Semi smooth graceful graph and construction of new graceful trees
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ABSTRACT
In this paper we paper we define smooth graceful labeling and semi smooth graceful labeling for a graph. We also prove that a grid graph \( P_n \times P_m \) is smooth graceful graph and a star \( K_{1,n} \) is semi smooth graceful graph. Using this we proved that star of a star, star of smooth graceful tree and path union of a smooth graceful tree are graceful trees. We also get graceful labeling for comb graph and simple lobster graph.

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Introduction
The graceful labeling introduced by A. Rosa [7] in 1967. He proved that cycle \( C_n \) is graceful, when \( n \equiv 0,3 \) (mod 4). Bloom and Golomb [1] proved that complete graph \( K_n \) is graceful, when \( n \equiv 4 \) (mod 4). For detail survey of graph labeling one can refer Gallian [2]. Jin et al. [4] proved that join sum of graceful trees is graceful. Liu et al. [6] proved that radical product of graceful trees is graceful.

Kaneria and Jariya [5] define smooth graceful labeling for a graph and proved that cycle \( C_n \) (\( n \equiv 0 \) (mod 4)), path \( P_n \) and complete bipartite graph \( K_{2,n} \) are smooth graceful graphs. Using this they also proved a graph obtained by \( C_n \) \( m \equiv 2 \) (mod 4), and \( C_n \) (\( n \equiv 0 \) (mod 4)) with a path of arbitrary length is graceful as well as a graph obtained by joining \( C_m \) (\( m \equiv 0 \) (mod 4)) and \( W_n \) with a path of arbitrary length is also graceful.

We begin with a simple undirected finite graph \( G = (V, E) \) on \( p \) vertices and \( q \) edges. For all terminologies and notation we follows Harary [3]. First of all we shall recall some definitions which are useful for this paper.

Definition – 1.1 : A function \( f \) is called graceful labeling of a graph \( G = (V, E) \) if \( f : V \to \{0, 1, 2, \ldots, q\} \) is injective and the induce function \( \tilde{f} : E \to \{1, 2, \ldots, q\} \) defined as \( \tilde{f}(e) = |f(u) - f(v)| \) is bijective for every edge \( e = (u, v) \in E \). A graph \( G \) is called graceful graph if it admits a graceful labeling.

Definition – 1.2 : Let \( G \) be a graph and \( G_1, G_2, \ldots, G_n \) be \( n \) copies of graph \( G \). Then the graph obtained by adding an edge from \( G_i \) to \( G_{i+1} \) (\( i = 1, 2, \ldots, n-1 \)) is called path union of \( n \) copies of the graph \( G \).

Definition – 1.3 : A graph obtained by replacing each vertex of star \( K_{1,n} \) by a graph \( G \) of \( n \) vertices is called star of \( G \) and it is denoted by \( G^* \). The graph \( G \) which replaced at the center of \( K_{1,n} \) we call the central copy of \( G^* \).

Definition – 1.4 : A bipartite graceful graph \( G \) with graceful labeling \( f \) is said to be smooth graceful graph if it admits an injective map \( g : V \to \{0, 1, \ldots, \binom{n-2}{2} + 1, \ldots, q+1\} \) such that its induce edge labeling map \( g^* : E \to \{1+1, 2+1, \ldots, q+1\} \) defined as \( g^*(e) = |g(u) - g(v)| \), for every edge \( e = (u, v) \in E \). Since for any edge \( e = (u, v) \in E \) in \( P_n \times P_m \), one of \( f(u) \) and \( f(v) \) is less than \( \frac{q}{2} \) and another is greater than or equal to \( \frac{q}{2} \). We must have

\[
g^*(e) = \begin{cases} |g(u) - g(v)| & \text{if } f(u) < \frac{q}{2} \\ |f(u) - f(v)| + 1 & \text{if } f(u) \geq \frac{q}{2} \end{cases}
\]

and its induce edge labeling function \( g^* : E(P_n \times P_m) \to \{1+1, 2+1, \ldots, q+1\} \) defined by

\[
g^*(e) = |g(u) - g(v)|, \quad \text{for every edge } e = (u, v) \in E \).
\]

Theorem – 2.1 : \( P_n \times P_m \) is smooth graceful graph.

