The Jordan Canonical Forms of Complex s-orthogonal and s-skew symmetric Matrices

S.Krishnamoorthy¹, and K.Jaikumar²

¹Department of Mathematics, Government Arts College (Autonomous), Kumbakonam, Tamil nadu, India.
²Department of Mathematics, Dharmapuram Adhinam Arts College Dharmapuram, Mayiladuthurai, Tamil nadu, India.

Abstract

We study the Jordan Canonical Forms of complex s-orthogonal and s-symmetric matrices, and consider some related results.

© 2014 Elixir All rights reserved.

Keywords

Jordan Canonical forms,
Complex s-orthogonal matrix,
Complex s-skew symmetric matrix.

Introduction

The study of secondary symmetric and secondary orthogonal matrices was initiated by Anna Lee [1] and [2]. In this paper we present some extended results of [3] in the context of s-orthogonal and s-skew symmetric matrices. We denote the space of \( n \times n \) matrices and complex matrices by \( \mathbb{M}_n \) and \( \mathbb{C} \) respectively. The secondary transpose of \( A \) is defined by \( A^s = VA^TV \), where ‘V’ is the fixed disjoint permutation matrix with units in its secondary diagonal.

Definition 1.1 [4]. Let \( A \in \mathbb{C} \)

a) The matrix \( A \) is called \( s \)-symmetric, if \( A^t = A \). That is \( A^tV = VA \).

b) The matrix \( A \) is called \( s \)-skew symmetric, if \( A^t = -A \). That is \( A^tV = -VA \).

c) The matrix \( A \) is called \( s \)-orthogonal, if \( AA^t = A^tA = I \). That is \( A^tVA = V \).

Basic Results

Our main objective is to present a new approach to the following classical characterization of the Jordan Canonical Forms of Complex s-orthogonal and s-skew symmetric matrices.

Theorem 2.1. A \( n \times n \) complex matrix is similar to a complex s-orthogonal matrix if and only if its Jordan Canonical Form can be expressed as a direct sum of matrices of only the following five types

(a). \( J_k(\lambda) \oplus J_k(\lambda^{-1}) \) for \( \lambda \in \mathbb{C} \setminus \{-1, 0, 1\} \) and any k,

(b). \( J_k(1) \oplus J_k(1) \) for any even k,

(c). \( J_k(-1) \oplus J_k(-1) \) for any even k,

(d). \( J_k(1) \) for any odd k, and

(e). \( J_k(-1) \) for any odd k.

Theorem 2.2. A \( n \times n \) complex matrix is similar to a complex s-skew symmetric matrix if and only if its Jordan Canonical Form can be expressed as a direct sum of matrices of only the following five types

(a). \( J_k(\lambda) \oplus J_k(-\lambda) \) for \( \lambda \in \mathbb{C} \setminus \{0\} \) and any k,

(b). \( J_k(0) \oplus J_k(0) \) for any even k, and

(c). \( J_k(0) \) for any odd k.

The Complex s-orthogonal

Lemma 3.1. Let \( A \in \mathbb{M}_n \) be nonsingular. The following are equivalent

(a). \( A \) is similar to a complex s-orthogonal matrix

(b). \( A \) is similar to a complex s-orthogonal matrix via a complex s-symmetric similarity
(c) there exists a nonsingular complex s-symmetric S such that \( A' = SAS^{-1} \) and
(d) there exists a nonsingular complex s-symmetric S such that \( A'SA = S \).

**Proof:** Assuming (a), suppose that X is nonsingular and \( XAX^{-1} = L \) is complex s-orthogonal. The algebraic polar decomposition ensures that there is a nonsingular complex s-symmetric G and a complex s-orthogonal Q such that \( X = GQ \).

Then \( L = XAX^{-1} = GQG^{-1}Q' \). so \( GAG^{-1} = Q'LQ \) is a product of complex s-orthogonal matrices and hence is complex s-orthogonal.

