b-chromatic number of some operations on Cycle and Path

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Abstract
A b-vertex coloring of a graph $G$ is a proper vertex coloring of $G$ such that each color class contains a vertex that has at least one vertex in every other color class in its neighborhood. The b-chromatic number of a graph $G$ is the largest integer $\phi(G)$ for which $G$ has a b-vertex coloring with $\phi(G)$ colors. This concept was introduced in [2] by Irving and Manlove by a certain partial ordering on all proper colorings in contrast to chromatic number $\chi(G)$, namely $\phi(G)$ is the minimum of colors used among all minimal elements of this partial ordering, while $\phi(G)$ is the maximum of colors used among all minimal elements of the same partial ordering. The b-chromatic number has been considered with respect to subgraphs in [10,11], while the b-chromatic number under graph operations was considered in [15] for the Cartesian product and in [8] for the other three standard products. Operations on graphs produce new ones from older ones. Here the paper deals with the b-chromatic number of adding parallel chords in Cycle, Union of Path with Cycle and its complement, deletion and addition of vertices and edges in a Cycle.

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1.1 Introduction
All graphs considered here are finite and simple. Notations and terminology not defined here will conform to those in [1]. For a graph $G$, let $V(G)$, $E(G)$, $p(G)$, $q(G)$ and $\mathcal{G}$, respectively, be the set of vertices, the set of edges, the order, the size and the complement of $G$. Let $G$ be a simple graph and suppose that we have a proper coloring of $G$ for which there exists a color class $c$ such that every vertex $v$ in $c$ is not adjacent to any vertex in at least one other color class; then we can separately change the color of each vertex in $c$ to obtain a proper coloring with fewer colors. Since then the b-chromatic number has drawn quite some attention among the scientific community. Already Irving and Manlove [2] have shown, that computing $\phi(G)$ is an NP-complete problem in general.

The b-chromatic number has drawn much attention in scientific area [5,6,7,8,9,10]. We can easily imagine the color classes as different communities, where every community $i$ has a representative that is able to communicate with all the others communities. Even though the b-chromatic number is a simple concept, it is hard to determine the exact values, even for known families of graphs. This lead to studies of lower and upper bounds, [13].

The b-chromatic number has been considered with respect to subgraphs in [11], while the b-chromatic number under graph operations was considered in [15] for the Cartesian product and in [8] for the other three standard products.

Operations on graphs produce new ones from older ones. Unary operations create a new graph from the old one. It creates a new graph from the original one by a simple or a local change, such as addition or deletion of a vertex or an edge, merging and splitting of vertices, edge contraction, etc.

Definition 1.1.1
A Chord of a cycle $C$ is an edge not in $C$ whose end vertices lie in $C$. The Disjoint union of graphs [31, 45] sometimes referred as simply graph union, which is defined as follows. Given two graphs $G_1$ and $G_2$, their union will be a graph such that $V(G_1 \cup G_2) = V(G_1) \cup V(G_2)$ and $E(G_1 \cup G_2) = E(G_1) \cup E(G_2)$.

Definition 1.1.2
The Complement $\mathcal{G}$ of a graph $G$ is defined as a simple graph with the same vertex set as $G$ and where two vertices $u$ and $v$ are adjacent only when they are not adjacent in $G$.

Definition 1.1.3
A closed walk with at least one edge in which no vertex except the terminal vertices appears more than once is called a cycle or circuit.

Definition 1.1.4
A cycle that has odd length is an odd cycle; otherwise it is an even cycle. A graph is acyclic if it contains no cycles; unicyclic if it contains exactly one cycle.

1.2 b-Chromatic Number of a Graph in Addition of Parallel Chords
1.2.1 Theorem
For any Cycle $C_n$, addition of parallel chords between non adjacent vertices holds the following statements:

- When $n$ is odd, there exists a unique 3 cycle and $\lceil \frac{n}{2} \rceil$ times 4 cycle.
- When $n$ is even, there exists exactly two 3 cycle and $\lfloor \frac{n}{2} \rfloor$ 2 times 4 cycle.

Proof
Let $C_n$ be a Cycle with $n$ vertices. Let $v_1, v_2, \ldots, v_n$ be the vertices and $e_1, e_2, \ldots, e_k$ be the edges of the Cycle $C_n$ with parallel chords, where $k$ is defined as

$$k = \left\{ \begin{array}{ll}
1 & \text{if } n \text{ is odd} \\
0 & \text{if } n \text{ is even}
\end{array} \right.$$
Case 1
When _n_ is an odd cycle, the mutually adjacent edges _e_i, e_{i+1}_ forms a 3 cycle and the remaining vertices are even, which forms \( \frac{n}{3} \) times 3 cycles. Thus, when _n_ is odd there exists a unique 3 cycle and \( \frac{n}{2} \) times 4 cycle.

**Example**

![Figure 1: C_7 with parallel chords](image)

**Theorem**
The Cycle C_7 with parallel chords has the b-Chromatic number four for every _n_ \( \geq 8 \).

**Example**

![Figure 2: C_10 with parallel chords](image)

1.3 b-Chromatic Number of a Graph when an Edge is removed
1.3.1 Theorem
For any Cycle \( (n \geq 5) \) with an edge \( e \in V(C_n) \), \( \varphi(C_n) = \varphi(C_n-e) \)

**Proof**
Let \( C_n \) be the Cycle of length _n_. Let \( v_1, v_2, ..., v_n \) be the vertices arranged in anticlockwise direction i.e. \( V(C_n) = \{v_1, v_2, ..., v_n\} \) and the edge set be denoted as \( E(C_n) = \{e_1, e_2, ..., e_n\} \). Here the vertex \( v_i \) is adjacent with the vertices \( v_{i+1} \) and \( v_{i+1} \) for \( i = 2, 3, ..., n-1 \). \( v_1 \) is adjacent with \( v_2, v_n \) and the vertex \( v_n \) is adjacent with \( v_{n-1} \) and \( v_1 \). We know that every Cycle is a connected graph with _n_ vertices. It is evident that b-Chromatic number of Cycle of length _n_ for \( n \geq 5 \) is 3. Suppose if we delete any edge from the Cycle, we obtain a Path graph of length \( n-1 \) with b-Chromatic number 3.

Therefore \( \varphi(C_n) = \varphi(C_n-e) \) for every \( n \geq 5 \).

**Example**

![Figure 4(a): \( \varphi(C_6)=3 \)](image)  
![Figure 4(b): \( \varphi(C_6-e)=3 \)](image)

1.3.2 Corollary
\( \varphi(C_n) \neq \varphi(C_n-e) \) for every \( n \leq 3 \).

**Example**

![Figure 5(a): C_3](image)  
![Figure 5(b): C_4-e](image)

1.3.3 Corollary
For any Path for \( n \geq 5 \), \( e \in V(P_n) \), \( \varphi(P_n) = \varphi(P_n-e) \)

1.4 b-Colouring of Adding a Pendant Vertex to each Vertex of a Cycle
1.4.1 Theorem
For any \( n \geq 6 \), \( \varphi(C_n \bullet K_1) = \varphi(W_n) \)

**Proof**
Let \( v_i \) for \( 1 \leq i \leq n \) are the vertices taken in the anticlockwise direction in the wheel graph \( W_n \) where \( v_n \) is the hub. It is clear that the vertex \( v_i \) is adjacent with the vertices \( v_{i-1} \) and \( v_{i+1} \) for \( i = 2, 3, ..., n-1 \), the vertex \( v_1 \) is adjacent with \( v_2 \) and \( v_{n-1} \), the vertex \( v_n \) is adjacent with all the vertices. Here every vertex except the hub is incident with three edges, so we assign four colours, which produces a maximum and b-chromatic colouring by the colouring procedure. Also we know that the b-chromatic number of any Cycle has three colours for \( n \geq 5 \). If we attach a pendant vertex to every vertex of Cycle \( C_n \), it is obvious that it has four colours for producing a b-chromatic colouring.
Therefore \( \varphi(C_n \bullet K_1) = \varphi(W_n) \)