Proof : Let \( v_{ij} \) (\( 1 \leq i \leq n, 1 \leq j \leq m \)) be vertices of the grid graph \( P_n \times P_m \). We know that \( P_n \times P_m \) is graceful if it admits a graceful labeling function \( f : V(P_n \times P_m) \to \{0, 1, 2, \ldots, q\} \) such that its induce edge labeling map \( g^* : E(P_n \times P_m) \to \{1+1, 2+1, \ldots, q+1\} \) defined by

\[
g^*(e) = |g(u) - g(v)|, \quad \text{for every edge } e = (u, v) \in E \).
\]

Since for any edge \( e = (u, v) \in E \) in \( P_n \times P_m \), one of \( f(u) \) and \( f(v) \) is less than \( \frac{q}{2} \) and another is greater than or equal to \( \frac{q}{2} \). We must have

\[
g^*(e) = \begin{cases} |g(u) - g(v)| & \text{if } f(u) < \frac{q}{2} \\ |f(u) - f(v)| + 1 & \text{if } f(u) \geq \frac{q}{2} \end{cases}
\]

and its induce edge labeling function \( g^* : E(P_n \times P_m) \to \{1+1, 2+1, \ldots, q+1\} \) defined by

\[
g^*(e) = |g(u) - g(v)|, \quad \text{for every edge } e = (u, v) \in E \).
\]

Therefore \( g^*(E(P_n \times P_m)) = \{1+1, 2+1, \ldots, q+1\} \), which gives \( g^* \) is bijection. Hence \( P_n \times P_m \) is a smooth graceful graph.
Theorem – 2.2 : $K_{1,n}$ is a semi smooth graceful graph.

Proof : Let $v_0, v_1, \ldots, v_n$ be vertices of $K_{1,n}$. We know that $K_{1,n}$ is a bipartite graceful graph with graceful labeling $f : V(K_{1,n}) \rightarrow \{0,1,2, \ldots, n\}$ defined by

$$f(v_i) = 1, \quad \forall i=0,1,2, \ldots, n$$

Now we define $g : V(K_{1,n}) \rightarrow \{0,1+1,2+1, \ldots, n+1\}$ such that its induce edge labeling map $g' : E(K_{1,n}) \rightarrow \{1+1,2+1, \ldots, n+1\}$ defined by

$$g(u) = f(u), \quad \text{when } u = v$$

$$= f(u) + l, \quad \text{when } u \notin \{v_0, v_1, \ldots, v_n\}$$

Now for any $e = (u,v) \in E(K_{1,n})$,

$$g'(e) = g'(u,v) = |g(u) - g(v)|, \quad \text{for some } i \in \{1, 2, \ldots, n\}$$

$$= |0 - (f(v)+1)| = |0 - (f(v)+1)| = |0 - f(v)| + 1 = f'(i) = f'(0,0) + 1$$

Therefore $g'(E) = \{1+1, 2+1, \ldots, n+1\}$ and so it is a bijection.

Hence $K_{1,n}$ is a semi smooth graceful graph.

Theorem – 2.3 : Star of $K_{1,n}$ is a graceful tree.

Proof : Let $G$ be a graph formed by star of $K_{1,n}$, i.e. $G$ is a graph obtained by each of vertices of $K_{1,n}$ by $K_{1,n}$ itself. Let $v_0, v_1, \ldots, v_n$ be vertices of the central copy of $K_{1,n}$. Let $v_0, v_1, \ldots, v_n$ be vertices of $i$th copy of $K_{1,n}, \forall i=1,2,\ldots, n+1$. We define $f : V(K_{1,n}) \rightarrow \{0,1,2, \ldots, q\}$ where $q = n^2 + 3n + 1$ as follows.

$$f(v_0,0) = 0;$$

$$f(v_{i,0}) = q + i - 1, \quad \forall i=1,2, \ldots, n;$$

$$f(v_{0,i}) = f(v_{0,0}) - 1;$$

$$f(v_{i,0}) = f(v_{0,i}) + (n+1), \quad \forall i=1,2, \ldots, n;$$

$$f(v_{i,j}) = f(v_{i-1,j} + (n+1)), \quad \text{if } f(v_{i,j}) < \frac{q}{2}$$

$$\quad = f(v_{i-1,j} + n+1), \quad \text{if } f(v_{i,j}) \geq \frac{q}{2}$$

Now we see that the difference of vertex labels for central copy $K_{1,n}$, with its other copies $K_{1,n}^i$ ($1 \leq i \leq n+1$) is precisely following sequence.

$$(n+1)^2, (n+1), 2(n+1), (n-1)(n+1), \ldots, \left[\frac{\sqrt{n+1}}{2}\right](n+1).$$

Now join each vertex of central copy $K_{1,n}^{0}$ with its other copies $K_{1,n}^i$ by an edge to corresponding vertices. Above defined labeling function $f$ give rise graceful labeling to $K_{1,n}$ and so $G$ is a graceful tree.