Assuming (b), suppose that \( A = GQG^{-1} \) for some complex s-symmetric G and complex s-orthogonal Q. Then

\[
A^{-1} = GQ'G^{-1} \text{ and } A' = G^{-1}Q'G = G^{-2}A^{-1}G^2,
\]

which is (c) with \( S = G^{-2} \).

Now assume (c) and write \( S = Y^*Y \) for some \( Y \in M_n \) so \( A' = SA^{-1}S^{-1} = Y^*YA^{-1}Y^{-1} \).

or \( (YAY^{-1})^r = Y^{-r}A^rY^{-1} = (YAY^{-1})^{-1} \): \( YAY^{-1} \) is therefore complex s-orthogonal and so (a) follows. The equivalence of (c) and (d) is clear.

**Lemma 3.2.** For any positive integer \( k \) and any \( \lambda \neq 0 \), \( J_k(\lambda) \oplus J_k(\lambda^{-1}) \) is similar to a complex s-orthogonal matrix.

**Lemma 3.3.** For any odd positive integer \( k \), each of \( J_k(1) \) and \( J_k(-1) \) is similar to a complex s-orthogonal matrix.

**Lemma 3.4.** Let \( r, k_1, \ldots, k_r \) be positive integers with \( k_i \) even, and suppose that \( k_1 > k_2 \geq \ldots \geq k_r \) if \( r > 1 \). Then neither \( J_{k_1}(1) \oplus \ldots \oplus J_{k_r}(1) \) nor \( J_{k_1}(-1) \oplus \ldots \oplus J_{k_r}(-1) \) is similar to a complex s-orthogonal matrix.

**Theorem 3.5.** Let \( r, k_1, \ldots, k_r \) and \( p, l_1, \ldots, l_p \) be positive integers with \( k_i \) and \( l_i \) even, suppose that \( k_1 > k_2 \geq \ldots \geq k_r \) if \( r > 1 \) and that \( l_1 > l_2 \geq \ldots \geq l_p \) if \( p > 1 \). Then \( J_{k_1}(1) \oplus \ldots \oplus J_{k_r}(1) \oplus J_{l_1}(-1) \oplus \ldots \oplus J_{l_p}(-1) \) is not similar to a complex s-orthogonal matrix.

**Lemma 3.6.** Let \( C \in M_{k_1} \) be similar to a complex s-orthogonal matrix. If \( B \oplus C \) is similar to a complex s-orthogonal matrix for some \( B \in M_n \), then \( B \) is similar to a complex s-orthogonal matrix.

**Theorem 3.7.** Let \( A \) be a complex s-orthogonal matrix. Then the even sized Jordan blocks of \( A \) corresponding to each of the Eigen values \(+1\) and \(-1\) are paired.

The s-skew symmetric

**Lemma 4.1.** A given \( A \in M_n \) is similar to a complex s-skew symmetric matrix if and only if there is a nonsingular s-symmetric \( S \) such that \( A' = -SAS^{-1} \).

**Lemma 4.2.** For any positive integer \( k \) and any \( \lambda \in \mathbb{C} \), \( J_k(\lambda) \oplus J_k(-\lambda) \) is similar to a s-skew symmetric matrix.

**Lemma 4.3.** For any odd positive integer \( k \), \( J_k(0) \) is similar to a s-skew symmetric matrix.

**Lemma 4.4.** Let \( r, k_1, \ldots, k_r \) be positive integers with \( k_i \) even, and suppose that \( k_1 > k_2 \geq \ldots \geq k_r \) if \( r > 1 \). Then neither \( J_{k_1}(0) \oplus \ldots \oplus J_{k_r}(0) \) is similar to a s-skew symmetric matrix.

**Lemma 4.5.** Let \( C \) be similar to a complex s-skew symmetric matrix. If \( B \oplus C \) is similar to a s-skew symmetric matrix, then \( B \) is also similar to a s-skew symmetric matrix.

**Theorem 4.6.** Let \( A \in M_n \) be s-skew symmetric. Then the even sized singular Jordan blocks of \( A \) are paired.

Reference: