]

(13)

For some point \( \hat{X}_{t_j} \) joining the two points \( X_{t_j} \) and \( X_{t_j}^- \) from (13) and condition (d) we get the following result

\[
\phi(s, x) \geq E^{(s,x)} \left[ g(s, x) + \phi(s + T_R, X(T_R)) \right]
\]

(14)

Now we use the limiting argument as \( R \to \infty \) to get

\[
\phi(s, x) \geq E^{(s,x)} \left[ g(X_s) \right] + \lim_{R \to \infty} E^{(s,x)} \left[ \phi(s + T_R, X(T_R)) \right]
\]

(15)

Using (iii) we get

\[
\phi(s, x) \geq E^{(s,x)} [g(X_s)] \geq \Phi(s, x)
\]

(16)

This proof the requirement in (7).

b- For the second part of the proof, consider \( D \) as in (iv) and when we apply conditions (iii) and (v) in (11) we get

\[
\phi(s, x) = E^{(s,x)} \left[ g(s, x) \right]
\]

(17)

This result together with (i) prove the requirements (9)

References