Figure 6(a): \( \varphi(C_6 \ K_1) = 4 \)

Figure 6(b): \( \varphi(W_n) = 4 \)

1.4.2 Results obtained by Removing Edges from the Complete Graph

- \( \varphi(K_1 - e) = \varphi(C_3) = \varphi(K_{1,n}), (n \geq 2) \)
- \( \varphi(K_1 - e) = \varphi(C_3), (n \geq 2) \)
- \( \varphi(K_1 - 2e) = \varphi(W_n), (n \geq 6) \)
- \( \varphi(K_1 - 3e) = \varphi(C_n), (n \geq 5) \)

1.5 b-Chromatic Number of Union of Path with Cycle

1.5.1 Theorem

For any Path \( P_n \) and the Cycle \( C_n \) with \( n \) vertices, the b-chromatic number of \( P_n \cup C_n \) is given by \( \varphi(P_n \cup C_n) = n - 1 \) for \( n \geq 2 \).

Proof

Let \( G = P_n \cup C_n \) be the graph obtained by the union of subgraph \( P_n \) and \( C_n \) of a graph has the vertex set \( V(P_n) \cup V(C_n) \) and edge set \( E(P_n) \cup E(C_n) \).

Consider \( G = P_n \cup C_n \) whose vertex set \( V(G) = \{v_1, v_2, v_3, \ldots, v_{2n}\} \). Here in \( P_n \cup C_n \), we see that the vertex \( v_i \) is adjacent with the vertices \( v_{i+1} \) and \( v_{i-1} \) for \( i = 2, 3, \ldots, n-1, n+1, \ldots, 2n-2 \). \( v_i \) is adjacent with \( v_{2n} \) and \( v_{i} \) is adjacent with the vertices \( v_{i+1}, v_{i-1}, v_{i+2n-1} \) and \( v_{i+2n} \).

Now consider the graph \( G = P_n \cup C_n \). By the definition of Complement, for any graph \( G \), the non-adjacent vertices are adjacent in its complement. Here \( P_n \cup C_n \) contains \( 2n - 2 \) vertices as in \( G \). Arrange the vertices of \( 2n - 2 \) namely \( v_1, v_2, v_3, \ldots, v_{2n-2} \) in clockwise direction. Assign a proper colouring to these vertices as follows. Consider the colour class \( C = \{c_1, c_2, c_3, \ldots, c_n\} \). First assign the colour \( c_i \) to the vertex \( v_i \) for \( i = 1, 2, \ldots, 2n - 2 \), it will not produce a b-chromatic colouring, due to the above mentioned non-adjacency condition. Hence to make the colouring as b-chromatic one, assign the colour \( c_{i+1} \) to the vertices \( v_i \) and \( v_{i+1} \) for \( i = 1, 2, 3, \ldots, 2n - 2 \). Now all the vertices \( v_i \) for \( i = 1, 2, \ldots, 2n - 2 \) realizes its own colour, which produces a b-chromatic colouring. Furthermore it is the maximum colouring possible.

Example

Figure 7(a): \( P_5 \cup C_5 \)

Figure 7(b): \( P_5 \cup C_5 \)

1.5.2 Theorem

\( \varphi(P_n \cup C_n) = 3 \) for every \( n \geq 3 \)

Proof

The result is trivial from the above theorem.

1.5.3 Theorem

For any Path \( P_n \) and Cycle \( C_m \) with \( n \) and \( m \) vertices respectively, then \( \varphi(P_n \cup C_m) = 3 \) for \( n \geq 2 \).

Example

Figure 8: \( P_5 \cup C_{10} \approx 3 \)

1.5.4 Result

For any integer \( n > 2 \), \( \varphi(C_n) = \varphi(C_n) \).
Example

Figure 9(a): $\varphi(C_6)=3$

Figure 9(b): $\varphi(C_5)=3$

References