Illustration – 2.4 : Stat of $K_{1,5}$ and its graceful labeling shown in figure – 1

Figure 1. A tree obtained by star of star ($K_{1,5}$) and its graceful labeling

Theorem – 2.5 : Path union of a smooth graceful tree is also a graceful tree.

Proof : Let $G$ be a path union of $t$ copies of a smooth graceful tree $T$, where $|V(T)|=p$ and $|E(T)|=q$. Let $f : V(T) \rightarrow \{0,1,2, \ldots, q\}$ be a smooth graceful labeling for $T$. Then its induce edge labeling function is $f' : E(T) \rightarrow \{1,2, \ldots, q\}$, which is bijection. Let $V(T) = \{v_1, v_2, \ldots, v_p\}$. Let $v_{i,1}, v_{i,2}, \ldots, v_{i,t}$ be vertices of $i$th copy of $G, \forall i=1,2, \ldots, t$. Then we define $g : V(G) \rightarrow \{0,1,2, \ldots, q\}$, where $q=nt+1$ as follows.

$$g(v_{i,1}) = f(v_{i,1});$$

$$= (q - q) + f(v_{i,1}), \quad \text{when } f(v_{i,1}) \leq \frac{q}{2}, \quad \forall i=1,2, \ldots, p.$$
$P_i$ join by the corresponding the middle vertex of each copy. So by last Theorem – 2.5 a simple lobster is a graceful tree.

**Illustration – 2.10**: A simple lobster of length 6 and its graceful labeling shown in figure – 4.

![Figure 4. A graph of a simple lobster of six length and its graceful labeling](image)

**Theorem – 2.11**: Let $T$ be smooth graceful tree. Then star of $T$ is also a graceful tree.

**Proof**: Let $G$ be a star of given smooth graceful tree $T$. Let $|V(T)|=p$, $|E(T)|=q$ and $V(T) = \{v_1, v_2, \ldots, v_p\}$. Let $f : V(T) \rightarrow \{0,1,2, \ldots , q\}$ be a smooth graceful labeling for $T$. Let $u_{0,i}$ ($1 \leq i \leq p$) be vertices of central copy $T^{(0)}$ of $G$ and $u_{l,i}$ ($1 \leq i \leq p$) be vertices of $l^{th}$ copy of $T$ in $G$, $\forall l=1, 2, \ldots , p$. We define a labeling function $g : V(G) \rightarrow \{0,1,2, \ldots , Q\}$, where $Q=(p+1)q+p$ as follows.

$$g(u_{0,i}) = f(v_i), \quad \text{when } f(v_i) < \left\lfloor \frac{Q}{2} \right\rfloor$$
$$g(u_{0,i}) = (Q - q) + f(v_i), \quad \text{when } f(v_i) \geq \left\lfloor \frac{Q}{2} \right\rfloor, \forall i=1, 2, \ldots, p;$$

$$g(u_{l,i}) = g(u_{l-1,i}) + (q+1), \quad \text{if } g(u_{l-1,i}) < \left\lfloor \frac{Q}{2} \right\rfloor$$
$$g(u_{l,i}) = (q+1) - (q+1), \quad \text{if } g(u_{l-1,i}) > \left\lfloor \frac{Q}{2} \right\rfloor, \forall i=1, 2, \ldots, p;$$

Now we see that the difference of vertex labels for the central copy $T^{(0)}$ with its other copies $T^{(i)}$ ($1 \leq i \leq p$) is precise the sequence

$$p(q+1), (q+1), (p-1)(q+1), 2(q+1), (p-2)(q+1), \ldots \ldots, \left\lfloor \frac{Q}{2} \right\rfloor(q+1).$$

Now join each vertex of central copy $T^{(0)}$ with it other copy $T^{(i)}$ by an edge to corresponding vertices. Above labeling function $g$ give rise graceful labeling to $G$ and so it is a graceful tree.

**Illustration – 2.12**: A star of a smooth graceful tree and its graceful labeling shown in figure – 5.

![Figure 5. A star of a smooth graceful tree and its graceful labeling](image)

**Concluding Remarks**

We have introduce a new graceful labeling namely semi smooth graceful labeling and proved that star $K_{1,n}$ ($n \in \mathbb{N}$) is a semi smooth graceful graph. Using this we have constructed some graceful trees by making star or path union of given graceful tree. Present work contribute few new results to the theory of graceful trees. The labeling pattern is demonstrated by suitable illustrations.

**References**


